2021 LEHIGH UNIVERSITY HIGH SCHOOL MATH CONTEST

1. (2 pts) How many right triangles have all side lengths integers and both legs of length a prime number?
2. (2 pts) Find a number k satisfying
$$\sqrt[3]{27\sqrt{3} - 34\sqrt{2}} = \sqrt{3} + k\sqrt{2}.$$
3. (2 pts) Evaluate the continued fraction $6 + \cfrac{1}{3 + \frac{1}{6 + \frac{1}{3 + \cdots}}}$.
4. (2 pts) How many ordered pairs (a, b) of positive integers satisfy
$$a^2 + b^2 = ab + 7?$$
5. (2 pts) Let x be a real number for which $x^2 - 3x + 1 = 0$. What is the value of $x^3 + \dfrac{1}{x^3}$?
6. (2 pts) Andy and Beth cross a lake in a straight path with the help of a one-seat kayak. Each can paddle at 7 km/hr and swim at 3 km/hr. They start from the same point at the same time, with Andy paddling and Beth swimming. After a while, Andy stops the kayak and starts swimming. The kayak doesn't move until Beth gets to it. Then she starts paddling the kayak. They arrive at the far side of the lake at the same time, 90 minutes after they began. For how many minutes was the kayak not being paddled?
7. (2 pts) Dawn has P packages, each containing 19 candies. If she divides all the candies equally among 7 friends, there are 4 candies left over. If she divides them equally among 11 friends there is 1 candy left over. What is the smallest possible value of P?

8. (3 pts) A bridge hand consists of 13 cards, randomly selected from a standard 52-card deck. When evaluating a hand, each Ace counts 4 points, each King counts as 3 points, each Queen 2 points, and each Jack 1 point. Given that a hand has exactly four cards that are Ace, King, Queen, or Jack, what is the probability that it has exactly 13 points?

9. (3 pts) Write as a reduced fraction
$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{98\cdot 99\cdot 100}.$$

10. (3 pts) In a class of 100 students, 90 are studying Spanish, 80 are studying French, 70 are studying German, and 60 are studying Italian. No student is studying all four languages. What is the number of students who are studying both Spanish and French?

11. (3 pts) The Gaussian integers are complex numbers whose real and imaginary parts are both integers. The solutions of
$$x^4+8x^3+14x^2-8x+145=0$$
are Gaussian integers. What is the largest value of a such that $a+bi$ is a solution of this equation for some b?

12. (3 pts) Find the smallest possible value of the expression $\left\lfloor \frac{a+b}{c} \right\rfloor + \left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor$ where a, b, and c are positive integers.

13. (3 pts) Let $f(k)=\lfloor k/3 \rfloor$. How many positive integers $n<1000$ have the property that none of n, $f(n)$, $f(f(n))$, and $f(f(f(n)))$ are divisible by 3?

14. (3 pts) Triangle ABC has perimeter 25. Points A' and B' lie on sides AC and BC, respectively, such that $A'B'$ is tangent to a circle inscribed inside the triangle and is parallel to AB. If $A'B'$ has length 2, what is the largest possible length of AB?

15. (3 pts) Let r_1, r_2, r_3, and r_4 be the roots of the polynomial $P(x) = 9x^4 - 3x^3 - 101x^2 + 195x - 100$. Compute

$$(r_1 + r_2 + r_3)(r_1 + r_2 + r_4)(r_1 + r_3 + r_4)(r_2 + r_3 + r_4).$$

16. (3 pts) Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4 + 4}$.

17. (3 pts) Let ABC be an isosceles triangle with $AB = AC$, and let ℓ be the line which bisects $\angle ABC$. The line through C parallel to ℓ intersects the circumcircle of triangle ABC again at D. Let P be the intersection of lines AD and ℓ, and suppose that $AP : PD = 2 : 3$. Find the value of $\cos(\angle BAC)$.

18. (3 pts) The number $989 \cdot 1001 \cdot 1007 + 320$ is a product of three primes. Write them, separated by commas, without parentheses?

19. (4 pts) How many 10-digit numbers can be formed using just the digits 1, 2, or 3 such that the number of occurrences of the digit 1 is even?

20. (4 pts) Let $\displaystyle\frac{1}{1 - x - x^2 - x^3} = \sum_{n=0}^{\infty} a_n x^n$. What is the sum of all positive integers n for which $a_{n-1} = n^2$?

21. (4 pts) Let x and y be positive integers such that $\dfrac{xy}{x+y} > 2$. What is the minimum possible value of $\dfrac{xy}{x+y}$?

22. (4 pts) Define a sequence by $a_1 = 2021$ and, for $n \geq 1$,

$$a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ is even} \\ a_n - 1 & \text{if } a_n \text{ is odd.} \end{cases}$$

What is the smallest value of n for which $a_n = 0$?

23. (4 pts) Tommy rolls a fair 6-sided die until a 6 occurs for the first time. If n is the number of rolls preceding the first 6, he wins n^2 dollars. On average, how many dollars does he win?

24. (4 pts) Beginning at a vertex, an ant moves from vertex to adjacent vertex on a regular octahedron, with all moves equally likely. What is the probability that, after 10 moves, it is back where it started?

25. (4 pts) Let $\phi(n)$ denote the number of positive integers less than or equal to n that are relatively prime to n. For how many positive integers n less than 2021 is n divisible by $\phi(n)$?

26. (5 pts) What is the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$

for all x for which the expression is defined?

27. (5 pts) Cyclic quadrilateral $ABCD$ has $AB = 4$, $BC = 1$, $CD = 2$, and $DA = 3$. The midpoints of BC and DA are P and Q, respectively. Find PQ^2. (Note: this is length squared.)

28. (5 pts) What is the number of ordered pairs (m, n) of integers, with $1 \leq m, n \leq 1000$ such that $m/(n+1) < \sqrt{2} < (m+1)/n$? Recall $\sqrt{2} \approx 1.4142\cdots$.

29. (5 pts) The integers from 1 to 12 are written in some order. A "move" consists of interchanging an even number with an odd number in the list. What is the smallest number n such that, for any initial order, the integers can always be brought to their usual order $1, 2, \ldots, 12$ in n moves?

30. (5 pts) Triangle ABC is inscribed in a circle of radius 1. For each side of the triangle, a circle is drawn outside the triangle but inside the circle, tangent to the side of the triangle at its midpoint and tangent to the circle. The radii of two of the circles are $\frac{2}{3}$ and $\frac{2}{11}$. What is the radius of the third circle?

SOLUTIONS, annotated by number of students (out of 137) with correct answer.

1. 0. [125] Since 4 cannot be a difference of squares, and the sum of two odd squares is 2 mod 4, hence is not a square, no such triangles are possible.
2. -2. [113] $(\sqrt{3}+k\sqrt{2})^3 = 3\sqrt{3}+9k\sqrt{2}+6k^2\sqrt{3}+2k^3\sqrt{2}$. We must have $3+6k^2 = 27$ so $k = \pm 2$, and $2k^3+9k = -34$ so $k = -2$.
3. $3+\sqrt{11}$. [103] It satisfies $6+\dfrac{1}{3+\frac{1}{x}} = x$, so $(3+\frac{1}{x})(x-6) = 1$, so $3x^2-18x-6 = 0$, from which the solution (> 6) is obtained from the quadratic formula.
4. 4. [89] We have $(a-b)^2+a^2+b^2 = 14$. Therefore, $\{|a-b|, a, b\} = \{1, 2, 3\}$. Hence, all possible pairs (a, b) are $(1, 3), (2, 3), (3, 1), (3, 2)$.
5. 18. [111] Dividing by x on both sides of the equation $x^2-3x+1 = 0$, we have $x+\frac{1}{x} = 3$. Therefore,
   $$x^3+\tfrac{1}{x^3} = \left(x+\tfrac{1}{x}\right)^3 - 3\left(x+\tfrac{1}{x}\right) = 3^3-3(3) = 18.$$
6. 36. [100] Let t_A be the number of hours when Andy paddles and Beth swims, t_B the number of hours when Beth paddles and Andy swims, and t_N the number of hours when neither paddles and both swim. Also, D is the distance (in km) across the lake. Then $7t_A+3t_B+3t_N = D$, $7t_B+3t_A+3t_N = D$, $7t_A+7t_B = D$, and $t_A+t_B+t_N = \frac{3}{2}$. We obtain $t_A = t_B$, then $10t_A+3t_N = 14t_A$. Hence $t_N = \frac{4}{3}t_A$ and then $\frac{10}{3}t_A = 90$ minutes so $t_A = 27$ minutes and $t_N = 36$ minutes.
7. 40. [98] The total number of candies must be of the form $7k+4$ and $11\ell+1$. Note that if you add 10 to both sides, you obtain a multiple of 7 and 11, hence the number of candies is 10 less than a multiple of 77. We want to know the smallest multiple of 19 which is 10 less than a multiple of 77. Each time you add 77, you increase the mod 19 value by 1. Starting at 67,

which is the smallest number 10 less than a multiple of 77, and is $\equiv -9 \bmod 19$, you would need to add 9 multiples of 77 to get to 0 mod 19. Thus you need $67 + 9 \cdot 77 = 760$ candies, so $P = 760/19$.

8. 32/455. [35] It suffices to consider just the four "face" cards in the hand. There $\binom{16}{4} = 1820$ ways to choose these. The ways to get 13 points are AAAJ (16 ways), AAKQ (96 ways), and AKKK (16 ways). For example, with AAAJ, four ways to say which suit had the J, and independently 4 ways to choose which suit did not have an Ace. The unreduced answer is 128/1820.

9. 4949/19800. [73] You need to know that $\frac{1}{x(x+1)(x+2)} = \frac{1}{2}\left(\frac{1}{x} - \frac{2}{x+1} + \frac{1}{x+2}\right)$. Then most of the intermediate terms cancel, leaving just $\frac{1}{2}(1 - \frac{1}{2} - \frac{1}{99} + \frac{1}{100}) = \frac{1}{2}(\frac{1}{2} - \frac{1}{9900})$.

10. 70. [60] Let U be the set of all students in this class, A be the set of students who are not studying Spanish, B be the set of students who are not studying French, C be the set of students who are not studying German, and D be the set of students who are not studying Italian. We have $U = A \cup B \cup C \cup D$ and $n(A) = 10, n(B) = 20, n(C) = 30, n(D) = 40$. Therefore,

$$100 = n(A \cup B \cup C \cup D) \leq n(A)+n(B)+n(C)+n(D) = 10+20+30+40 = 100.$$

Therefore, A, B, C, D are mutually disjoint. Hence, the set of students who are studying both Spanish and French is $C \cup D$, and so $n(C \cup D) = n(C) + n(D) = 30 + 40 = 70$.

11. 1. [64] The solutions come in pairs $a_j \pm b_j i$ satisfying $x^2 - 2a_j + (a_j^2 + b_j^2) = 0$. 145 can be factored as $1 \cdot 145$ or $29 \cdot 5$. Since ± 1 and $\pm i$ are not solutions, we rule out the first, so one pair of solutions is either $(1 \text{ or } -1) \pm 2i$ or $(2 \text{ or } -2) \pm i$, and the other pair is either $(5 \text{ or } -5) \pm 2i$ or $(2 \text{ or } -2) \pm 5i$. We must have $2a_1 + 2a_2 = -8$. This can occur either with solutions $-5 \pm 2i$ and $1 \pm 2i$, or with solutions $-2 \pm 5i$ and $-2 \pm i$. The first way would give factorization $(x^2 + 10x + 29)(x^2 - 2x + 5)$, which is correct,

while the second way would give $(x^2+4x+29)(x^2+4x+5)$, which isn't. Thus the two values of a are -5 and 1.

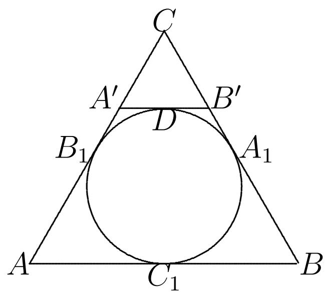
12. 4. [49] Note that the above expression is equal to 4 if $(a,b,c)=(3,4,4)$. Suppose there exists a 3-tuple (a,b,c) such that $\left\lfloor\frac{a+b}{c}\right\rfloor+\left\lfloor\frac{b+c}{a}\right\rfloor+\left\lfloor\frac{c+a}{b}\right\rfloor\le 3$. Without loss of generality, let $a\le b\le c$. We have $\frac{b+c}{a}\ge 2$, $\frac{c+a}{b}\ge 1$, and $\frac{a+b}{c}\ge 0$. Hence, $\left\lfloor\frac{a+b}{c}\right\rfloor=0$, $\left\lfloor\frac{b+c}{a}\right\rfloor=2$, and $\left\lfloor\frac{c+a}{b}\right\rfloor=1$. In particular, $a+b<c$, $b+c<3a$, and $c+a<2b$. Therefore, $(a+b)+(b+c)+(c+a)<c+3a+2b$, or equivalently, $c<a$, contradiction! Therefore, the minimum value of the given expression is 4.

13. 192. [42] These are the numbers whose base-3 expansion does not have a 0 among its last four digits. For any positive integer m, out of the 81 numbers from $81m+1$ to $81m+81$, there are 16 such numbers, of which the smallest is $81m+27+9+3+1$. Since $999/81$ equals 12 with remainder 27, there are $12\cdot 16$ such numbers.

14. 10. [31] Let A_1, B_1, C_1, and D be the points of tangency of the circle to the lines BC, AC, AB, and $A'B'$, respectively. Then $AC_1=AB_1$, $BC_1=BA_1$, $A'B_1=A'D$, and $B'D=B'A_1$. If $x=AB$, then the perimeter of $A'B'C$ equals

$$\begin{aligned}
& A'C+CB'+B'D+A'D \\
= \;& (AC-AB_1-B_1A')+(BC-A_1B-A_1B')+A_1B'+B_1A' \\
= \;& AC+BC-(AC_1+C_1B) \\
= \;& 25-2x.
\end{aligned}$$

Since triangles ABC and $A'B'C$ are similar, we obtain $2/x=(25-2x)/25$. Thus $2x^2-25x+50=0$, so $x=10$ or 2.5.

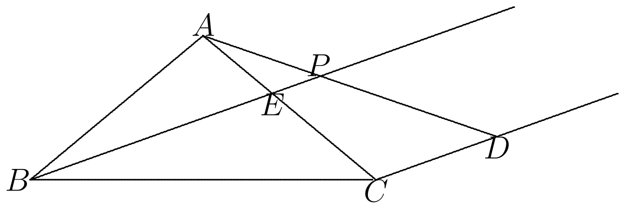


15. $-416/81$. [56] Consideration of the first two terms shows that the sum of the roots is $\frac{1}{3}$, so the desired quantity is $(\frac{1}{3}-r_4)(\frac{1}{3}-r_3)(\frac{1}{3}-r_2)(\frac{1}{3}-r_1)$. This equals $\frac{1}{9}P(\frac{1}{3}) = -\frac{416}{81}$. (In fact, the roots are 1, -4, $\frac{5}{3}$, and $\frac{5}{3}$.)

16. $3/8$. [65]

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n}{n^4+4} &= \sum_{n=1}^{\infty}\frac{n}{(n^2-2n+2)(n^2+2n+2)}\\ &= \frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{n^2-2n+2}-\frac{1}{n^2+2n+2}\right)\\ &= \frac{1}{4}\left[\sum_{n=1}^{\infty}\frac{1}{(n-1)^2+1}-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2+1}\right]\\ &= \frac{1}{4}\left(\frac{1}{0^2+1}+\frac{1}{1^2+1}\right)=\frac{3}{8}.\end{aligned}$$

17. $-1/8$. [21] Let ℓ intersect side AC at E. Since $PE \parallel DC$, we have $\frac{AP}{PD} = \frac{AE}{EC}$. By Angle Bisector Theorem, we have $\frac{AE}{EC} = \frac{AB}{BC}$. Therefore, $\frac{AB}{BC} = \frac{AP}{PD} = \frac{2}{3}$, so let $AB = AC = 2k$ and $BC = 3k$. By the Law of Cosines on $\triangle ABC$, we get $(3k)^2 = (2k)^2 + (2k)^2 - 2(2k)(2k)\cos(\angle BAC)$. Therefore, $\cos(\angle BAC) = -1/8$. Note that the circumcircle was not relevant. D could have been any point on the line through C.



18. 991, 997, 1009. [74] Setting $X = 991$, we obtain

$$\begin{aligned} 989 \cdot 1001 \cdot 1007 + 320 &= (X-2)(X+10)(X+16) + 320 \\ &= X^3 + 24X^2 + 108X - 320 + 320 \\ &= X(X+6)(X+18) \\ &= 991 \cdot 997 \cdot 1009. \end{aligned}$$

19. 29525. [76] Let a_n be the number of n-digit numbers can be formed using the digits $1, 2, 3$ such that the number of 1's is even. Among such numbers, if the first digit is 2 or 3, then the remaining $(n-1)$-digit number has an even number of 1's. However, if the first digit is 1, then the remaining $(n-1)$-digit number has an odd number of 1's. Therefore, $a_n = 2a_{n-1} + (3^{n-1} - a_{n-1}) = a_{n-1} + 3^{n-1}$ for all $n > 1$. Therefore,

$$a_n = 3^{n-1} + 3^{n-2} + \ldots + 3^1 + a_1 = \frac{3^n - 3}{2} + 2 = \frac{3^n + 1}{2}.$$

Hence, $a_{10} = \dfrac{3^{10}+1}{2} = 29525$.

20. 10. [15] The equation reduces to

$$1 = a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1 - a_0)x^3 + (a_4 - a_3 - a_2 - a_1)x^4 + \cdots.$$

Thus $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, and for $n \geq 3$, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. The sequence $a_0, a_1, \ldots$ begins $1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \ldots$. Thus $a_{n-1} = n^2$ for $n = 1$ and $n = 9$. To see that this cannot happen again, one can note that subsequent ratios a_n/a_{n-1} are greater than 1.5, while subsequent values of $(n+1)^2/n^2$ are less than 1.5.

21. 2.1 or 21/10. [69] The answer is the reciprocal of the value of $\frac{1}{x}+\frac{1}{y}$ closest to $\frac{1}{2}$ but less than $\frac{1}{2}$. For each x, find the smallest y such that $\frac{1}{y}<\frac{1}{2}-\frac{1}{x}$. For $x=3,4,5,6,\geq 7$, the values of y are $7,5,4,4,3$. Choosing $\{x,y\}=\{3,7\}$ gives the desired solution.

22. 19. [50] The binary expansion of $2021=(2^{11}-1)-26$ has $(11-3)$ 1's and three 0's. A 0 in the binary expansion is removed in one step, and a 1 is removed in two steps, except that the last 1 only takes one step. So $2\cdot 8+3-1$ steps are required, making $a_{19}=0$.

23. 55. [39] We wish to determine the value of $\sum_{n=0}^{\infty} n^2\left(\frac{5}{6}\right)^n\frac{1}{6}$. Squaring the formula $\frac{1}{1-p}=\sum p^n$ yields $\left(\frac{1}{1-p}\right)^2=\sum np^{n-1}$, and multiplying again by $\frac{1}{1-p}$, we obtain $\left(\frac{1}{1-p}\right)^3=\sum\frac{n(n+1)}{2}p^{n-1}$. Thus $\sum n^2p^n=2p\left(\frac{1}{1-p}\right)^3-p\left(\frac{1}{1-p}\right)^2$. Our desired sum is $\frac{1}{6}\left(\frac{5}{3}\cdot 6^3-\frac{5}{6}\cdot 6^2\right)=60-5$.

24. 171/1024. [29] Let a_n, b_n, and c_n denote the probabilities that after n steps, the ant will be on (a) the vertex where it started, (b) the union of the four adjacent vertices, and (c) the opposite vertex. Then $a_n=b_{n-1}/4$, $b_n=a_{n-1}+\frac{b_{n-1}}{2}+c_{n-1}$, and $c_n=b_{n-1}/4$. Combining yields $b_n=\frac{b_{n-1}}{2}+\frac{b_{n-2}}{2}$. Solving the characteristic equation $x^2=\frac{1}{2}x+\frac{1}{2}$ yields $x=1$ and $-\frac{1}{2}$, so $b_n=\alpha\cdot 1^n+\beta(-\frac{1}{2})^n$ for some numbers α and β. We have $a_0=1$, $b_0=c_0=0$, so $b_1=1$, so $\alpha=\frac{2}{3}$ and $\beta=-\frac{2}{3}$, thus $b_n=\frac{2}{3}(1-(-\frac{1}{2})^n)$. Finally $a_{10}=\frac{1}{4}\cdot\frac{2}{3}(1+\frac{1}{512})=\frac{513}{3\cdot 1024}$.

25. 41. [16] First, we will prove the following lemma:

Lemma. If n is divisible by $\phi(n)$, then $n=1$ or $n=2^p3^q$ for some $p\geq 1$ and $q\geq 0$.

Proof. Clearly, $n = 1$ works. If $n > 1$ is odd, then $\phi(n)$ is even since $\gcd(k, n) = 1$ whenever $\gcd(n-k, n) = 1$. Therefore, n is not divisible by $\phi(n)$ since $2 \mid \phi(n)$ but $2 \nmid n$.

Consider the case when n is even. If $n = 2^k$ for some positive integer k, then $\phi(n) = 2^{k-1}$, so n is divisible by $\phi(n)$. Next, write $n = 2^k p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$, where $k, t, a_1, \dots, a_t \geq 1$ and $p_1, p_2, \dots, p_t$ are distinct odd primes. By the Euler's product formula, we have

$$\phi(n) = n\left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{1}{p_1}\right)\left(1 - \tfrac{1}{p_2}\right)\dots\left(1 - \tfrac{1}{p_t}\right) = n \cdot \tfrac{(p_1-1)(p_2-1)\dots(p_t-1)}{2p_1p_2\dots p_t}.$$

If $\phi(n)|n$, then $(p_1 - 1)(p_2 - 1)\dots(p_t - 1) \mid 2p_1p_2\dots p_t$. Note that $2^t \mid (p_1-1)(p_2-1)\dots(p_t-1)$ since $p_i - 1$ is even for each i. However, $2p_1p_2\dots p_t$ is divisible by 2, but not by 4. Therefore, $t = 1$. In particular, $p_1 - 1 \mid 2p_1$. Hence $p_1 = 3$. Therefore, $n = 2^k 3^a$ for some $k, a \geq 1$. ∎

Hence, we need to count the number of ordered pairs (p, q) of integers such that $2^p 3^q < 2021$ and $p \geq 1, q \geq 0$. Note that $q \leq 6$.

- If $q = 0$, then $p \in \{1, 2, \dots, 10\}$.
- If $q = 1$, then $p \in \{1, 2, \dots, 9\}$.
- If $q = 2$, then $p \in \{1, 2, \dots, 7\}$.
- If $q = 3$, then $p \in \{1, 2, \dots, 6\}$.
- If $q = 4$, then $p \in \{1, 2, 3, 4\}$.
- If $q = 5$, then $p \in \{1, 2, 3\}$.
- If $q = 6$, then $p = 1$.

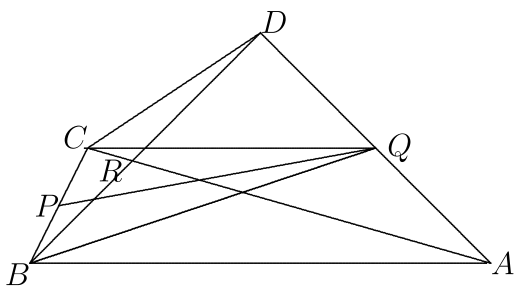
Therefore, the answer is $10 + 9 + 7 + 6 + 4 + 3 + 1 + 1 = 41$. (don't forget $n = 1$!)

26. $2\sqrt{2} - 1$. [14] Let $y = \sin x + \cos x$. Then $y^2 = 2 \sin x \cos x + 1$, so $\sin x \cos x = \dfrac{y^2 - 1}{2}$. Then $\tan x + \cot x = \dfrac{1}{\sin x \cos x} = \dfrac{2}{y^2 - 1}$, and $\sec x + \csc x = \dfrac{\sin x + \cos x}{\sin x \cos x} = \dfrac{2y}{y^2 - 1}$. Thus the desired expression is $f(y) = \left| y + \dfrac{2 + 2y}{y^2 - 1} \right| = \left| y + \dfrac{2}{y - 1} \right|$ for $y \neq 1$. By AM-GM, the minimum of $x + \frac{2}{x}$ is $2\sqrt{2}$, occurring when $x = \sqrt{2}$. Letting $x = y - 1$, we find that for $y > 1$, the minimum value of

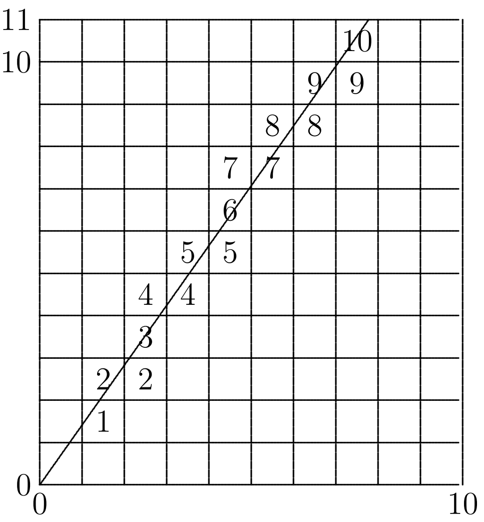
$f(y)$ is $2\sqrt{2}+1$, occurring when $y = \sqrt{2}+1$, while for $y = -z$ with $z > 0$, then $f(y) = z + \dfrac{2}{z+1}$ has minimum $2\sqrt{2}-1$ occurring when $z = \sqrt{2}-1$. For $0 \le y < 1$, $f(y) \ge 2$.

27. 291/35. [2] Let R be the intersection point of the diagonals. Because the quadrilateral is cyclic, triangles ABR and DCR are similar, as are ADR and BCR. Thus if $CR = x$, then $BR = 2x$, $DR = 3x$, and $AR = 6x$. Ptolemy's theorem implies $(x+6x)(2x+3x) = 2\cdot 4 + 1\cdot 3$, so $x^2 = \frac{11}{35}$. By a property of medians,

$$\begin{aligned} PQ^2 &= \tfrac{1}{4}(2BQ^2 + 2CQ^2 - 1) = \tfrac{1}{2}\cdot\tfrac{1}{4}(32 + 2BD^2 - 9) + \tfrac{1}{2}\cdot\tfrac{1}{4}(2AC^2 + 8 - 9) - \tfrac{1}{4} \\ &= \tfrac{1}{8}\cdot 20 + \tfrac{1}{4}(BD^2 + AC^2) = \tfrac{5}{2} + \tfrac{1}{4}((5x)^2 + (7x)^2) \\ &= \tfrac{5}{2} + \tfrac{1}{4}\cdot 74\cdot\tfrac{11}{35} = 291/35. \end{aligned}$$



28. 1706. [2] The inequality is satisfied if and only if the line $y = \sqrt{2}x$ passes through the unit square whose lower left corner is at (n, m). It passes through all 1000 horizontal strips with bottom at $y = m$ for $1 \le m \le 1000$. For every vertical strip through which it passes, it will pass through two squares at some level. There will be 707 such strips, since $\frac{1001}{\sqrt{2}} \approx 707.8$. There would be $1000 + 707$ such intersections, except that we exclude the square with lower corner at $(0, 1)$. The diagram below using 10 instead of 1000 should clarify this.



29. 14. [0] Think of an arrangement as a set of 12 ordered pairs (i, i'), where i' is the integer in the ith position. A "cycle" is a subset of these of the form $\{(i_1, i_2), (i_2, i_3), \ldots, (i_r, i_1)\}$ for some $r \geq 1$. A mixed cycle is one which contains both even and odd numbers. The goal is to get it to 12 length-1 cycles.

With no more than 3 moves, you can get it so that all cycles are mixed. From a mixed cycle, you can do a move which splits off a length-1 cycle, and either decreases by 1 the absolute value of the difference between the number of even entries and odd entries, or if this was 0, changes it to 1. Thus a length-r cycle can be split into r length-1 cycles in $r - 1$ moves. Thus $3 + 11$ moves will always work.

If you start with

$$\{(2,4),(4,6),(6,8),(8,10),(10,12)\} \cup \{(1,3),(3,1)\} \cup \{(5,7),(7,5)\} \cup \{(9,11),(11,9)\},$$

then interchanging the 3 and 4 changes the initial cycle so that the $(2,4)$ is replaced by $(2,3),(3,1),(1,4)$. Thus the three moves required to make it so that all cycles are mixed makes it a length-12 cycle, and 11 moves are required to split it completely.

30. [5] 1/33. Let O be the center of the initial circle, and D the point of tangency of the circle of radius $\frac{2}{3}$, the midpoint of BC. Then

$$BC^2 = 4BD^2 = 4(OB^2 - OD^2) = 4(1 - (\tfrac{4}{3} - 1)^2) = \tfrac{32}{9}.$$

Similarly, if E is the midpoint of AC and the point of tangency of the circle of radius $\frac{2}{11}$, then

$$AC^2 = 4(OC^2 - OE^2) = 4(1 - (1 - \tfrac{4}{11})^2) = 4(1 - \tfrac{49}{121}) = \tfrac{288}{121}.$$

Now we obtain

$$\cos BOC = \tfrac{OB^2+OC^2-BC^2}{2OB\cdot OC} = -\tfrac{7}{9} \text{ and } \cos COA = \tfrac{OC^2+OA^2-AC^2}{2OC\cdot OA} = -\tfrac{23}{121}.$$

Hence $\sin BOC = \frac{4\sqrt{2}}{9}$ and $\sin COA = \frac{84\sqrt{2}}{121}$. Next we obtain

$$\cos AOB = \cos(BOC - COA) = \cos BOC\cdot\cos COA + \sin BOC\cdot\sin COA = \tfrac{833}{1089}.$$

Now $AB^2 = OA^2 + OB^2 - 2OA\cdot OB\cos AOB = \frac{512}{1089}$. Now, with F the midpoint of AB, we have $OF^2 = OA^2 - \frac{1}{4}AB^2 = \frac{961}{1089}$. So $OF = \frac{31}{33}$ and the desired radius is $\frac{1}{2}(1 - OF)$.