MATH 23, SPRING 2002, HOUR TEST # 1

SOLUTION

1. The given plane can be taken to have normal vector $\mathbf{N} =< 1, -3, 6 >$, and our plane can be taken to have the same normal vector since it is supposed to be parallel to the given plane. Therefore, our plane will have equation $(x-4) - 3(y-3) + 6(z+1) = 0$. Just like Exercises 23 - 25, in Section 12.5 (p. 819).

2. The approach, here, is that the desired distance is $|comp_{\mathbf{n}}\mathbf{b}|$, where $\mathbf{n}$ is the normal vector to our plane and $\mathbf{b}$ is the vector corresponding to the directed line segment from an arbitrary point in our plane, to the given point $P(10, 10, 1)$. So $\mathbf{n}$ can be taken to be $< 3, -2, 6 >$. For the arbitrary point in our plane we can $x = y = z = 1$ (other choices are possible, but this seems simplest), so we take $\mathbf{b} =< 9, 9, 0 >$, and we have $D = \dfrac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|}$, i.e., $D = \dfrac{27 - 18 + 0}{9\sqrt{2}} = \dfrac{1}{\sqrt{2}}$. Like Exercises 59 - 62, in Section 12.5, p. 820.

3. The approach here is to take $\mathbf{v}_1$ to be the vector corresponding to the directed line segment from P_1 to P_2, take $\mathbf{v}_2$ to be the vector corresponding to the directed line segment from P_1 to P_3, take $\mathbf{N} = \mathbf{v}_1 \times \mathbf{v}_2$ to be the normal vector to our plane, and take the point to be P_1 (for example ... other choices are possible). So, we have $\mathbf{v}_1 =< 3, -4, -1 >$, $\mathbf{v}_2 =< 2, -3, 0 >, \mathbf{N} = \mathbf{i}(0-3) - \mathbf{j}(0+2) + \mathbf{k}(-9+8) = < -3, -2, -1 >$, so our plane will have equation $-3(x-1) - 2(y-3) - (z-2) = 0$. Like Exercises 27 - 29 in Section 12.5, p. 819.

4. The equation of the line gives us $x = 1 + t,\ y = 1 - t,\ z = 3t$, so, substituting these into the equation of the plane, we get $(1+t) - 3(1-t) - 3t = 2$, i.e. $-2+t = 2$, so $t = 4$, and so (substituting back into the equation of the line) $x = 5,\ y = -3,\ z = 12$. Like Exercises 35 - 38, Section 12.5, p. 819.

5. (10 points) Multiplying both sides of the equation by ρ, we get $\rho^2 = 6\rho cos(\phi)$, or $x^2+y^2+z^2 = 6z$, or $x^2+y^2+(z^2-6z) = 0$, or $x^2+y^2+(z^2-6z+9) = 9$, or, finally, $x^2+y^2+(z-3)^2 = 3^2$, so this is the sphere of radius 3 centered at $(0,0,3)$. Like Exercise 42 in Section 12.7, p. 831.

6. The tangent line can be taken to have direction vector $\mathbf{r}'(0)$ (**and YES, it is absolutely essential to evaluate at 0, it is meaningless to leave** $\mathbf{r}'(t)$**, even if one takes care to use a different letter for the parameter of the tangent line)** and pass through the point $\mathbf{r}(0)$. Since $\mathbf{r}'(t) = \; < e^t, 2, 3t^2 >$, $\mathbf{r}'(0) =< 1,2,0 >$, and $\mathbf{r}(0) =< 1,1,0 >$, so the tangent line has parametric equations $x = 1+s$ $y = 1+2s$, $z = 0$. Like Exercises 23 - 26, in Section 13.2, p. 848.

7 & 8. The things that are easy to calculate are as follows:

(1) $\mathbf{v}(t) = \mathbf{r}'(t) =< cos(t), -sin(t), 3t^2 >$,

(2) $v(t) = |\mathbf{v}(t)| = \sqrt{(cos(t))^2 + (sin(t))^2 + 9t^4} = \sqrt{1+9t^4}$ (while (mercifully) no points were deducted for not simplifying $(cos(t))^2 + (sin(t))^2 = 1$, points **WERE** deducted for simplifying it horribly incorrectly to be 0).

(3) $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) =< -sin(t), -cos(t), 6t >$,

(4) $\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{v(t)} = \dfrac{< cos(t), -sin(t), 3t^2 >}{\sqrt{1+9t^4}}$.

For the remaining items, $\mathbf{N}(t)$, $\kappa(t)$, and the tangential and normal components of acceleration, $a_T(t)$ and $a_N(t)$, probably the most efficient way of proceeding is as follows.

(5) Compute $a_T(t)$ according to the formula $a_T(t) = \dfrac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{v(t)}$,

(6) Compute $a_N(t)$ according to the formula $a_N(t) = \dfrac{|\mathbf{v}(t)\times\mathbf{a}(t)|}{v(t)}$,

(7) Compute $\kappa(t)$ according to the formula $\kappa(t) = \dfrac{a_N(t)}{(v(t))^2}$.

This yields (substituting results from (1) - (4) into (5) - (7)):

(5*) $a_T(t) = \dfrac{-cos(t)sin(t) + cos(t)sin(t) + 18t^3}{\sqrt{1+9t^4}} = \dfrac{18t^3}{\sqrt{1+9t^4}}$,

(6*) $a_N(t) = \dfrac{|\mathbf{i}(-6tsin(t)+3t^2cos(t)) - \mathbf{j}(6tcos(t)+3t^2sin(t)) + \mathbf{k}(-((cos(t))^2) - (sin(t))^2)|}{\sqrt{1+9t^4}} =$

$\dfrac{|< (-6tsin(t)+3t^2cos(t)), (-6tcos(t)-3t^2sin(t)), -1 >|}{\sqrt{1+9t^4}} =$

(after some simplifying algebra, using, among other things, once again, that $(cos(t))^2 + (sin(t))^2 = 1$)

$= \dfrac{\sqrt{9t^4+36t^2+1}}{\sqrt{1+9t^4}}$,

(7*) $\kappa(t) = \dfrac{\sqrt{9t^4+36t^2+1}}{(1+9t^4)^{1.5}}$.

It remains to find $\mathbf{N}(t)$, the hardest part of the problem. Based on the above, the most efficient route is to use that

$$(8)\quad \mathbf{a}(t) = a_T(t)\mathbf{T}(t) + a_N(t)\mathbf{N}(t).$$

This can be reorganized as

$$(9)\quad \mathbf{N}(t) = \frac{\mathbf{a}(t) - a_T(t)\mathbf{T}(t)}{a_N(t)}.$$

Substituting from (3), (4), (5*) and (6*) into (9), and using that $a_T(t)\mathbf{T}(t) = \dfrac{18t^3\mathbf{v}(t)}{(1+9t^4)}$, we get:

$$(9*)\mathbf{N}(t) = \frac{(1+9t^4) < -sin(t), -cos(t), 6t > -18t^3 < cos(t), -sin(t), 3t^2 >}{\sqrt{1+9t^4}\sqrt{9t^4+36t^2+1}}.$$

Other approaches involve doing more differentiation, and less vector algebra. Thus, one could use that $a_T(t) = (v(t))'$ **which is NOT the same thing as** $|\mathbf{v}'(t)|$, so $a_T(t) = (\sqrt{1+9t^4})' =$ (as before) $1/2(1+9t^4)^{-(1/2)}36t^3$, and that $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{((v(t))^{-1}\mathbf{v}(t))'}{|((v(t))^{-1}\mathbf{v}(t))'|}$ = the unit vector in direction of $-((v(t))^{-2}(18t^3)(v(t))^{-1}\mathbf{v}(t) + (v(t))^{-1}\mathbf{a}(t)$ = the unit vector in direction of $-18t^3(1+9t^4)^{-(3/2)} < cos(t), -sin(t), 3t^2 > +(1+9t^4)^{-(1/2)} < -sin(t), -cos(t), 6t >$. Then one could even use that $a_T(t) = \mathbf{a}(t)\cdot\mathbf{T}(t)$ and $a_N(t) = \mathbf{a}(t)\cdot\mathbf{N}(t)$, and that $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{v(t)}$. There are still other combinations of these two "least differentiation" and "most differentiation" approaches, but most of the steps involved should have occurred somewhere above. It would also have been possible to use that $\mathbf{B}(t)$ is the unit vector in the direction of $\mathbf{v}(t)\times\mathbf{a}(t)$ and that $\mathbf{N}(t) = \mathbf{B}(t)\times\mathbf{T}(t)$, but this involves computing two cross products rather than just one.

This combination of problems was like a combination of Exercise 36 in Section 13.3 on p. 856 (harder than 35, since the speed was not constant) and Exercises 32, 33 in Section 13.4, on p. 866.