A Log-Robust Optimization Approach to Portfolio Management

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Outline

Motivation – The LogNormal Model

- Black and Scholes (1973).
- If there is no correlation, random stock price of asset i at time T, $S_i(T)$, is given by:

$$\ln \frac{S_i(T)}{S_i(0)} = \left(\mu_i - \frac{\sigma_i^2}{2} \right) T + \sigma_i \sqrt{T} Z_i.$$

where Z_i obeys a standard Gaussian distribution, i.e., $Z_i \sim N(0, 1)$, and:

T : the length of the time horizon,
$S_i(0)$: the initial (known) value of stock i,
μ_i : the drift of the process for stock i,
σ_i : the infinitesimal standard deviation of the process for stock i,

- Widely used in industry, especially for option pricing.

Motivation (Cont'd)

- Other distributions have been investigated by:
  - Fama (1965),
  - Blattberg and Gonedes (1974),
  - Kon (1984),
  - Jansen and deVries (1991),
  - Richardson and Smith (1993),
  - Cont (2001).
- In real life, the distribution of stock prices have fat tails (Jansen and deVries (1991), Cont (2001))

Motivation (Cont'd)

- Jansen and deVries (1991) states:

  " Numerous articles have investigated the distribution of share prices, and find that the returns are **fat-tailed.** Nevertheless, there is still controversy about the amount of probability mass in the tails, and hence about the most appropriate distribution to use in modeling returns. This controversy has proven **hard to resolve.**"

- The Gaussian distribution in the Log-Normal model leads the manager to take more risk than he is willing to accept.

Motivation (Cont'd)

- Numerous studies suggest that the continuously compounded rates of return are indeed the true drivers of uncertainty.
- There does not seem to be one good distribution for these rates of return.
- Managers want to protect their portfolio from adverse events.
- This makes **robust optimization** particularly well-suited for the problem at hand.

Robust Optimization:

- Models random variables as uncertain parameters belonging to known intervals.
- Optimizes the worst-case objective.
- All (independent) random variables are not going to reach their worst case simultaneously! They tend to cancel each other out. (Law of large numbers.)
- Key to the performance of the approach is to take the worst case over a "reasonable uncertainty set."
- Tractability of max-min approach depends on the ability to rewrite the problem as one big maximization problem using strong duality.
- Setting of choice: objective **linear** in the uncertainty.

Robust Optimization (Cont'd)

- Theory of Robust Optimization:
  - Ben-Tal and Nemirovski (1999),
  - Bertsimas and Sim (2004).
- Applications to Finance:
  - Bertsimas and Pachamanova (2008).
  - Fabozzi et. al. (2007).
  - Pachamanova (2006).
  - Erdogan et. al. (2004).
  - Goldfarb and Iyengar (2003).

Robust Optimization (Cont'd)

- All the researchers who have applied robust optimization to portfolio management before us have modeled the **returns** $S_i(T)$ as the uncertain parameters.
- This matters because of the nonlinearity (exponential term) in the asset price equation.
- To the best of our knowledge, we are the first ones to apply robust optimization to the true drivers of uncertainty.

Contributions

- We incorporate randomness on the continuously compounded rates of return using range forecasts and a budget of uncertainty.
- We maximize the worst-case portfolio value at the end of the time horizon in a one-period setting.
- We derive a tractable robust formulation, specifically, a linear programming problem, with only a moderate increase in the number of constraints and decision variables.
- We gain insights into the worst-case scaled deviations and the structure of the optimal strategy.

Portfolio Management with Independent Assets

We use the following notation:

$$
\begin{aligned}
n &: \quad \text{the number of stocks,} \\
T &: \quad \text{the length of the time horizon,} \\
S_i(0) &: \quad \text{the initial (known) value of stock } i, \\
S_i(T) &: \quad \text{the (random) value of stock } i \text{ at time } T, \\
w_0 &: \quad \text{the initial wealth of the investor,} \\
\mu_i &: \quad \text{the drift of the process for stock } i, \\
\sigma_i &: \quad \text{the infinitesimal standard deviation of the} \\
&\quad\;\; \text{process for stock } i, \\
\tilde{x}_i &: \quad \text{the number of shares invested in stock } i, \\
x_i &: \quad \text{the amount of money invested in stock } i.
\end{aligned}
$$

Problem Formulation

- Assumptions:
  - Short sales are not allowed.
  - All stock prices are independent.
- Example: manager invests in asset classes such as gold and real estate.
- In the traditional Log-Normal model, the random stock price i at time T, $S_i(T)$, is given by:

$$\ln \frac{S_i(T)}{S_i(0)} = \left(\mu_i - \frac{\sigma_i^2}{2}\right) T + \sigma_i \sqrt{T} Z_i.$$

- Z_i obeys a standard Gaussian distribution, i.e., $Z_i \sim N(0, 1)$.

Problem Formulation (Cont'd)

- We model Z_i as uncertain parameters with nominal value of zero and known support$[-c, c]$ for all i.

$$Z_i = c\,\tilde{z}_i,$$

- $\tilde{z}_i \in [-1, 1]$ represents the *scaled deviation* of Z_i from its mean, which is zero.

- Budget of uncertainty constraint:

$$\sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma,$$

Problem Formulation (Cont'd)

The robust portfolio management problem can be formulated as:

$$
\begin{aligned}
\max_{x} \quad & \min_{\tilde{z}} \sum_{i=1}^{n} \tilde{x}_i S_i(0) \exp\left[(\mu_i - \frac{\sigma_i^2}{2}) T + \sigma_i \sqrt{T} c \tilde{z}_i \right] \\
& \text{s.t.} \ \sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma, \\
& \qquad |\tilde{z}_i| \leq 1 \ \forall i, \\
\text{s.t.} \quad & \sum_{i=1}^{n} \tilde{x}_i S_i(0) = w_0. \\
& \tilde{x}_i \geq 0 \ \forall i,
\end{aligned}
$$

Problem Formulation (Cont'd)

With $x_i = S_i(0)\,\tilde{x}_i$ amount of money invested in stock i at time 0 for all i:

$$\begin{aligned}
\max_{x} \quad & \min_{\tilde{z}} \sum_{i=1}^{n} x_i \exp\left[(\mu_i - \frac{\sigma_i^2}{2})T + \sigma_i \sqrt{T} c \tilde{z}_i\right] \\
& \quad \text{s.t.} \sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma, \\
& \quad\quad\quad |\tilde{z}_i| \leq 1 \ \forall i, \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0. \\
& x_i \geq 0 \ \forall i.
\end{aligned}$$

The problem is **linear** in the asset allocation and nonlinear but **convex** in the scaled deviations.

Tractable Reformulation

Theorem (Optimal wealth and allocation)

(i) The optimal wealth in the robust portfolio management problem is: $w_0 \exp(F(\Gamma))$*, where* F *is the function defined by:*

$$
\begin{aligned}
F(\Gamma) = \max_{\eta,\chi,\xi} \quad & \sum_{i=1}^{n} \chi_i \ln k_i - \eta\,\Gamma - \sum_{i=1}^{n} \xi_i \\
s.t. \quad & \eta + \xi_i - \sigma_i \sqrt{T}\, c\, \chi_i \geq 0, \ \forall i, \\
& \sum_{i=1}^{n} \chi_i = 1, \\
& \eta \geq 0,\ \chi_i,\ \xi_i \geq 0,\ \forall i.
\end{aligned}
$$

(ii) The optimal amount of money invested at time 0 *in stock* i *is* $\chi_i\, w_0$*, for all* i*.*

Structure of the optimal allocation

Theorem (Structure of the optimal allocation)

Assume assets are ordered in decreasing order of the stock returns without uncertainty $k_i = \exp((\mu_i - \sigma_i^2/2)T)$ (i.e., $k_1 > \cdots > k_n$), and it is strictly suboptimal to invest in only one stock.
There exists an index j such that he optimal asset allocation is given by:

$$x_i = \begin{cases} \dfrac{1/\sigma_i}{\sum_{a=1}^{j} 1/\sigma_a} w_0, & i \leq j, \\ 0, & i > j. \end{cases}$$

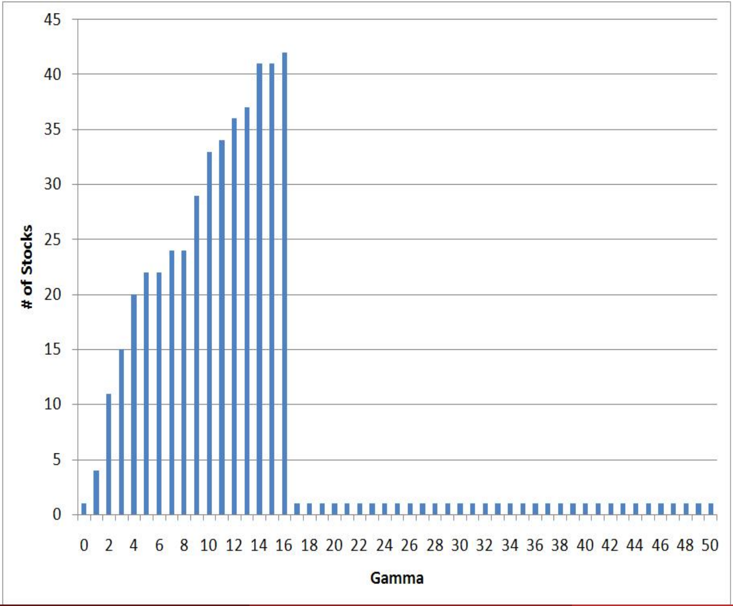
Remarks

- $x_i\,\sigma_i$ is constant for all the assets the manager invests in.
- The robust optimization **selects the number of assets** j the manager will invest in.
- When the manager invests in all assets, the allocation is similar to Markovitz's allocation but the σ_i have a different meaning.
- Assume it is strictly suboptimal to invest in only one stock. Then the scaled deviations for the assets the manager invests in **never reach their bounds,** i.e., $0 < z_i < 1$ for i such that $x_i > 0$ at optimality.

Diversification

- How does the budget of uncertainty Γ affect diversification?
- When assets are uncorrelated, $x_i = \frac{\eta}{\sigma_i \sqrt{Tc}} w_0$ for $x_i > 0$.
- η decreases with Γ, but we must still have $\sum_{i=1}^{n} x_i = w_0$.
- This means j increases with Γ, until η becomes zero and we invest in the stock with the highest worst-case return only.

Number of stocks in optimal portfolio vs Γ



Portfolio Management with Correlated Assets

- The behavior of stock prices, is replaced by:

$$\ln \frac{S_i(T)}{S_i(0)} = \left(\mu_i - \frac{\sigma_i^2}{2}\right) T + \sqrt{T} Z_i,$$

  where the random vector Z is normally distributed with mean $\mathbf{0}$ and covariance matrix $\mathbf{Q}$.

- We define:

$$\mathbf{Y} = \mathbf{Q}^{-1/2}\mathbf{Z},$$

  where $\mathbf{Y} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ and $\mathbf{Q}^{1/2}$ is the square-root of the covariance matrix $\mathbf{Q}$, i.e., the unique symmetric positive definite matrix $\mathbf{S}$ such that $\mathbf{S}^2 = \mathbf{Q}$.

Formulation (Cont'd)

The robust optimization model becomes:

$$
\begin{aligned}
\max_{x} \quad & \min_{\tilde{y}} \quad \sum_{i=1}^{n} x_i \exp\left[\left(\mu_i - \sigma_i^2/2\right) T + \sqrt{T} c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\, \tilde{y}_j\right)\right] \\
& \text{s.t.} \quad \sum_{j=1}^{n} |\tilde{y}_j| \le \Gamma, \\
& \qquad\quad |\tilde{y}_j| \le 1, \;\; \forall j, \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0, \\
& x_i \ge 0, \;\; \forall i.
\end{aligned}
$$

Theorem (Optimal wealth and allocation)

(i) The optimal wealth in the robust portfolio management problem with correlated assets is: $w_0 \exp(F(\Gamma))$, *where* F *is the function defined by:*

$$
\begin{aligned}
F(\Gamma) = \max_{\eta,\chi,\xi} \quad & \sum_{i=1}^{n} \chi_i \ln k_i - \eta\,\Gamma - \sum_{i=1}^{n} \xi_i \\
\text{s.t.} \quad & \eta + \xi_i - \sqrt{T}\, c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\, \chi_j \right) \geq 0, \;\; \forall i, \\
& \eta + \xi_i + \sqrt{T}\, c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\, \chi_j \right) \geq 0, \;\; \forall i, \\
& \sum_{i=1}^{n} \chi_i = 1, \\
& \eta \geq 0, \; \chi_i, \, \xi_i \geq 0, \;\; \forall i.
\end{aligned}
$$

(ii) The optimal amount of money invested at time 0 *in stock* i *is* $\chi_i\, w_0$*, for all* i*.*

Diversification

- We observe that the number of stocks invested in increases and then decreases with Γ, before it becomes optimal to invest in the stock with the highest worst-case return.
- The decrease for high levels of aversion to ambiguity is due to the correlation.

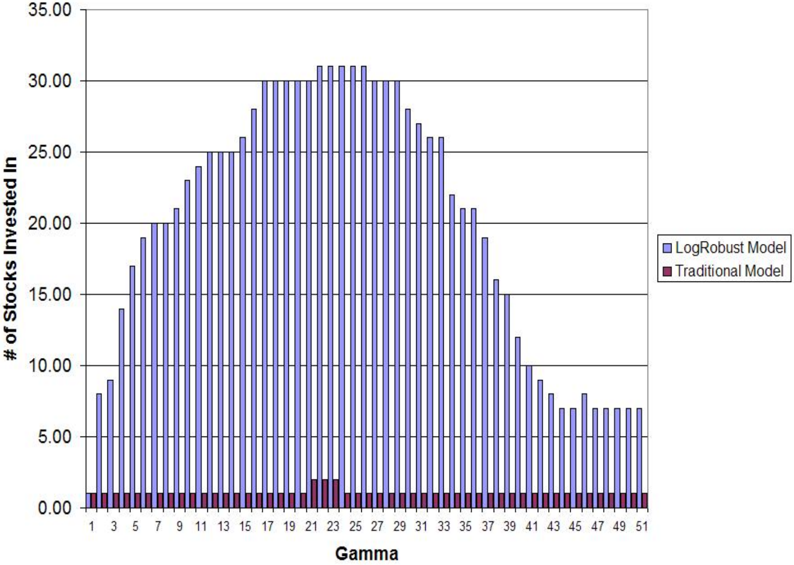
Numerical Experiments

Goal: to compare the proposed Log-robust approach with the robust optimization approach that has been traditionally implemented in portfolio management.

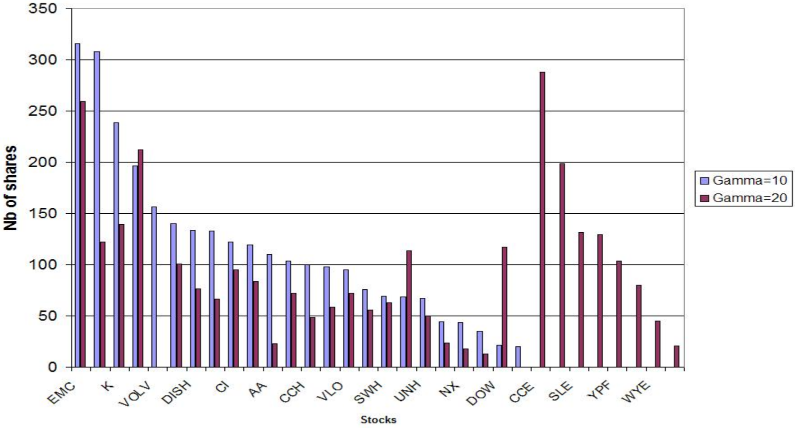
We will see that:

- The Log-robust approach yields far greater diversification in the optimal asset allocation.
- It outperforms the traditional robust approach, when performance is measured by percentile values of final portfolio wealth, if:
  - The budget of uncertainty parameter is relatively small, or
  - The percentile considered is low enough.
- This means that the Log-robust approach shifts the **left** tail of the wealth distribution to the right, compared to the traditional robust approach; how much of the whole distribution ends up being shifted depends on the choice of the budget of uncertainty.

Number of stocks in optimal portfolio vs Γ



Number of shares in optimal Log-robust portfolio for $\Gamma = 10$ and $\Gamma = 20$

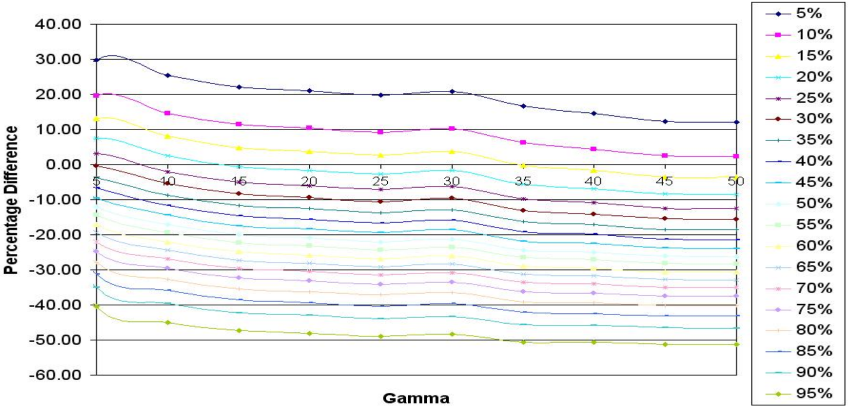


Numerical Experiments (Cont'd)

Γ	Traditional	Log-Robust	Relative Gain
5	70958.81	107828.94	51.96%
10	70958.81	104829.93	47.73%
15	70958.81	102502.79	44.45%
20	70958.81	101707.00	43.33%
25	70958.81	100905.96	42.40%
30	70958.81	101763.58	43.41%
35	70958.81	98445.23	38.74%
40	70958.81	96120.18	35.46%
45	70958.81	94253.62	32.83%
50	70958.81	94032.09	32.52%

Table: 99% VaR as a function of Γ for Gaussian distribution.

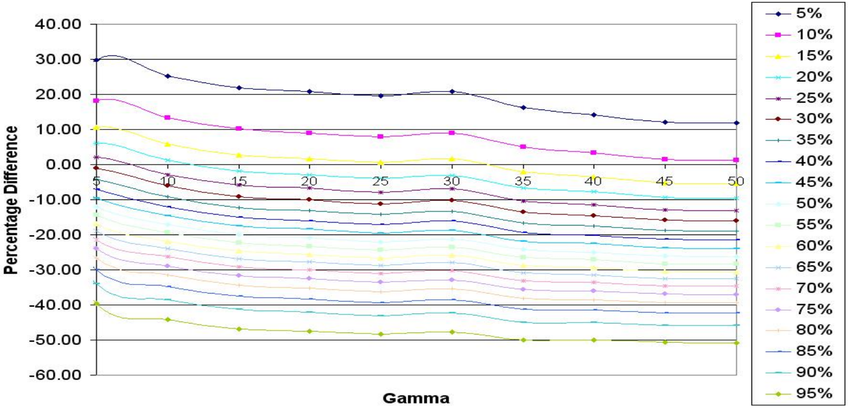
Relative gain of the Log-robust model compared to the Traditional robust model - Gaussian Distribution



Γ	Traditional	Log-Robust	Relative Gain
5	68415.97	108234.32	58.20%
10	68415.97	105146.66	53.69%
15	68415.97	102961.66	50.49%
20	68415.97	102124.75	49.27%
25	68415.97	101294.347	48.06%
30	68415.97	102206.73	49.39%
35	68415.97	98508.69	43.98%
40	68415.97	95940.01	40.23%
45	68415.97	93841.05	37.16%
50	68415.97	93562.59	36.76%

Table: 99% VaR as a function of Γ for Logistic distribution.

Relative gain of the Log-robust model compared to the Traditional robust model - Logistic Distribution



Conclusions

- We have presented an approach to uncertainty in stock prices returns that does not require the knowledge of the underlying distributions.
- It builds upon observed dynamics of stock prices while addressing limitations of the Log-Normal model.
- It leads to a tractable linear robust formulation.
- The model is more aligned with the finance literature than the traditional robust model that does not address the true uncertainty drivers.
- The model exhibits better performance for the ambiguity-averse manager, in particular due to increased diversification.
- We believe the Log-robust approach holds much potential in portfolio management under uncertainty.