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# On the zero-divisor-cup-length of spaces of oriented isometry classes of planar polygons

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ABSTRACT

Using information about the rational cohomology ring of the space $M(\ell_1, \ldots, \ell_n)$ of oriented isometry classes of planar $n$-gons with the specified side lengths, we obtain bounds for the zero-divisor-cup-length (zcl) of these spaces, which provide lower bounds for their topological complexity (TC). In many cases our result about the cohomology ring is complete and we determine the precise zcl. We find that there will usually be a significant gap between the bounds for TC implied by zcl and dimensional considerations.



## 1. Introduction

The topological complexity, TC($X$), of a topological space $X$ is, roughly, the number of rules required to specify how to move between any two points of $X$. A "rule" must be such that the choice of path varies continuously with the choice of endpoints. (See [3, §4].) Information about the cohomology ring of $X$ can be used to give a lower bound for TC($X$).

Let $\ell = (\ell_1, \ldots, \ell_n)$ be an $n$-tuple of positive real numbers. Let $M(\ell)$ denote the space of oriented $n$-gons in the plane with successive side lengths $\ell_1, \ldots, \ell_n$, where polygons are identified under translation and rotation. Thus

$$M(\ell) = \{(z_1, \ldots, z_n) \in (S^1)^n : \sum \ell_i z_i = 0\}/SO(2).$$

If we think of the sides of the polygon as linked arms of a robot, then TC($M(\ell)$) is the number of rules required to program the robot to move between any two configurations.

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Let $[n] = \{1, \ldots, n\}$ throughout. We say that $\ell$ is *generic* if there is no subset $S \subset [n]$ for which $\sum_{i \in S} \ell_i = \sum_{i \notin S} \ell_i$. For such $\ell$, $M(\ell)$ is an orientable $(n-3)$-manifold [3, Thm. 1.3] and hence, by [3, Cor. 4.15], satisfies

$$\mathrm{TC}(M(\ell)) \leq 2n - 5. \tag{1.1}$$

A lower bound for topological complexity is obtained using the zero-divisor-cup-length of $X$, $\mathrm{zcl}(X)$, which is the maximum number of elements $\alpha_i \in H^*(X \times X)$ satisfying $m(\alpha_i) = 0$ and $\prod_i \alpha_i \neq 0$. Here $m : H^*(X) \otimes H^*(X) \to H^*(X)$ denotes the cup product pairing with rational coefficients, and $\alpha_i$ is called a zero divisor. Throughout the paper, all cohomology groups have coefficients in $\mathbb{Q}$, unless specified to the contrary. In [4, Thm. 7], it was shown that

$$\mathrm{TC}(X) \geq \mathrm{zcl}(X) + 1. \tag{1.2}$$

In this paper, we obtain some new information about the rational cohomology ring $H^*(M(\ell))$ when $\ell$ is generic to obtain lower bounds for $\mathrm{zcl}(M(\ell))$ and hence for $\mathrm{TC}(M(\ell))$. Frequently, our description of the cohomology ring is complete (64 out of 134 cases when $n = 7$), and we can give the best lower bound implied by ordinary cohomological methods. However, unlike the situation for isometry classes of polygons, i.e., when polygons are also identified under reflection, this lower bound is usually significantly less than $2n - 5$.

Indeed, for the space of isometry classes of planar polygons,

$$\overline{M}(\ell) = \{(z_1, \ldots, z_n) \in (S^1)^n : \sum \ell_i z_i = 0\}/O(2),$$

the mod-2 cohomology ring was completely determined in [9], and in [1] and [2] we showed that for several large families of $\ell$,

$$2n - 6 \leq \mathrm{TC}(\overline{M}(\ell)) \leq 2n - 5,$$

the latter because $\overline{M}(\ell))$ is also an $(n-3)$-manifold when $\ell$ is generic. Note that for motions in the plane, $M(\ell)$ would seem to be a more relevant space than $\overline{M}(\ell)$. For the spaces $M(\ell)$ considered here, rational cohomology often gives slightly stronger bounds than does mod-2 cohomology.

In Section 2, we describe what we can say about the rational cohomology ring $H^*(M(\ell))$. In Section 3, we obtain information about $\mathrm{zcl}(M(\ell))$ and hence $\mathrm{TC}(M(\ell))$. Theorems 3.1 and 3.2 give upper and lower bounds for $\mathrm{zcl}(M(\ell))$. See Table 3.12 for a tabulation when $n = 8$. In Section 4, we give an example, due to the referee, in which there are what we call "exotic products" in the cohomology ring. The possibility of these prevents us from making stronger zcl estimates.

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## 2. The rational cohomology ring $H^*(M(\ell))$

We assume throughout that $\ell_1 \leq \cdots \leq \ell_n$. It is well-understood [9, Prop. 2.2] that the homeomorphism type of $M(\ell)$ is determined by which subsets $S$ of $[n]$ are *short*, which means that $\sum_{i \in S} \ell_i < \frac{1}{2} \sum_{i=1}^{n} \ell_i$. For generic $\ell$, a subset which is not short is called *long*.

Define a partial order on the power set of $[n]$ by $S \leq T$ if $S = \{s_1, \ldots, s_\ell\}$ and $T \supset \{t_1, \ldots, t_\ell\}$ with $s_i \leq t_i$ for all $i$. This order will be used throughout the paper, applied also to multisets. As introduced in [10], the *genetic code* of $\ell$ is the set of maximal elements (called *genes*) in the set of short subsets of $[n]$ which contain $n$. The homeomorphism type of $M(\ell)$ is determined by the genetic code of $\ell$. Note that if $\ell = (\ell_1, \dots, \ell_n)$, then all genes have largest element $n$. We introduce the new terminology that if $\{n, i_r, \dots, i_1\}$ is a gene, then $\{i_r, \dots, i_1\}$ is called a *gee*. (Gene without the $n$.) We define a *subgee* to be a set of positive integers which is $\le$ a gee under the above ordering.

The following result was proved in [5, Thm. 6].

**Theorem 2.1.** *The rational cohomology ring $H^*(M(\ell))$ contains a subalgebra generated by classes $V_1, \dots, V_{n-1} \in H^1(M(\ell))$ whose only relations are that if $S = \{s_1, \dots, s_k\}$ with $s_1 < \dots < s_k$, then $V_S := V_{s_1} \cdots V_{s_k}$ satisfies $V_S = 0$ iff $S$ is not a subgee of $\ell$.*

In other words, the nonzero monomials in the $V_i$'s correspond exactly to the subgees. Of course, $V_i^2 = 0$, since $\dim(V_i)$ is odd.

It is well-known (e.g. [7, Expl. 2.3]) that if the genetic code of $\ell$ is $\langle\{n, n-3, n-4, \dots, 1\}\rangle$, then $M(\ell)$ is homeomorphic to $(S^1)^{n-3} \sqcup (S^1)^{n-3}$. We will exclude this case from our analysis and use the following known result, in which, as always, $\ell = (\ell_1, \dots, \ell_n)$.

**Proposition 2.2.** *([7, Rmk. 2.8]) If the genetic code of $\ell$ does not equal $\langle\{n, n-3, \dots, 1\}\rangle$, then all genes have cardinality less than $n-2$, and $M(\ell)$ is a connected $(n-3)$-manifold.*

From now on, let $m = n - 3$ denote the dimension of $M(\ell)$, and let $W_\emptyset$ denote the orientation class of $H^m(M(\ell))$. We obtain

**Theorem 2.3.** *A basis for $H^*(M(\ell))$ consists of the classes $V_S$ of Theorem 2.1 such that $S$ is a subgee of $\ell$, together with classes $W_S \in H^{m-|S|}(M(\ell))$, for exactly the same $S$'s, satisfying that*

$$V_S W_{S'} = \delta_{S,S'} W_\emptyset \quad \text{if } |S'| = |S|.$$

*Also $V_S V_{S'} = V_{S \cup S'}$ if $S$ and $S'$ are disjoint and $S \cup S'$ is a subgee of $\ell$, while $V_S V_{S'} = 0$ otherwise. Finally, $W_S W_{S'} = 0$ whenever $|W_S| + |W_{S'}| = m$.*

**Proof.** By [3, Thm. 1.7], for all $i$, the $i$th Betti number of $M(\ell)$ equals the number of $V_S$'s described in Theorem 2.1 of degree $i$ plus the number of such $V_S$'s of degree $m - i$. By Theorem 2.1, our classes $V_S$ are linearly independent in $H^*(M(\ell))$ and all products $V_S V_{S'}$ are zero except those listed in our set. By Lemma 2.4, the nonsingularity of the Poincaré duality pairing implies that there are classes $W_S$ which pair with the classes $V_S$ and with each other in the claimed manner, and the Betti number result implies that there are no additional classes. □

The following elementary lemma was used in the preceding proof. This lemma is applied to $U = H^i(M(\ell))$, $U' = H^{m-i}(M(\ell))$, $\{u_1, \dots, u_k\}$ the set of $V_S$'s in $H^i(M(\ell))$, and $\{u'_{k+1}, \dots, u'_t\}$ the set of $V_S$'s in $H^{m-i}(M(\ell))$.

**Lemma 2.4.** *Suppose $U$ and $U'$ are $t$-dimensional vector spaces over $\mathbb{Q}$ and $\phi : U \times U' \to \mathbb{Q}$ is a nonsingular bilinear pairing. Suppose $\{u_1, \dots, u_k\} \subset U$ is linearly independent, as is $\{u'_{k+1}, \dots, u'_t\} \subset U'$, and $\phi(u_i, u'_j) = 0$ for $1 \le i \le k < j \le t$. Then there exist bases $\{u_1, \dots, u_t\}$ and $\{u'_1, \dots, u'_t\}$ of $U$ and $U'$ extending the given linearly-independent sets and satisfying $\phi(u_i, u'_j) = \delta_{i,j}$.*

**Proof.** For $1 \le i \le k$, let $\psi_i : U \to \mathbb{Q}$ be any homomorphism for which $\psi_i(u_j) = \delta_{i,j}$ for $1 \le j \le k$. By nonsingularity, there is $u'_i \in U'$ such that $\phi(u, u'_i) = \psi_i(u)$ for all $u \in U$. To see that $\{u'_1, \dots, u'_t\}$ is linearly independent, assume $\sum c_\ell u'_\ell = 0$. Applying $\phi(u_i, -)$ implies that $c_i = 0$, $1 \le i \le k$, while linear independence of $\{u'_{k+1}, \dots, u'_t\}$ then implies that $c_{k+1} = \cdots = c_t = 0$. Nonsingularity now implies that there are classes $u_i$ for $i > k$ such that $\phi(u_i, u'_j) = \delta_{i,j}$ for all $j$, and linear independence of the $u_i$'s is immediate. $\square$

Results similar to Theorem 2.3 and Proposition 2.5 below were also presented, in slightly different situations, in [8, Rmk. 10.3.20] and [6, Prop. A.2.4].

Let $s$ denote the size of the largest gee of $\ell$. The only $V_S$'s occur in gradings $\le s$, and so the only $W_S$'s occur in grading $\ge m-s$. If $2(m-s) \le m-1$ (i.e., $m \le 2s-1$), then there can be nontrivial products of $W_S$'s, about which we apparently have little control.

The following simple result gives excellent information about products of $V$ classes times $W$ classes. In particular, if $m \ge 2s$, the entire ring structure is determined! See Corollary 2.8. When $m = 4$, this is the case for 64 of the 134 equivalence classes of $\ell$'s, as listed in [11].

**Proposition 2.5.** *Let $\rho_i(T)$ denote the number of elements of $T$ which are greater than $i$. Modulo polynomials in $V_1, \dots, V_{n-1}$,*

$$V_i W_S \equiv \begin{cases} (-1)^{\rho_i(T)} W_T & \text{if } S = T \sqcup \{i\} \\ 0 & \text{if } i \notin S. \end{cases} \tag{2.6}$$

*In particular, if $s$ is the maximal size of gees and $m - |S| \ge s$, then*

$$V_i W_S = \begin{cases} (-1)^{\rho_i(T)} W_T & \text{if } S = T \sqcup \{i\} \\ 0 & \text{if } i \notin S. \end{cases} \tag{2.7}$$

**Proof.** Write $V_i W_S = \sum \alpha_P V_P + \sum \alpha'_Q W_Q$ with $\alpha_P, \alpha'_Q \in \mathbb{Q}$. If $V_T$ is any monomial in grading $|S|-1$, then

$$(-1)^{\rho_i(T)} \delta_{S, T\cup\{i\}} W_\emptyset = V_T V_i W_S = \sum_Q \alpha'_Q \delta_{T,Q} W_\emptyset = \alpha'_T W_\emptyset,$$

as all monomials in the $V$'s are 0 in grading $m$. The first result follows immediately.

The second part follows since $|V_i W_S| = m - |S| + 1$ and all polynomials in the $V$'s are 0 in grading $> s$. $\square$

**Corollary 2.8.** *If $m \ge 2s$, where $s$ is the maximal gee size, then the complete structure of the algebra $H^*(M(\ell))$ is given by Theorem 2.3 and (2.7).*

We offer the following illustrative example, in which we have complete information about the product structure. Here we begin using the notation introduced in [10] of writing genes (and gees) which are sets of 1-digit numbers by just concatenating those digits.

**Example 2.9.** Suppose the genetic code of $\ell$ is $\langle 9421, 95 \rangle$. Then a basis for $H^*(M(\ell))$ is:

| | |
|---|---|
| 0 | $1$ |
| 1 | $V_1, V_2, V_3, V_4, V_5$ |
| 2 | $V_1V_2, V_1V_3, V_1V_4, V_2V_3, V_2V_4$ |
| 3 | $V_1V_2V_3, V_1V_2V_4, W_{123}, W_{124}$ |
| 4 | $W_{12}, W_{13}, W_{14}, W_{23}, W_{24}$ |
| 5 | $W_1, W_2, W_3, W_4, W_5$ |
| 6 | $W_\emptyset$. |

The only nontrivial products of $V$'s are those indicated. All products of $W$'s are 0. The multiplication of $V_i$ by $W_S$ is given, up to sign, by removal of the subscript $i$, if $i \in S$, else 0.

In the above example, $m = 6$ and $s = 3$. It is quite possible that a similarly nice product structure might hold in various cases in which $m < 2s$. When it does not, we refer to nonzero products of $W$'s or cases in which (2.7) does not hold as *exotic products*. In Section 4, we present an example, due to the referee, in which nontrivial exotic products occur.

One important class of examples in which $s \leq 2m$ and so Corollary 2.8 applies is the space $M_{2k+1}$ of equilateral $(2k+1)$-gons. Here $\ell = (1, \dots, 1)$ and the genetic code is $\langle\{2k+1, 2k, \dots, k+2\}\rangle$, and so $s = k-1$ and $m = 2k-2$.

## 3. Zero-divisor-cup-length

In this section we study the zero-divisor-cup-length $\mathrm{zcl}(M(\ell))$, where $\ell = (\ell_1, \dots, \ell_n)$, $\ell$ is generic, and its genetic code does not equal $\langle\{n, n-3, \dots, 1\}\rangle$. We also discuss the implications for topological complexity. Recall that $m = n - 3$.

Our first result is an upper bound, which will sometimes be sharp. One sharp example is when the genetic code is 742.

**Theorem 3.1.** *If $s$ is the largest cardinality of the gees of $\ell$, then* $\mathrm{zcl}(M(\ell)) \leq 2s + 2$.

**Proof.** For $u \in H^*(M(\ell))$, let $\overline{u} = u \otimes 1 - 1 \otimes u$. We first consider products of the form $\prod \overline{u_i}$. A product of $a$ $\overline{V_i}$'s and $b$ $\overline{W_S}$'s has grading $\geq a + b(m-s)$. If $a > 2s$, then $\prod \overline{V_i} = 0$, so we may assume that $a \leq 2s$. If $a + b \geq 2s + 3$, then $b \geq 3$ and

$$a + b(m-s) \geq 2s + 3 - b + b(m-s) = bm + 1 - (b-2)(s+1) \geq bm + 1 - (b-2)m = 2m + 1,$$

and so the product must be 0. We have used that $s \leq m - 1$ by Proposition 2.2.

Now we consider the possibility of more general zero divisors. Let $\alpha_j$ denote a zero divisor which contains a term $A \otimes B$ in which the total number of $V$-factors (resp. $W$-factors) in $AB$ is $p_j$ (resp. $q_j$) with $p_j + q_j \geq 2$. Its grading is $\geq p_j + q_j(m-s)$. A product of $a$ $\overline{V_i}$'s, $b$ $\overline{W_S}$'s, and $c$ $\alpha_j$'s, with $a + b + c \geq 2s + 3$ will be 0 if $a + \sum p_j > 2s$, so we may assume $a + \sum p_j \leq 2s$. This product, with $c \geq 1$, has grading

$$\begin{aligned}
&\geq a + b(m-s) + \sum p_j + (m-s)\sum q_j \\
&\geq a + b(m-s) + \sum p_j + (m-s)(2c - \sum p_j) \\
&\geq a + (b+2c)(m-s) + (m-s-1)(a-2s) \\
&= (m-s-1)(a+b+2c-2s) + a + b + 2c \\
&\geq (3+c)(m-s-1) + 2s + 3 + c \\
&= 2m + (c+1)(m-s) \\
&> 2m,
\end{aligned}$$

and hence is 0. □

Next we give our best result for lower bounds. Recall that the partial order described just before Theorem 2.1 is applied also to multisets.

**Theorem 3.2.**

a. *If $G$ and $G'$ are gees of $\ell$, not necessarily distinct, and there is an inequality of multisets $G \cup G' \geq [k]$, then*

$$\mathrm{zcl}(M(\ell)) \geq \begin{cases} k+2 & k \equiv m\ (2) \\ k+1 & k \not\equiv m\ (2). \end{cases}$$

b. *If there are no exotic products in $H^*(M(\ell))$, then (a) is sharp in the sense that if*

$$k_0 := \max\{k : \exists \text{ gees } G \text{ and } G' \text{ of } \ell \text{ with } G \cup G' \geq [k]\},$$

*then*

$$\mathrm{zcl}(M(\ell)) = \begin{cases} k_0+2 & k_0 \equiv m\ (2) \\ k_0+1 & k_0 \not\equiv m\ (2). \end{cases}$$

The result in (b) says that zcl is the smallest integer $> k_0$ with the same parity as $m$.

Note that (b) holds if $m \geq 2s$, where $s$ denotes the maximum size of the gees of $\ell$. In the example $M_{2k+1}$ mentioned at the end of Section 2, we obtain $\mathrm{zcl} = 2k$, hence $2k+1 \leq \mathrm{TC}(M_{2k+1}) \leq 4k-3$, so there is a big gap here.

**Proof.** Under the hypothesis of (a), there is a partition $[k] = S \sqcup T$ with $G \geq S$, and $G' \geq T$. Then the following product of $k+1$ zero-divisors is nonzero:

$$\prod_{i\in S} \overline{V_i} \cdot \overline{W_S} \cdot \prod_{j\in T} \overline{V_j}.$$

Indeed, this product contains the nonzero term $W_\emptyset \otimes V_T$, and this term cannot be cancelled by any other term in the expansion, since the only way to obtain $W_\emptyset$ is as $V_U W_U$ for some set $U$. The stronger result when $k \equiv m$ (mod 2) is obtained using

$$\prod_{i\in S} \overline{V_i} \cdot \overline{W_S} \cdot \prod_{j\in T} \overline{V_j} \cdot \overline{W_T},$$

which is nonzero by Remark 3.8.

Part (a) implies $\geq$ in (b). We will prove $\leq$ by showing that, under the assumption that there are no exotic products, if there is a nonzero product of $k+1$ zero divisors with $k \equiv m$ (mod 2), then there are gees $G$ and $G'$ of $\ell$ such that $G \cup G' \geq [k]$. This says that if $\mathrm{zcl} \geq k+1$ with $k \equiv m$ (mod 2), then $k_0 \geq k$. Thus if $\mathrm{zcl} \geq \begin{cases} k_0+3 & k_0 \equiv m \\ k_0+2 & k_0 \not\equiv m, \end{cases}$ then $k_0 \geq k_0 + 2$ (resp., $k_0+1$), a contradiction.

We begin by considering the case when all the zero divisors are of the form $\overline{V_i}$ or $\overline{W_S}$. Since products of $W$'s are 0, there cannot be more than two $\overline{W}$'s. The case of no $\overline{W}$'s is easiest and is omitted. Denote $\overline{V}_S := \prod_{i\in S} \overline{V_i}$. Note the distinction: $\overline{W_S} = W_S \otimes 1 + 1 \otimes W_S$, whereas $\overline{V}_S = \prod_{i\in S}(V_i \otimes 1 + 1 \otimes V_i)$, with the usual convention that the entries of $S$ are listed in increasing order.

For the case of one $\overline{W}$, assume $\overline{V}_{T_1}\overline{V}_{T_2}\overline{W_S} \neq 0$ with $T_1 \subset S$, $T_2$ and $S$ disjoint, and $|T_1 \cup T_2| \geq k$. Since $W_S \neq 0$, $S \subset G$ for some gee $G$. The product expands, up to $\pm$ signs on terms, as

$$\sum_{T'\subset T_1} V_{T_2}V_{T'} \otimes W_{S-T'} + W_{S-T'} \otimes V_{T_2}V_{T'}.$$

For this to be nonzero, we must have $V_{T_2} \neq 0$, and so $T_2 \leq G'$ for some gee $G'$. Thus

$$[k] \leq T_1 \cup T_2 \leq G \cup G'.$$

For the case of two $\overline{W}$'s, we may assume that

$$\overline{V}_{E_1}\overline{V}_{E_2}\overline{V}_{E_3}\overline{W_{D_1\cup D_3}}\ \overline{W_{D_2\cup D_3}} \neq 0, \tag{3.3}$$

with $D_1$, $D_2$, and $D_3$ disjoint, $E_i \subset D_i$, and $|E_1 \cup E_2 \cup E_3| = k-1$ with $k \equiv m$ mod 2. Note that we cannot have a factor $\overline{V}_{E_4}$ with $E_4$ disjoint from $D_1 \cup D_2 \cup D_3$, since

$$\overline{W_{D_1\cup D_3}}\ \overline{W_{D_2\cup D_3}} = -W_{D_1\cup D_3} \otimes W_{D_2\cup D_3} \pm W_{D_2\cup D_3} \otimes W_{D_1\cup D_3},$$

and the product of $\overline{V}_{E_4}$ with this would be 0.

Since $W_{D_i\cup D_3} \neq 0$ for $i = 1, 2$, we have

$$E_i \cup E_3 \subset D_i \cup D_3 \leq G_i$$

for gees $G_i$. Thus

$$G_1 \cup G_2 \geq D_1 \cup D_2 \cup D_3 \supset E_1 \cup E_2 \cup E_3 \geq [k-1],$$

and so $G_1 \cup G_2 \geq [k]$ unless each $D_i = E_i$, $1 \leq i \leq 3$. But in this case the LHS of (3.3) is 0 by Lemma 3.4, since $|D_1 \cup D_2 \cup D_3| \not\equiv m$ (mod 2) in this case. This completes the proof when all zero divisors are of the form $\overline{V_i}$ or $\overline{W_S}$.

Let $R = H^*(M(\ell))$. In Lemma 3.9, we show that any product $P$ of $z$ zero divisors can be written as $\sum \alpha_i P_i$, where $\alpha_i \in R \otimes R$ and $P_i$ is a product of $z$ factors of the form $\overline{V}_i$ or $\overline{W_S}$. If $P \neq 0$, then some $P_i$ must be nonzero, and so by the above argument there exist gees $G$ and $G'$ as claimed. □

The following two lemmas were used in the preceding proof.

**Lemma 3.4.** *If there are no exotic products in $H^*(M(\ell))$, and $D_1$, $D_2$, and $D_3$ are pairwise disjoint subgees, then*

$$\overline{V}_{D_1}\overline{V}_{D_2}\overline{V}_{D_3}\overline{W_{D_1\cup D_3}}\,\overline{W_{D_2\cup D_3}} \neq 0 \text{ iff } |D_1 \cup D_2 \cup D_3| \equiv m \pmod 2.$$

**Proof.** The notation

$$\langle S, T, e\rangle := W_S \otimes W_T + (-1)^e W_T \otimes W_S,$$

with $S$ and $T$ disjoint, will be useful in this proof. We begin with the observation that if $k \notin S \cup T$, then

$$\overline{V_k}\langle S \cup \{k\}, T, e\rangle = \pm\langle S, T, e + |W_T| + 1\rangle. \tag{3.5}$$

Indeed, since $|V_k| = 1$, we have

$$\begin{aligned}&(V_k \otimes 1 - 1 \otimes V_k)(W_{S\cup\{k\}} \otimes W_T + (-1)^e W_T \otimes W_{S\cup\{k\}})\\ &= V_k W_{S\cup\{k\}} \otimes W_T - (-1)^{|W_T|}(-1)^e W_T \otimes V_k W_{S\cup\{k\}}.\end{aligned}$$

The $\pm$, which is not important, is $(-1)^{\rho_k(S)}$ from Proposition 2.5. Similarly,

$$\overline{V}_k\langle S, T\cup\{k\}, e\rangle = \pm\langle S, T, e+|W_S|+1\rangle. \tag{3.6}$$

Since products of $W$'s are 0 by assumption, we have

$$\overline{W_{D_1\cup D_3}}\,\overline{W_{D_2\cup D_3}} = -\langle D_1\cup D_3, D_2\cup D_3, |W_{D_1\cup D_3}|\cdot|W_{D_2\cup D_3}|\rangle.$$

Now apply (3.6) $|D_2|$ times to this to eliminate the elements of $D_2$, each time adding $|W_{D_1\cup D_3}|+1$ to the third component of $\langle -,-,-\rangle$, obtaining

$$\overline{V}_{D_2}\overline{W_{D_1\cup D_3}}\,\overline{W_{D_2\cup D_3}} = \pm\langle D_1\cup D_3, D_3, |W_{D_1\cup D_3}|\cdot|W_{D_2\cup D_3}|+|D_2|(|W_{D_1\cup D_3}|+1)\rangle.$$

Let $d_i=|D_i|$ and note that $|W_{D_i}|=m-d_i$. Now apply (3.5) $d_1$ times, eliminating the elements of $D_1$. We obtain that the expression in the lemma equals

$$\pm\overline{V}_{D_3}\ \langle D_3, D_3, f\rangle \tag{3.7}$$

with

$$f=(m-d_1-d_3)(m-d_2-d_3)+d_2(m-d_1-d_3+1)+d_1(m-d_3+1)\equiv m+d_1+d_2+d_3 \pmod 2.$$

Thus (3.7) equals 0 if $m+d_1+d_2+d_3$ is odd, while if $m+d_1+d_2+d_3$ is even, it equals

$$\pm 2\overline{V}_{D_3}(W_{D_3}\otimes W_{D_3}) = \pm 2(W_\phi\otimes W_{D_3}+\text{other terms})\neq 0. \qquad \square$$

**Remark 3.8.** Note that the backwards implication in Lemma 3.4 is true without the assumption of no exotic products because the only additional terms will involve just products of $V$'s, and these cannot cancel $W_\phi\otimes W_{D_1}$.

**Lemma 3.9.** *If $R=H^*(M(\ell))$ has no exotic products, then every zero divisor of $R\otimes R$ is in the ideal spanned by elements of the form $\overline{V}_i$ and $\overline{W_S}$.*

**Proof.** The vector space $R\otimes R$ is spanned by monomials of three types: (1) $W_S\otimes W_T$; (2) $V_S\otimes V_T$; and (3) $V_S\otimes W_T$ and $W_T\otimes V_S$. If a zero divisor is written as $Z_1+Z_2+Z_3$, where $Z_i$ is of type $i$, then each $Z_i$ must be a zero divisor, since their images under multiplication $m$ are, respectively 0, $V$'s, and $W$'s. We show that each type of zero divisor is in the claimed ideal.

(1) Every monomial $W_S\otimes W_T$ is a zero divisor and can be written as $-(W_S\otimes 1)(W_T\otimes 1-1\otimes W_T)$.

(2) There are three types of zero divisors of this type. (a) One of the form $V_iV_S\otimes V_iV_T$ equals $\pm(V_iV_S\otimes V_T)(V_i\otimes 1-1\otimes V_i)$. (b) If $V_SV_T=0$ in $R$, then $V_S\otimes V_T$ is a zero divisor and equals $-(V_S\otimes 1)(V_T\otimes 1-1\otimes V_T)$, and one easily shows $V_T\otimes 1-1\otimes V_T$ is in the ideal by induction on $|T|$. (c) If $V_S\neq 0$ and $S=S_i\sqcup T_i$ so that $V_{S_i}V_{T_i}=(-1)^{\varepsilon_i}V_S$, then $\sum c_iV_{S_i}\otimes V_{T_i}$ is a zero divisor if $\sum(-1)^{\varepsilon_i}c_i=0$. We prove by induction on $|S|$ that these zero divisors have the required form. WLOG, assume that $1\in S$. For every $i$ with $1\in T_i$, let $\widetilde{T}_i=T_i-\{1\}$, and write

$$V_{S_i}\otimes V_{T_i}=\pm(V_1\otimes 1-1\otimes V_1)V_{S_i}\otimes V_{\widetilde{T}_i}\pm V_1V_{S_i}\otimes V_{\widetilde{T}_i}.$$

In this way, the given zero divisor can be written as a sum of terms of the desired form plus a sum of terms with $V_1$ in the left factor of each. These latter terms can be written as $V_1\otimes 1$ times a sum with smaller $|S|$, and this can be written in the desired form by the induction hypothesis.

**Table 3.10**
Special cases.

| $\ell$ | Gen. code | Space | zcl | TC |
|---|---|---|---|---|
| $(1,1,1,1,1,1,1,6)$ | $\langle 8\rangle$ | $S^5$ | 1 | 2 |
| $(0,1,1,1,1,1,1,5)$ | $\langle 81\rangle$ | $S^4\times T^1$ | 3 | 4 |
| $(0,0,1,1,1,1,1,4)$ | $\langle 821\rangle$ | $S^3\times T^2$ | 3 | 4 |
| $(0,0,0,1,1,1,1,3)$ | $\langle 8321\rangle$ | $S^2\times T^3$ | 5 | 6 |
| $(0,0,0,0,1,1,1,2)$ | $\langle 84321\rangle$ | $T^5$ | 5 | 6 |

(3) There are zero divisors of the form

$$\sum_i c_i V_{S_i}\otimes W_{T\cup S_i}+\sum_j d_j W_{T\cup S_j}\otimes V_{S_j},$$

with $S_i$ and $S_j$ disjoint from $T$, $V_{S_i}W_{T\cup S_i}=(-1)^{\varepsilon_i}W_T$, $W_{T\cup S_j}V_{S_j}=(-1)^{\varepsilon_j}W_T$, and $\sum(-1)^{\varepsilon_i}c_i+\sum(-1)^{\varepsilon_j}d_j=0$. We claim that each term $V_{S_i}\otimes W_{T\cup S_i}$ is equivalent, mod terms of the desired form, to $1\otimes W_T$, and similarly for $W_{T\cup S_j}\otimes V_{S_j}$. Thus the given zero divisor is equivalent, mod things of the desired form, to a multiple of $W_T\otimes 1-1\otimes W_T$.

The claim is proved by induction on $|V_{S_i}|$, noting that if $s$ is the smallest element of $S_i$, then

$$V_{S_i}\otimes W_{T\cup S_i}=(V_s\otimes 1-1\otimes V_s)(V_{S_i-\{s\}}\otimes W_{T\cup S_i})\pm V_{S_i-\{s\}}\otimes W_{T\cup S_i\cup\{s\}}. \quad\square$$

Our zcl results depend only on the gees and the parity of $n$, and not on the value of $n$. (Recall $m=n-3$.) However the possible gees depend on $n$. Of course, the numbers which occur in the gees must be less than $n$, but also, if $G$ and $G'$ are gees (not necessarily distinct), then we cannot have $[n-1]-G'\le G\cup\{n\}$, for then $G\cup\{n\}$ would be both short and long. Thus, for example, 8531 is an allowable gene, but 7531 is not, since $642<7531$ but $7642\not<8531$.

There are 2469 equivalence classes of nonempty spaces $M(\ell)$ with $n=8$. Genes for these are listed in [11]. We perform an analysis of what we can say about the zcl and TC of these. Since $n=8$, each satisfies $\mathrm{TC}(M(\ell))\le 11$ by (1.1). As we discuss below in more detail, for most of them we can assert that $\mathrm{zcl}(M(\ell))\ge 7$, and so $\mathrm{TC}(M(\ell))\ge 8$. For most of them we can only assert lower bounds for zcl, due to the possibility of exotic products. We emphasize that the following analysis pertains to the case $n=8$.

As discussed in Proposition 2.2 and the paragraph which preceded it, there is only one $\ell$ with a gee of size 5. This $M(\ell)$ is homeomorphic to $T^5\sqcup T^5$ with topological complexity 6. This is a truly special case, as it is the only disconnected $M(\ell)$.

Other special cases which we wish to exclude from the analysis below are those in Table 3.10, which are completely understood by elementary means. Sides of "length 0" stand for sides of very small length. The identification as spaces is from [7, Prop. 2.1] and [10, Expl. 6.5], with $T^k$ a $k$-torus. The upper bound for TC follows from [3, Prop. 4.41 and Thm. 4.49], and the lower bound from Theorem 3.2(a) and (1.2). We thank the referee for suggesting this table.

There are 768 $\ell$'s whose largest gee has size 4. For all of them, we can deduce only $\mathrm{zcl}(M(\ell))\ge 7$, using Theorem 3.2(a) and the following result.

**Proposition 3.11.** *Suppose $G$ and $G'$ are subsets of $[7]$, not necessarily distinct, with neither strictly less than the other and with $\max(|G|,|G'|)=4$. Assume also that it is not the case that $G=G'=4321$, and it is not the case that $[7]-G'\le G\cup\{8\}$. Then $G\cup G'\ge[5]$ but $G\cup G'\not\ge[6]$.*

**Proof.** The first conclusion follows easily from the observation that if $G=4321$, then $5\in G'$. For the second, if $G\cup G'\ge[6]$ then applying $\cup G'$ to the false statement $[7]-G'\le G\cup\{8\}$ would yield a true statement, and the ordering that we are using for multisets has a cancellation property for unions. $\square$

**Table 3.12**
Number of types of 8-gon spaces.

| $s$ | zcl | # |
|---|---|---|
| 1 | 3 | 6 |
| 2 | 5 | 120 |
| 3 | 5, 6, 7 or 8 | 116 |
| 3 | 7 or 8 | 1453 |
| 4 | 7, 8, 9 or 10 | 768 |

There are 1569 $\ell$'s whose largest gee has size 3. By Theorem 3.1, these all satisfy zcl $\leq 8$. For these, we again cannot rule out exotic products, so we cannot use Theorem 3.2(b) to get sharp zcl results. Of these, 929 have a gee $G \geq 531$, and this satisfies $G \cup G \geq [5]$, hence zcl $\geq 7$. In addition to these, there are 524 with distinct gees satisfying $G \cup G' \geq [5]$. Combining these with the 768 $\ell$'s with some $|\text{gee}| = 4$, we find that 2221 of the 2469 $\ell$'s with $n = 8$ satisfy $\text{zcl}(M(\ell)) \geq 7$. There are another 116 $\ell$'s with largest gee of size 3 for which we can only assert zcl $\geq 5$. An example of a genetic code of this type is $\langle 8421, 843, 862, 871 \rangle$.

There are 120 $\ell$'s whose largest gee has size 2. For these, exotic products are not possible and we can assert the precise value of zcl. Of these, 85 have a gee $G \geq 42$ and since $G \cup G \geq [4]$, they have $\text{zcl}(M(\ell)) = 5$. In addition to these, there are 10 having distinct gees satisfying $G \cup G' \geq [4]$ and so again zcl $= 5$. There are 25 others for which we only have $G \cup G' \geq [3]$, but still zcl $= 5$. Finally, there are 6 $\ell$'s with largest gee of size 1. These satisfy $\text{zcl}(M(\ell)) = 3$.

In Table 3.12, we summarize what we can say about zcl when $n = 8$, omitting the six special cases described earlier. Keep in mind that

$$1 + \text{zcl} \leq \text{TC} \leq 11.$$

In the table, $s$ denotes the size of the largest gee, and # denotes the number of distinct homeomorphism classes of 8-gons having the property.

For general $m$ $(= n - 3)$, the largest gees (with one exception) have size $s = m - 1$, and so Theorem 3.1 allows the *possibility* of zcl as large as $2m$, which would imply $\text{TC} = 2m + 1$ by (1.1). However, this would require many nontrivial exotic products. By an argument similar to Proposition 3.11, all we can assert from Theorem 3.2(a) is zcl $\geq m + 2$ (when $s = m - 1$). If $s \leq [m/2]$, then we can determine the precise zcl, which can be as large as $2s + 2$, so we can obtain $m + 1$ or $m + 2$ as zcl, yielding a lower bound for TC only roughly half the upper bound given by (1.1).

## 4. An example with a nontrivial exotic product

The following example, provided by the referee, suggests that additional geometric information may be needed in finding exotic products and sharper zcl bounds.

**Theorem 4.1.** *Let $X = M(\ell)$ with genetic code $\langle 632 \rangle$. There are exotic products in $H^*(X; \mathbb{Q})$. The (rational) zcl of $X$ is 6, and* $\text{TC}(X) = 7$.

**Proof.** The space $X$ is homeomorphic to the connected sum of two 3-tori by [7, (2) in Expl. 2.11]. The length vector $\ell$ could be taken to be $(1, 1, 1, 3, 3, 4)$, although this is irrelevant to the proof. An elementary argument is presented in [8, Prop. 4.2.1] that there is a ring isomorphism in positive dimensions $H^*(X) \approx (H^*(T^3) \oplus H^*(T^3))/(a_1 a_2 a_3 - b_1 b_2 b_3)$ with any coefficients. Here $a_i$ and $b_i$ are the generators of the first cohomology groups of the two 3-tori. We have $a_i^2 = 0 = b_i^2$ and $a_i b_j = 0$.

If there were no exotic products in its rational cohomology, then, by Theorem 3.2(b), since $m = 3$ and $k_0 = 3$, $\text{zcl}(X)$ would equal 5. However, in $H^*(X \times X; \mathbb{Q})$, we have

$$\bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{b}_1 \bar{b}_2 \bar{b}_3 = a_1 a_2 a_3 \otimes b_1 b_2 b_3 + b_1 b_2 b_3 \otimes a_1 a_2 a_3.$$

Since $a_1a_2a_3 = b_1b_2b_3$, this equals 2 times the top class of $H^*(X \times X; \mathbb{Q})$. Thus the rational zcl equals 6, and $\mathrm{TC}(X) = 7$ by (1.1) and (1.2).

An isomorphism between the $(a, b)$- and $(V, W)$-presentations is given by $V_i = a_i + b_i$, $W_{1,2} = a_3$, $W_{1,3} = b_2$, $W_{2,3} = a_1$, $W_1 = b_2b_3$, $W_2 = a_1a_3$, and $W_3 = b_1b_2$. One can check that this satisfies Theorem 2.3, but has exotic products such as $W_{1,2}W_{2,3} = -W_2$ and $V_2W_{1,2} = -W_1 + V_2V_3$. □

## References

[1] D.M. Davis, Topological complexity of some planar polygon spaces, Bol. Soc. Mat. Mexicana (2016), http://dx.doi.org/10.1007/s40590-016-0093-y, in press.
[2] D.M. Davis, Topological complexity of planar polygon spaces with small genetic code, Forum Math. (2016), in press.
[3] M. Farber, Invitation to Topological Robotics, European Math. Society, 2008.
[4] M. Farber, Topological complexity of motion planning, Discrete Comput. Geom. 29 (2003) 211–221.
[5] M. Farber, J.-C. Hausmann, D. Schütz, On the conjecture of Kevin Walker, J. Topol. Anal. 1 (2009) 65–86.
[6] V. Fromm, The topology of spaces of polygons, thesis, Durham Univ., 2011.
[7] J.-C. Hausmann, Geometric descriptions of polygon and chain spaces, Contemp. Math. 438 (2007) 47–57.
[8] J.-C. Hausmann, Mod Two Homology and Cohomology, Springer Universitext, 2014.
[9] J.-C. Hausmann, A. Knutson, The cohomology rings of polygon spaces, Ann. Inst. Fourier (Grenoble) 48 (1998) 281–321.
[10] J.-C. Hausmann, E. Rodriguez, The space of clouds in Euclidean space, Exp. Math. 13 (2004) 31–47.
[11] J.-C. Hausmann, E. Rodriguez, addendum to [10], www.unige.ch/math/folks/hausmann/polygones/.