Special Topics in Relaxation in Glass and Polymers

Lecture 4: Differential Equations

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Outline

- Fourier transform solution of differential equations
  - Brief review
  - Ordinary differential equation
  - Partial differential equation (heat equation)
- Laplace transform solution of differential equations
  - Brief review
  - Spring-mass equation
  - System of equations
- Quiz (multiple-choice)

Resources

- Erwin Kreysig, *Advanced Engineering Mathematics*, 5th edition.
  - Chapters 5 (Laplace Transformation) & 12: Complex Numbers. Complex Analytic Functions
- Gilbert Strang, *Introduction to Applied Mathematics*
  - Section 4.3: Fourier Integrals
- George W. Scherer, *Relaxation in Glass and Composite*, 1992
  - Appendix A: Laplace Transform
- Stanley J. Farlow, *Partial Differential Equations for Scientists and Engineers,* Dover, 1993
  - Lesson 12: The Fourier Transform and Its Application to PDEs

Fourier Transform Method

Some more historical background

In 1811, Fourier submitted to the French Academy of Sciences a revised version of the rejected 1807 paper. The new version won a prize but was still refused publication because some details were deemed unclear by the reviewers.

The Fourier transform actually appeared in writings of Cauchy and Laplace, starting around 1782.

Most of the 1822 results plus new results were finally published by Fourier in 1822.

-Peter V. O'Neil, *Advanced Engineering Mathematics*

Lithograph of Joseph Fourier, 1768-1830, by Jules Boilly, 1823, in the Academy of Sciences, Paris.

Fourier Transform Method

First let's review some of the formulas we've seen and add a few more

$\underline{f(x)}$	$\underline{\hat{f}(k)}$
$\delta(x)$	$\int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx = 1$
$\dfrac{d^n g(x)}{dx^n}$	$(ik)^n \hat{g}(k)$
$(g * h)(x) = \int_{-\infty}^{\infty} g(x-y) h(y) dy$	$\hat{g}(k) \hat{h}(k)$
? $\int_a^x g(t) dt$	$\dfrac{1}{ik} \hat{g}(k) + c\delta(k)$?
$e^{-a\lvert x\rvert}$	$\dfrac{2a}{a^2+k^2}, \quad a > 0$
e^{-ax^2}	$\sqrt{\dfrac{\pi}{a}}\, e^{-\frac{k^2}{4a}}, \quad \mathrm{Re}(a) > 0$

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Relaxation Processes in Glass
and Polymers, Lecture 4

Fourier Transform Method

Is this formula correct?

$$\widehat{\int_a^x g(t)\,dt} = \frac{1}{ik}\hat{g}(k) + c\delta(k)$$

Let $h(x) = \int_a^x g(t)\,dt$**, and invoke the Fundamental Theorem of Calculus, which tells us that** $h'(x) = g(x)$**. It's also true that** $(h+c)'(x) = g(x)$ **for any constant** c**. We know by the Fourier transform rules that**

$$\widehat{(h+c)'} = ik\widehat{(h+c)}$$

so

$$ik\widehat{(h+c)} = \hat{g}$$

Therefore

$$\hat{h}(k) = \frac{\hat{g}(k) - \hat{c}}{ik} = \frac{\hat{g}(k)}{ik} - 2\pi c\delta(k) = \frac{\hat{g}(k)}{ik} + c_1\delta(k)$$

Fourier Transform Method for Solving Differential Equations

- **Primarily used for linear differential equations with constant coefficients**
- **General step in applying method:**
  - **Choose the appropriate form of the transform***
  - **Apply FT to the differential equation**
  - **Solve the resulting equation in the transform domain**
  - **Invert the transform solution to find the solution of the original problem**

*** Depending on the range of independent variable, other forms of the Fourier transform may be desirable. For partial differential equations, the form of the boundary conditions may influence the decision.**

For example, if the domain is $0 < x < \infty$, then a *Fourier Cosine* or *Sine Transform* may be better for the problem. These are defined as

$$\hat{f}_c(k) = \int_0^\infty f(x)\cos(kx)\,dx \quad \text{and} \quad \hat{f}_s(k) = \int_0^\infty f(x)\sin(kx)\,dx$$

(we'll focus on problems making use of the FT as we originally defined it)

Fourier Transform Method

Example 1: An ordinary differential equation

Consider the differential equation

$$-\frac{d^2u}{dx^2} + a^2u = h(x), \quad -\infty < x < \infty$$

Take the Fourier transform term by term, using linearity and the derivative rule:

$$-(ik)^2\hat{u}(k) + a^2\hat{u}(k) = \hat{h}(k)$$

Solve for $\hat{u}(k)$**:**

$$\hat{u}(k) = \frac{\hat{h}(k)}{a^2 + k^2}$$

Take the inverse transform to find $u(x)$**:**

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{h}(k)}{a^2 + k^2} e^{ikx}\, dk$$

Could we write the solution in terms of $h(k)$ **rather than** $\hat{h}(k)$**?**

Fourier Transform Method

Example 1 (continued)

Look once more at the solution in the transform domain

$$\hat{u}(k) = \frac{\hat{h}(k)}{a^2 + k^2}$$

This is the product of $\hat{h}(k)$ **and the function** $\dfrac{1}{a^2 + k^2}$**, which we'll call** $\hat{G}(k)$.

Now we can write $\hat{u}(k) = \hat{G}(k)\hat{h}(k)$ **, so at this point we know that** $u(x)$ **is a convolution.**

If we can find $G(x)$**, then we can write** $u(x)$ **as an integral in** $h(x)$.

We know that $\widehat{e^{-a|x|}} = \dfrac{2a}{a^2 + k^2}, \quad a > 0$ **so** $G(x) = \dfrac{e^{-a|x|}}{2a}$

and we can write $u(x)$ **in terms of** $h(k)$ **as**

$$u(x) = \frac{1}{2a} \int_{-\infty}^{\infty} e^{-a|x-y|} h(y)\, dy$$

Fourier Transform Method

Example 2: A partial differential equation – the heat equation

Heat flow in an infinite rod with initial temperature $\phi(x)$ is governed by the initial value problem

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, 0 < t < \infty$$

$$u(x,0) = \phi(x), \quad -\infty < x < \infty$$

Taking the Fourier transform term by term with respect to x will turn the partial differential equation into a first order ordinary equation.

Applying FT to the pde and the initial condition:

$$\frac{d\hat{u}}{dt} = -c^2 k^2 \hat{u}$$

$$\hat{u}(k,0) = \hat{\phi}(k)$$

Keep in mind that $\hat{u}$ actually depends on $\underline{k}$ and $\underline{t}$, but we can treat the problem in the transform domain as an ode.

Fourier Transform Method

Example 2 (continued)

Now we solve the transformed problem: $\dfrac{d\hat{u}}{dt} = -c^2k^2\hat{u}$, $\hat{u}(k,0) = \hat{\phi}(k)$ **for** $\hat{u}$.

The solution is

$$\hat{u}(k,t) = \hat{\phi}(k)e^{-\alpha^2k^2t}$$

The final step is to find the inverse transform of $\hat{u}$.

We can use the convolution rule, writing $\hat{u}(k,t) = \hat{\phi}(k)\hat{h}(k,t)$, **if the inverse transform of** $\hat{h}(k,t) = e^{-c^2k^2t}$ **can be found.**

We know that

$$\widehat{e^{-ax^2}} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{k^2}{4a}}, \quad \mathrm{Re}(a) > 0$$

We can write this formula as $\sqrt{\dfrac{a}{\pi}}\ \widehat{e^{-ax^2}} = e^{-\frac{k^2}{4a}}, \quad \mathrm{Re}(a) > 0$

and recognize that by setting $\dfrac{1}{4a} = c^2t$, **we have** $h(x,t)$.

Fourier Transform Method

Example 2 (continued)

Summarizing what we just found:

We know that the Fourier transform of the solution of our initial value problem has the form

$$\hat{u}(k,t) = \hat{\phi}(k)\hat{h}(k,t)$$

with $$\hat{h}(k,t) = e^{-c^2k^2t} = \sqrt{\frac{a}{\pi}}\sqrt{\frac{\pi}{a}}e^{-c^2k^2t} = \sqrt{\frac{a}{\pi}}\;\widehat{e^{-ax^2}}$$

as long as we set $\frac{1}{4a} = c^2 t$ **, i.e.** $a = \frac{1}{4c^2t}$

Therefore, $$h(x,t) = \sqrt{\frac{a}{\pi}}e^{-ax^2} = \sqrt{\frac{1}{4\pi c^2 t}}e^{-\frac{x^2}{4c^2t}} = \frac{1}{2c\sqrt{\pi t}}e^{-\frac{x^2}{4c^2t}}$$

and finally, $$u(x,t) = \phi(x) * h(x,t) = \int_{-\infty}^{\infty}\phi(x-y)\frac{1}{2c\sqrt{\pi t}}e^{-\frac{y^2}{4c^2t}}dy$$

Comment: What we refer to as $h(x,t)$ **is often called** $G(x,t)$**, for** ***Green's*** **function, and it has interesting interpretations from both physical and mathematical perspectives.**

Exercise Set 1

a) Knowing that the Fourier transform of $\frac{dg(x)}{dx}$ is $ik\hat{g}(k)$ and that the Fourier transform of $cos(ax)$ is $\pi(\delta(k - a) + \delta(k + a))$, find the inverse Fourier transform of $3ik\pi(\delta(k - 2) + \delta(k + 2))$.

b) If $u(x)$ solves the differential equation $\frac{du(x)}{dx} + 3u(x) = \delta(x)$, find $\hat{u}(k)$, the Fourier transform of $u(x)$.

Laplace Transform Method

Some more historical background

The transform actually originates with Leonhard Euler (1707-1783), the Swiss mathematician. The Italian-French mathematical physicist Joseph Louis Lagrange (1736-1813) used similar integrals for work in probability theory. This work influenced Laplace.

- Paul J. Nahin, Behind the Laplace transform,
IEEE Spectrum, March 1991.

Pierre-Simon Laplace (1749–1827).
Posthumous portrait by Madame
Feytaud, 1842.

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Laplace Transform Method

Reviewing of some common formulas:

$\underline{f(t)}$	$\underline{F(s)}$
$g'(t) = \frac{dg}{dt}$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$\delta(t - a)$	e^{-as}
$\frac{g(t)}{t}$	$\int_s^\infty G(v)dv$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$(g * h)(t) = \int_0^t g(t-y)h(y)dy$	$G(s)H(s)$
t	$\frac{1}{s^2}$
$tg(t)$	$-G'(s)$
e^{-at}	$\frac{1}{s+a}$

Correction:

Laplace Transform Method for Solving Differential Equations

- **Primarily used for linear differential equations with constant coefficients (same as Fourier Transform method**
- **General step in applying method:**
  - **Transform the differential equation from the time domain to the frequency domain**
  - **Solve the resulting algebraic equations for the transform solution**
  - **Invert the transform solution to find the solution of the original problem**

Laplace Transform Method

Example 1: An initial value problem for a spring-mass system

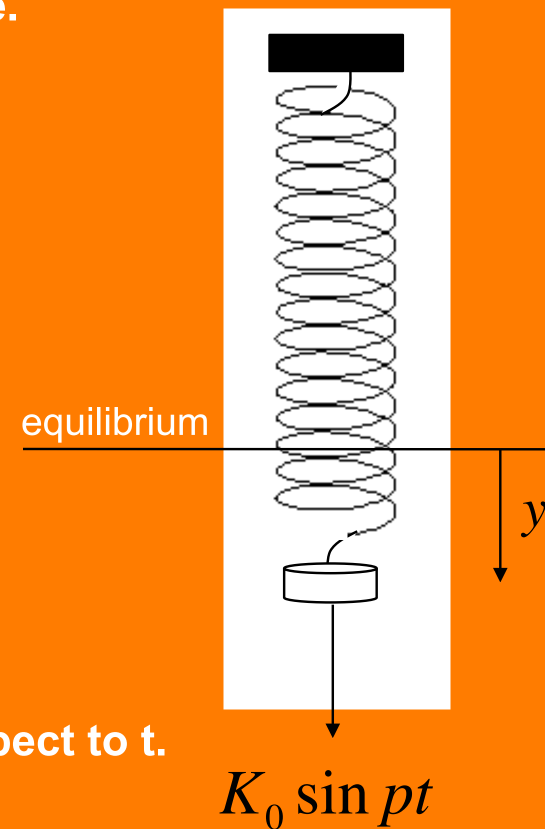
Consider the forced oscillations of a body attached at the lower end of an elastic spring whose upper end is fixed as shown in the picture.

The position of the mass with respect to the equilibrium position is governed by the initial value problem

$$my'' + ky = K_0 \sin pt$$

$$y(0) = 0$$

$$y'(0) = 0$$



The mass of the body is m

The driving force is $K_0 \sin pt$

The spring modulus is k

Prime notation (') denotes derivative with respect to t.

The main part of the spring was drawn as a helix in Maple with the commands

```
with(plots);
spacecurve([2*cos(t), 2*sin(t), t], t = 0 .. 32*Pi, numpoints = 1000, color = black, thickness = 1);
```

Laplace Transform Method

Example 1 (continued)

Introduce the variables $K = \frac{K_0}{m}$ **and** $\omega_0 = \sqrt{\frac{k}{m}}$ **and the original equation**

$$my'' + ky = K_0 \sin pt$$

simplifies to

$$y'' + \omega_0^2 y = K \sin pt$$

Taking the Laplace transform of this equation term-by-term results in the *subsidiary* equation

$$s^2 Y(s) + \omega_0^2 Y(s) = K \frac{p}{s^2 + p^2}$$

which could potentially have two more terms if the initial conditions are nonzero.

Solving for Y(s):

$$Y(s) = \frac{Kp}{\left(s^2 + \omega_0^2\right)\left(s^2 + p^2\right)}$$

Laplace Transform Method

Example 1 (continued)

This looks like the product of two transforms, especially if we write

$$Y(s) = \frac{Kp}{\left(s^2+\omega_0^2\right)\left(s^2+p^2\right)} = \frac{K}{\omega_0}\frac{\omega_0}{\left(s^2+\omega_0^2\right)}\frac{p}{\left(s^2+p^2\right)}$$

and we can find the inverse transform using the convolution rule:

$$y(t) = \frac{K}{\omega_0}\sin\omega_0 t * \sin pt = \frac{K}{\omega_0}\int_0^t \sin\omega_0(t-y)\sin py\,dy$$

Integration is simpler if we combine the terms in the integral, leaving:

$$y(t) = \frac{K}{2\omega_0}\int_0^t\{\cos[\omega_0(t-y)-py] - \cos[\omega_0(t-y)+py]\}dy$$

$$= \frac{K}{2\omega_0}\int_0^t\{\cos[\omega_0 t-(p+\omega_0)y] - \cos[\omega_0 t+(p-\omega_0)y]\}dy$$

Laplace Transform Method

Example 1 (continued)

Integrating $$y(t) = \frac{K}{2\omega_0}\int_0^t \{\cos[\omega_0 t - (p+\omega_0)y] - \cos[\omega_0 t + (p-\omega_0)y]\}\,dy$$

results in $$y(t) = \frac{K}{p^2-\omega_0^2}\left(\frac{p}{\omega_0}\sin\omega_0 t - \sin pt\right)$$

and we're done, UNLESS $p^2 - \omega_0^2 = 0$

Suppose $p = \omega_0$**. Then simplifying the integrand, and evaluating the integral results in the solution**

$$y(t) = \frac{K}{2\omega_0^2}\left(\sin\omega_0 t - \omega_0 t\cos\omega_0 t\right)$$

Now, instead of the superposition of two harmonic oscillations, we have the amplitude growing as t increases. This is called *resonance.*

Laplace Transform Method

Example 2: An initial value problem with variable coefficients in the differential equation

Solve this IVP:

$$t\,y'' + (4t-2)y' - 4y = 0$$

$$y(0) = 1$$

First consider the Laplace transform of $t\,y''$:

$$\mathcal{L}\{t\,y''\} = -\frac{d}{ds}\mathcal{L}\{y''\} = -\frac{d}{ds}\left(s^2Y - sy(0) - y'(0)\right) = -(2sY + s^2Y' - y(0))$$

Similarly

$$\mathcal{L}\{t\,y'\} = -\frac{d}{ds}\mathcal{L}\{y'\} = -\frac{d}{ds}(sY - y(0)) = -Y - sY'$$

So the differential equation becomes

$$-(2sY + s^2Y' - y(0)) - 4(Y + sY') - 2(sY - y(0)) - 4Y = 0$$

Gathering terms results in the equation

$$(s^2 + 4s)Y' + (4s+8)Y = 3$$

Laplace Transform Method

Example 2 (continued)

We have a first order differential equation in $Y(s)$**:**

$$Y' + \frac{(4s+8)}{(s^2+4s)}Y = \frac{3}{(s^2+4s)}$$

This is a linear first order differential equation that can be solved using an *integrating factor.*

The solution turns out to be $Y(s) = \frac{s}{(s+4)^2} + \frac{6}{(s+4)^2} + \frac{C}{s^2(s+4)^2}$

where C is an arbitrary constant.

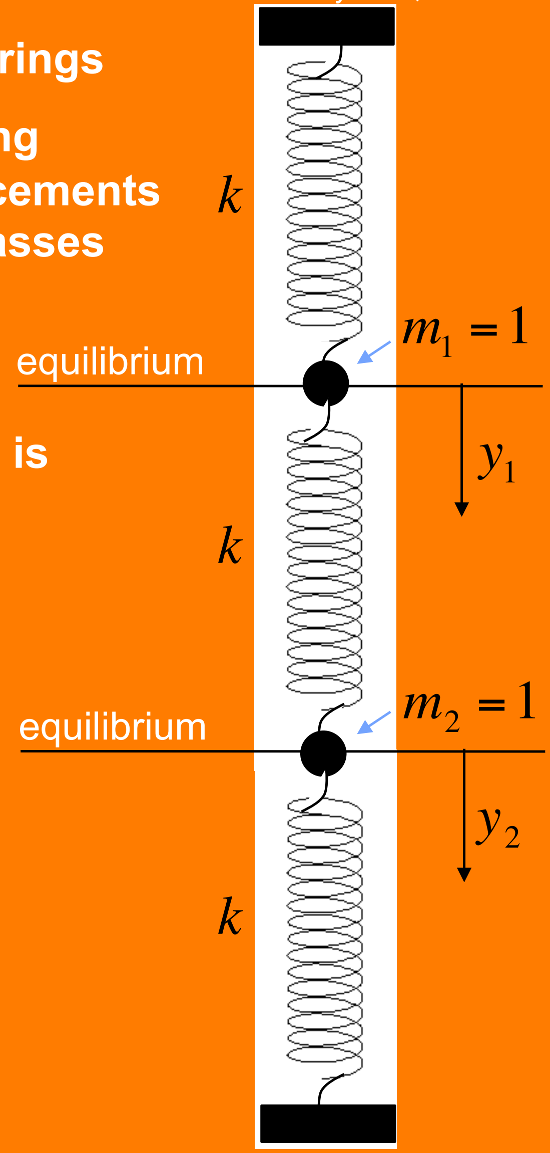
With some additional work (e.g. using the method of *partial fractions* before finding the inverse transform), the solution is found to be

$$y(t) = e^{-4t} + 2te^{-4t} + C\left[-\frac{1}{32} + \frac{1}{16}t + \frac{1}{16}te^{-4t} + \frac{1}{32}e^{-4t}\right]$$

Laplace Transform Method

Example 3: A system of equations for 2 masses on three springs

Consider the system shown in the picture, where each spring has spring modulus k, and the masses and displacements from equilibrium are as specified in the picture. Masses of the springs and damping effects are neglected.



The initial value problem to solve for this system of springs is

$$y_1'' = -ky_1 + k(y_2 - y_1)$$

$$y_2'' = -k(y_2 - y_1) - ky_2$$

$$y_1(0) = 0, \quad y_2(0) = 1$$

$$y_1'(0) = \sqrt{3k}, \quad y_2'(0) = -\sqrt{3k}$$

Laplace Transform Method

Example 3 (continued)

Applying Laplace transforms term by term, incorporating the initial conditions, results in the algebraic system

$$s^2Y_1 - s - \sqrt{3k} = -kY_1 + k(Y_2 - Y_1)$$
$$s^2Y_2 - s + \sqrt{3k} = -k(Y_2 - Y_1) - kY_2$$

To complete the solution procedure:

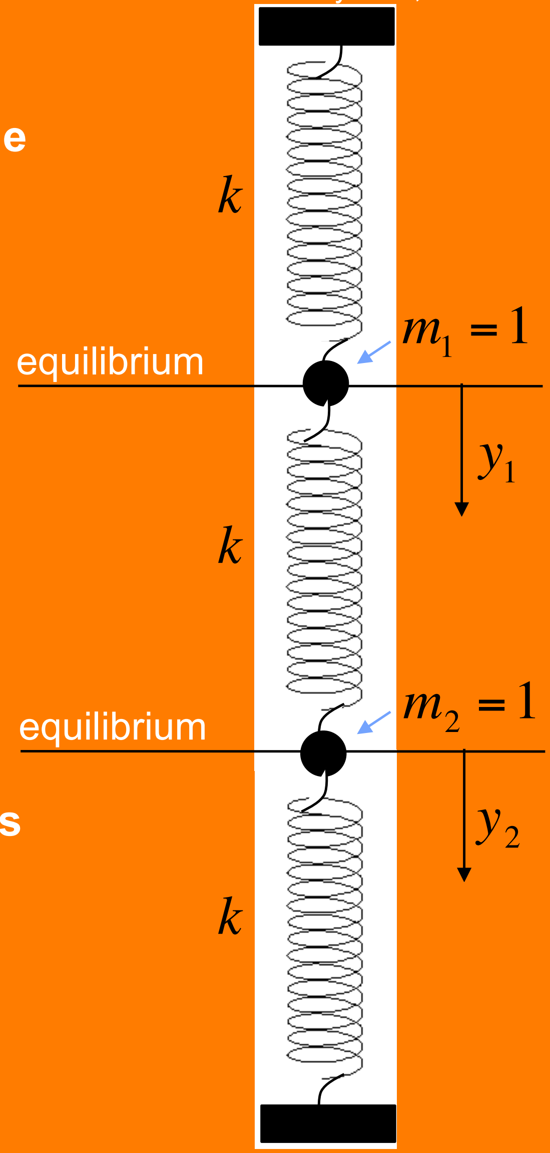
- Combine terms to write as a matrix system in the two transforms

- Solve the matrix system

- Find the inverse transforms, using partial fractions or convolution

Final answer:

$$y_1(t) = \cos\sqrt{k}\,t + \sin\sqrt{3k}\,t$$
$$y_2(t) = \cos\sqrt{k}\,t - \sin\sqrt{3k}\,t$$



Exercise Set 2

a) Knowing that the Laplace transform of $e^{at}g(t), \quad a > 0,$ **is** $G(s-a)$ **and that the Laplace transform of** t **is** $\frac{1}{s^2}$**, find the inverse Laplace transform of** $\frac{2}{(s-3)^2}$**.**

b) If $y(t)$ **solves the initial value problem** $y'' + y = 2\sin t,\ y(0) = 0,\ y'(0) = 0$**, find the Laplace transform** $Y(s)$ **of** $y(t)$**. When you type your answer you can use '^' for powers, e.g.** $\frac{1}{s^2}$ **can be written as** $1/s\text{^}2$ **or** $s\text{^}(-2)$**.**