EXPLICIT MOTION PLANNING RULES IN CERTAIN POLYGON SPACES

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Abstract. This is an addendum to our paper [1]. It seems somewhat relevant, but perhaps distracting. We give an optimal, explicit set of motion planning rules in a polygon space closely related to the polygon space studied in [1].

1. Introduction

In [1], we studied the algebraic and differential topology of the space

$$K_n = (S^1)^n/(z_1, \ldots, z_{n-1}, z_n) \sim (\overline{z}_1, \ldots, \overline{z}_{n-1}, -z_n). \tag{1.1}$$

We are particularly interested in determining its topological complexity, because it is homeomorphic to the space $\overline{M}(\epsilon^{n-1}, 1, 1, 1, 2)$ of isometry classes of planar polygons with the prescribed side lengths. Here $0 < \epsilon < \frac{1}{n-1}$ occurs $n-1$ times. All we can say is that $n \le \mathrm{TC}(K_n) \le 2n-5$. Here we consider motion planning in a closely related space of polygons.

Let $M(\epsilon^{n-1}, 1, 1, 1, 2)$ denote the space of planar polygons with the prescribed side lengths, identified under *oriented* isometry. Then the double cover $M(\epsilon^{n-1}, 1, 1, 1, 2) \to \overline{M}(\epsilon^{n-1}, 1, 1, 1, 2)$ which identifies a polygon with its reflection across the long edge corresponds to the double cover $T^n \to K_n$. The n-torus is well known to satisfy $\mathrm{TC}(T^n) = n+1$, with easily-described motion planning rules. Using [2], we give here $n+1$ explicit motion planning rules between polygons in $M(\epsilon^{n-3}, 1, 1, 1, 2)$ corresponding to the simple motion planning rules for the torus.

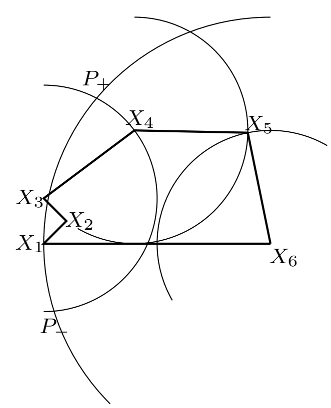
Date: June 26, 2017.
Key words and phrases. topological complexity, planar polygon spaces.
2000 *Mathematics Subject Classification*: 55M30, 70B15.

2. Description of polygons

Let $\ell = (\epsilon^{n-1}, 1, 1, 1, 2)$. A polygon in $M(\ell)$ or $\overline{M}(\ell)$ with successive vertices $X_1, \ldots, X_{n+3}$ can be placed so that $X_1 = (0, 0)$ and $X_{n+3} = (2, 0)$. Edges X_iX_{i+1}, $1 \le i \le n-1$, can be chosen as arbitrary vectors of length ϵ. These correspond to the first $n-1$ factors of T^n. The distance from X_n to X_{n+3} is a real number r satisfying $1 < r < 3$. Following [2], we choose X_{n+1} and X_{n+2} as follows.

Identify S^1 as $S := [-1, 1] \times \{-1, 1\}/(\pm 1, -1) \sim (\pm 1, 1)$. Let $C(\mathbf{x}, r)$ denote the circle of radius r centered at $\mathbf{x}$. Vertex X_{n+1} lies on the arc of $C(X_n, 1)$ which lies inside $C(X_{n+3}, 2)$. Parametrize this arc linearly from bottom (P_-) to top (P_+) as t goes from -1 to 1. For $[(t_1, t_2)] \in S$, X_{n+1} is the point on the arc with parameter value t_1. If $t_1 \ne \pm 1$, $C(X_{n+1}, 1)$ intersects $C(X_{n+3}, 1)$ at two points, one lying above the segment $X_{n+1}X_{n+3}$ and the other below it. Let X_{n+2} be the point above (resp. below) the segment if $t_2 = 1$ (resp. -1). We also say that X_{n+1}-X_{n+2}-X_{n+3} is an "up" (resp. "down") linkage. If $t_1 = \pm 1$, then $C(X_{n+1}, 1)$ and $C(X_{n+3}, 1)$ intersect at one point, which is chosen for X_{n+2}.

Note that conjugating the first $n-1$ S^1-factors, while negating the last one, corresponds to reflecting the polygon about its long side. The following figure illustrates the polygon associated to $(z_1, z_2, z_3) \in T^3$ with $z_1 = e^{i\pi/4}$, $z_2 = e^{3i\pi/4}$, $z_3 \approx [(.6, 1)]$, with $\epsilon \approx 0.3$.



3. Motion planning rules

Recall that the $n+1$ motion planning rules for T^n are that in each factor move along the shorter arc if the points are not antipodal and counterclockwise if they are.

The domains of continuity are sets having a fixed number of antipodal components. These motions can be done either simultaneously in all components, or sequentially.

We wish to tell how to move from a polygon with vertices $(X_1, \dots, X_{n+3})$ to polygon $(X_1, X_2', \dots, X_{n+2}', X_{n+3})$. For both of them, $X_1 = (0,0)$ and $X_{n+3} = (2,0)$. The polygons are associated to points $(z_1, \dots, z_{n-1}, [t_1, t_2])$ and $(z_1', \dots, z_{n-1}', [t_1', t_2'])$ in $T^{n-1} \times S$ as described in the previous section. We will do the motion for the first $n-1$ components first, as they are simpler.

We rotate the edges X_iX_{i+1} for $1 \leq i \leq n-1$ according to the rule for the torus (the shorter way if $z_i' \neq -z_i$, else counterclockwise). This can be done either simultaneously or sequentially. During this motion, the vertex X_n will be moving to X_n', causing the arc from P_- to P_+ to change smoothly. While this takes place, we maintain the parameter values $[t_1, t_2]$ from the initial polygon; as the arc moves, X_{n+1} stays the same fraction of the way along it, and the linkage X_{n+1}-X_{n+2}-X_{n+3} stays either "up" or "down" (or straight if $t_2 = \pm 1$). Following this motion, we will be at $(X_1, X_2', \dots, X_n', X_{n+1}'', X_{n+2}'', X_{n+3})$, where $(X_n', X_{n+1}'', X_{n+2}'', X_{n+3})$ has the initial parameter values $[t_1, t_2]$, and we wish to move it to $(X_n', X_{n+1}', X_{n+2}', X_{n+3})$ with parameter values $[t_1', t_2']$, without moving X_n'. There are two cases, corresponding to antipodal or not on the circle.

Case 1: Suppose that X_{n+1}' and X_{n+2}' are not the reflections of X_{n+1}'' and X_{n+2}'' across the segment $X_n'X_{n+3}$. If both are "up" linkages or both are "down" linkages (or one is straight), then, maintaining the sign of the linkage, move from X_{n+1}'' to X_{n+1}'. This will automatically move X_{n+2}'' to X_{n+2}'. If the linkages have opposite sign (i.e., $t_2' \neq t_2$), then without loss of generality assume that

$$d(X_{n+1}'', P_+) + d(P_+, X_{n+1}') < d(X_{n+1}'', P_-) + d(P_-, X_{n+1}'). \tag{3.1}$$

(These will not be equal by the "not reflections" assumption.) Then move from X_{n+1}'' to P_+ using its linkage sign (i.e., t_2), and then from P_+ to X_{n+1}' using linkage sign t_2'. If the opposite inequality occurs in (3.1), move similarly through P_-.

Case 2: Suppose that X_{n+1}' and X_{n+2}' are the reflections of X_{n+1}'' and X_{n+2}'' across the segment $X_n'X_{n+3}$. If X_{n+1}''-X_{n+2}''-X_{n+3} is an "up" linkage (i.e., $t_2 = 1$), move X_{n+1}'' down to P_-, maintaining the "up" orientation, and then switch to a "down" orientation as you move up from P_- to X_{n+1}'. If X_{n+1}''-X_{n+2}''-X_{n+3} is a "down" linkage, move X_{n+1}'' up to P_+, maintaining the "down" orientation, and then switch to an "up" orientation as you move down from P_+ to X'_{n+1}. The key point for continuity here is that if you were moving from P_+ to P_-, you get the same path regardless of whether you think of the initial orientation as being up or down. It will move through "up" linkages. Similarly, motion from P_- to P_+ will be through "down" linkages either way you think about it.

References

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[2] ———, *Real projective space as a planar polygon space*, Morfismos **19** (2015) 1–6.

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