SOLUTIONS TO 1999 EXAM

1. 1/2. It is $1/\sqrt{4}$.
2. 12/5. The area of the right triangle equals $3 \cdot 4/2$ and also equals 5/2 times the desired altitude.
3. 54. $f(3) = 2 \cdot 3^3$.
4. 52. $D = 3Q$ and $10D + 25Q = 715$. This yields $Q = 715/55 = 13$ and $D = 39$.
5. 30. Triangle $ADE$ is isosceles with vertex angle $90 + 60$. Thus angle $AED$ equals 15, and $CED$ equals $60 - 15 - 15$.
6. $-1$. Must have $(3x+1)/x = 2$. Thus $x = -1$.
7. 1606/4995. If $x$ equals the number, then $1000x = 321.5\overline{215}$, and subtracting $x$ yields $999x = 321.2$. Thus $x = 3212/9990 = 1606/4995$. Since the denominator is $5 \cdot 3^3 \cdot 37$, and the numerator is not divisible by 3, 5, or 37, this is reduced to lowest terms.
8. $-1$. $\log(1/8) = -\log 8$.
9. 6. Let $A = 9778895$. Then the desired expression equals $(A-1)(A+1)+(A+1)(A+4) - A(A+3) - (A-1)(A+3) = 6$.
10. 54. Each win multiplies your holding by 3/2 and each loss multiplies it by 1/2. Your final amount will be $128 \cdot (\frac{3}{2})^3 \cdot (\frac{1}{2})^3$, regardless of the order of wins and losses.
11. 0.6. It equals $f(1.4) - f(1) = f(0.4) - f(0) = f(-0.6) = 0.6$.
12. 86. $y$ must be a positive odd integer $\leq 171$.
13. 48. In 1 second, Joe does 1/80 of a lap, while Jim does $\frac{1}{30} - \frac{1}{80} = \frac{5}{240} = \frac{1}{48}$ of a lap.
14. $3\sqrt{13}$. One way to work the problem is to use the law of cosines. Another is to use coordinate geometry. Set the center of the clock at (0,0), the end of the hour hand at (9,0), and the end of the minute hand at $(6, 6\sqrt{3})$. The distance between these end points is $\sqrt{3^2 + 36 \cdot 3} = 3\sqrt{1+12}$.
15. $(28, 21)$. Since $(x-y)(x+y) = 7^3$, we must have $x - y = 7$ and $x + y = 49$ or else $x - y = 1$ and $x + y = 343$. Thus $x = 28$ and $y = 21$, or else $x = 172$ and $y = 171$.
16. 112. There are $8 \cdot 7/2$ pairs of circles, and each pair has 4 tangents, two external and two internal. The answer is $28 \cdot 4$.
17. $(2, 2\sqrt{2})$. The smallest value occurs when $C = A$ or $B$, in which case $s = 2$. The largest value occurs when $ABC$ is an isosceles right triangle with hypotenuse $AB$ of length 2. Then $AB = AC = \sqrt{2}$.
18. 6. The exponent of 2 in 17! is $8 + 4 + 2 + 1$ (this is the number of even numbers, number of multiples of 4, multiples of 8, and multiples of 16 which are $\leq 17$), and the exponent of 3 is $5 + 1$. Thus 17! is divisible by $12^6 = 2^{12}3^6$ but not by $12^7$.
19. $\frac{1}{3}(\pi - \frac{3}{4}\sqrt{3})$. The altitude of the triangle is 3/2, since the altitudes meet at the center of the circle in a point 2/3 of the way along each altitude. Thus the sidelength of the triangle is $\sqrt{3}$, and its area is $\frac{1}{2}\frac{3}{2}\sqrt{3}$. The desired answer is $\frac{1}{3}(A_{\text{cir}} - A_{\text{tri}})$.
20. 1. We must have $a + b = 4$, $1 + ab + c = 6$, $1 + ad + c = 4$, and $a + d = 2$. Thus $b - d = 2$ and $a(b-d) = 2$.

21. 5. $A^B = 1$ if $B = 0$ or $A = 1$ or $A = -1$ and $B$ is an even integer. We will have $B = 0$ if $x = 1$ or 2. We will have $A = 1$ if $x = -1 \pm \sqrt{2}$. We will have $A = -1$ if $x = -1$, in which case $B = 6$.

22. 8/49. Square both sides of the equation $\frac{1}{1-x} = (1 + x + x^2 + \cdots)$, obtaining $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$. Let $x = \frac{1}{8}$, and multiply both sides by $\frac{1}{8}$, obtaining that the desired sum equals $1/(8\frac{7}{8}\frac{7}{8}) = 8/49$.

23. 7. We must have $225 \cdot 224 = 10 \cdot n!$. We obtain $n! = 5040$.

24. $-31.5$. If $a$ is the initial term and $d$ the common difference, then $40a + 780d = 300$ and $40(a + 40d) + 780d = 3500$. Subtracting yields $d = 2$, and substituting this into the first equation yields $a = -1260/40$.

25. 4. The graph of $y = x^2 + 8x + 12$ is a parabola with roots at $-2$ and $-6$. Thus its minimum occurs at $x = -4$ with minimum value $y = -4$. Thus $y = x^2 + 8x + 12$ equals 4 for two values of $x$ and $-4$ for one value of $x$. Hence $|x^2 + 8x + 12| = 4$ for three values of $x$.

26. $-2i,-2,-3$. Since the conjugate of a root of a real polynomial must also be a root, $-2i$ is a root and the polynomial is divisible by $x^2 + 4$. Dividing it by $x^2 + 4$ yields $x^2 + 5x + 6 = (x + 2)(x + 3)$.

27. 0. A number is congruent to its digital sum mod 9. If the digital sum of $n$ is congruent to 0, 3, or 6, then the digital sum of $n^2$ is congruent to 0. If the digital sum of $n$ is congruent to $\pm 1$ (resp. $\pm 2$) (resp. $\pm 4$) the the digital sum of $n^2$ is congruent to 1 (resp. 4) (resp. 7). Thus the digital sum of $n^2$ is never congruent to 5.

28. 31. There is 1 path going down from the top C. There are 4 paths starting at the C on either end of the second row, corresponding to which row do you cut over to the middle column. There are 6 paths starting at the C on either side of the third row, corresponding to the two places that you change column. There are 4 paths starting at the C on either side of the fourth row, corresponding to the column in which you change row. Finally, there is 1 path starting at the C on either side of the bottom row. The total number of paths is $1 + 2(4 + 6 + 4 + 1)$.

29. 50. Jeeves saved 10 minutes of driving in each direction. Thus he picked her up at 5:50. So she had been walking for 50 minutes.

30. 180. Each step triples the number of noncircular regions. Thus the number of circles added is $2(1 + 3 + 3^2 + 3^3 + 3^4) = 242$. The number of added circles which touch $C$ doubles at each step, and hence is $2 + 2^2 + 2^3 + 2^4 + 2^5 = 62$.

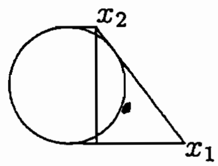
31. 35/46. Let $A$ be the first person selected. The other two can be chosen in $24 \cdot 23/2$ ways. The number of these ways in which no two (of the three) are adjacent is $22 \cdot 21/2 - 21$. Here the first term is the ways of choosing people not next to $A$, and the second removes the ways in which those two were adjacent. The answer is $10 \cdot 21/(12 \cdot 23) = 35/46$.

32. 3. Recall that the size of an angle of a regular $n$-gon is $(n - 2)/n$ straight angles. We wish to find pairs of fractions $\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \ldots$ whose ratio is 3:2. Note that $\frac{3}{2} \cdot \frac{1}{3} = \frac{2}{4}$ gives one, $\frac{3}{2} \cdot \frac{2}{4} = \frac{6}{8}$ gives another, while $\frac{3}{2} \cdot \frac{3}{5} = \frac{18}{20}$ gives a third, but 3/2 times any of the remaining fractions is $\geq 1$, and hence cannot equal one of these fractions.

33. $(8, 27)$. Write the equations as $x^{4/3}(x^{2/3}+y^{2/3}) = 2^4 13$ and $y^{4/3}(x^{2/3}+y^{2/3}) = 3^4 13$. Divide the equations to get $(y/x)^{4/3} = (3/2)^4$ and so $y = \frac{27}{8}x$. Substitute this into the first equation to get $\frac{13}{4}x^2 = 2^4 13$ and so $x = 8$.

34. 35. Expand $(7-1)^{83} + (7+1)^{83}$. Half the terms cancel out. Most of the others are divisible by $7^2$. We are left with $2 \cdot 83 \cdot 7$, which has remainder 35 when divided by 49.

35. $x_1 x_2 = 1$. In the right triangle in the diagram below, the base is $x_1 - x_2$, the height is 2, and the hypotenuse is $x_1 + x_2$ (because tangents to a circle from an exterior point are equal). The Pythagorean theorem readily simplifies to $x_1 x_2 = 1$.




36. 2. $\frac{1}{1999} = \frac{1}{2000}\frac{1}{1-.0005} = .0005 + .0005^2 + .0005^3 + \cdots$. One computes that $5^9 = 1953125$, $5^{10} = 9765625$, and $5^{11} = 48828125$. When the powers of .0005 are added, the only ones contibuting to the 36th decimal place are the 9th, 10th, and 11th, which line up as below:

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          *
    1953125
         9765625
            48828125
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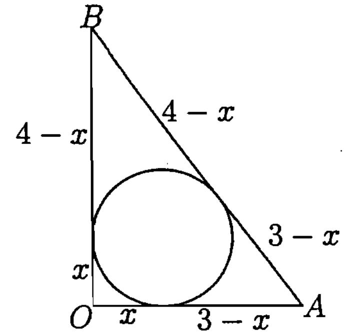
37. 11/26. Let Sam be the name of the first person to roll. On any roll, the probability that a 1, 2, 5, or 6 is rolled is 2/3. The probability that Sam is the first person to roll a 1, 2, 5, or 6 is $\frac{2}{3}(1 + \frac{1}{3^3} + \frac{1}{3^6} + \cdots) = \frac{9}{13}$. Given that he rolls a 1, 2, 5, or 6, the probability that he rolls a 5 or 6 (and hence wins) is 1/2. Thus the probability that Sam wins by rolling a 5 or 6 before the other players have rolled a 1, 2, 5, or 6 is $\frac{9}{13} \cdot \frac{1}{2}$. If Sam is not the first person to roll a 1, 2, 5, or 6, (which happens with probability 4/13), then the probability that the first person who rolled the 1, 2, 5, or 6 rolled a 1 or 2 (and hence lost) is 1/2. In this case, (which happens with probability $\frac{4}{13} \cdot \frac{1}{2}$), Sam and the other player each have a 50% chance of winning, since on any given roll a person's chance of winning or losing are equal. Thus the probability that Sam wins is $\frac{9}{26} + \frac{4}{13}\frac{1}{4} = \frac{11}{26}$

38. $(1, 1, \sqrt{3})$. Triangle $OAB$ is in the $xy$-plane. The intersection of the sphere with the $xy$-plane is the inscribed circle of this triangle. If $(x, x, 0)$ is the center of this inscribed circle, then $(3-x) + (4-x) = 5$, and so $x = 1$. The center of the sphere is at $(1, 1, z)$

at distance 2 from the point $(1,0,0)$, which lies on the sphere. Thus $z = \sqrt{3}$.




39. 3. We must have $100a + b = (a+b)^2$. This can be manipulated to $2500 - 99b = (a+b-50)^2$. If $z = a+b-50$, then $(50-z)(50+z) = 99b$. It is easy to check (by trying the multiples of 11) that the only ways to have two positive numbers summing to 100 with product divisible by 99 are $(55,45)$ and $(99,1)$. So $a+b$ can be 55, 45, 99, or 1, but 1 is too small, so $(a+b)^2$ can be 3025, 2025, or 9801.

40. 12/13. Let $B = (0,0)$ be the center of the large circle, $A = (-1,0)$ and $C = (3,0)$ the centers of the circles of radius 3 and 1, respectively, and $D$ the center of the circle whose radius $r$ we desire. The semiperimeter of triangle $ABD$ is $\frac{1}{2}(3+r+4-r+1) = 4$, and the semiperimeter of triangle $BDC$ is $\frac{1}{2}(1+r+4-r+3) = 4$. Let $A_1$ and $A_2$ denote the areas of triangles $ABD$ and $BDC$, respectively. Then

$$3 = A_2/A_1 = \sqrt{\tfrac{1}{2}4(3-r)r\cdot 1} \;/\; \sqrt{\tfrac{1}{2}4(1-r)r\cdot 3} = \sqrt{(3-r)/(3-3r)}.$$

Thus $9(3-3r) = 3-r$, and so $r = 12/13$.