1. What is the sum of the digits of $(1010101)^2$?
2. Find two numbers whose sum is 52 such that one is three times as large as the other.
3. What is the area of a circle inscribed in a square of side-length 3?
4. A student takes three exams. The second has twice as many questions as the first, and the third has three times as many questions as the first. The student answers exactly 75% of the questions correctly on the first exam, exactly 81% on the second exam, and exactly 85% on the third exam. Out of all the questions on the three exams, what percent did he answer correctly?

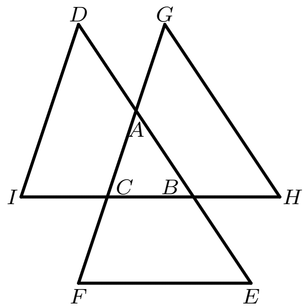
5. How many members of the set $\{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots, \frac{99}{7}\}$ are integer multiples of $\frac{2}{5}$?
6. List all positive solutions of $x^3 + x^2 - 2x - 2 = 0$.
7. Find all possible ordered pairs (A, B) of digits for which the decimal number $\underline{7}\,\underline{\mathrm{A}}\,\underline{8}\,\underline{\mathrm{B}}$ is divisible by 45.
8. A sequence satisfies $a_1 = 3$, $a_2 = 5$, and $a_{n+2} = a_{n+1} - a_n$ for $n \geq 1$. What is the value of a_{2013}?
9. Find the ordered triple (a, b, c) of positive integers which satisfies both $(a + b)(a + c) = 77$ and $(a + b)(b + c) = 42$.
10. The set of points (x, y) which satisfy $|x| \leq 1$, $|y| \leq 1$, and $|y| = |x| + x$ consists of several line segments. What is the sum of the lengths of these segments?
11. What is the smallest positive integer with more than 13 positive divisors?
12. How many positive integers less than 1200 have no repeating digits; i.e., no digit occurs more than once.

13. The sum of the first ten terms of a nonzero geometric series is 244 times the sum of the first five terms. What is the common ratio?

14. Five circles of equal radius are placed inside a square of side length 1 in such a way that no two intersect in more than one point. What is the largest possible radius for these circles?

15. You take six steps, each time moving either one step forward or one step back, with probability $\frac{1}{2}$ of each each time. What is the probability that you end up back where you started?

16. In the diagram below, $DA = AB = BE$, $GA = AC = CF$, and $IC = CB = BH$. If $EF = 5$, $DI = 6$, and $GH = 7$, what is the area of triangle ABC?



17. Assume that for every person the probability that they have exactly one child is $1/4$, the probability that they have exactly two children is $1/2$, and the probability that they have exactly three children is $1/4$. What is the probability that a person will have exactly four grandchildren?

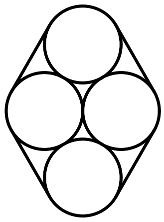
18. If $x = 2 + \sqrt{3}$, find an integer or fraction equal to $x^4 + \dfrac{1}{x^4}$.

19. Let $S = \{3, 4, 5, 6, 8, 9, 10, 11, 12\}$. What is the sum of the elements in S which are divisors of 7021500420?

20. What is the largest base-10 number whose base-3 and base-4 expansions have the same number of digits?

21. Find the area of triangle ABC satisfying the following properties. $AC = 13$ and $CM = 4\sqrt{10}$, where M is the midpoint of AB. The altitude from C does not meet AB itself, but it meets the extension of AB, and this altitude has length 12.

22. Four circles of radius 1 are tangent to one another as in the diagram below, except that clearly the top and bottom circles are not tangent to one another. A tight band is wrapped around the four circles as in the diagram. What is its length?



23. Diameter AB of circle with center O has length 2. From the midpoint Q of OA, a perpendicular is drawn intersecting the circle at P. Find the radius of the circle which can be inscribed in triangle APB.

24. Find the maximum value of $x + y$ taken over all ordered pairs (x, y) of real numbers which satisfy $x^2 + y^2 = 4$ and $x + \sqrt{5}y \geq 2\sqrt{5}$.

25. The bases of a trapezoid have lengths 12 and 15. Find the length of the segment parallel to the bases which passes through the point of intersection of the diagonals and extends from one side to the other.

26. How many ordered pairs of integers (x, y) satisfy $x+y-xy = 49$?

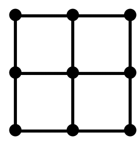
27. Compute the number of ordered 4-tuples (a, b, c, d) of positive integers such that $a + b + c + d = 14$.

28. For $n \geq 1$, let d_n denote the length of the line segment connecting the two points where the line $y = x + n + 1$ intersects the parabola $8x^2 = y - \frac{1}{32}$. Compute the sum $\displaystyle\sum_{n=1}^{1000} \frac{1}{n \cdot d_n^2}$.

29. Let $P(x)$ be a monic polynomial of degree 3. (*Monic* here means that the coefficient of x^3 is 1.) Suppose that the remainder when $P(x)$ is divided by x^2-5x+6 equals 2 times the remainder when $P(x)$ is divided by $x^2 - 5x + 4$. If $P(0) = 100$, what is $P(5)$?

30. What is the range of values of $a + \frac{3-b}{a}$ taken over all ordered pairs (a, b) of real numbers for which $x^2 - ax + b$ has two distinct positive roots? Use interval notation to express your answer.

31. A bug starts at a corner vertex of the graph below and at each step moves along an adjacent edge, with equal probability for any adjacent edge. It continues until it gets to the center vertex. What is the expected (average) number of steps required until the first time it gets to the center vertex?



32. List all positive integers which have exactly 6 positive divisors, the sum of whose reciprocals is 2.

33. List all numbers which can be written as $x + y$ where x and y are positive integers satisfying $x^4 + y^4 = 15266$.

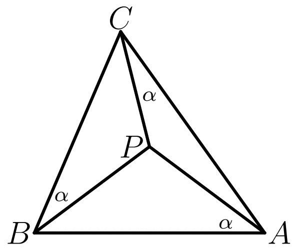
34. Compute all integers r such that a circle of radius r with diameter AC has a point B on the circle such that $AB = 30$ and BC is a positive integer.

35. Out of all positive integers n which have exactly 2107 positive divisors, what is the largest number of positive divisors that n^2 could have?

36. What is the smallest positive integer that cannot be written as the sum of 11 or fewer factorials? Note that the factorials need not be distinct.

37. Let a, b, and c be distinct nonzero real numbers such that $\frac{1+a^3}{a} = \frac{1+b^3}{b} = \frac{1+c^3}{c}$. Determine all possible values of $a^3 + b^3 + c^3$.

38. In the triangle below, angles PAB, PBC, and PCA are each equal to the same value α. The sides of triangle ABC have lengths 7, 8, and 9. Find $\tan(\alpha)$.



39. What is the largest positive integer n, not a multiple of 10, such that removing the last two digits of n^2 leaves a perfect square?

40. Let $P(x)$ be a polynomial of degree 10 such that $P(2^i) = i$ for $0 \leq i \leq 10$. What is the coefficient of x^1 in $P(x)$?

Solutions to 2013 test, annotated with the number of the 72 people who scored at least 20 answering the question correctly.

1. 16. [69] The product is 1020304030201.
2. 13 and 39. [71] We have $x + 3x = 52$ so $x = 52/4 = 13$.
3. $9\pi/4$. [72] The radius of the circle is 3/2.
4. 82 or 82%. [70] If there were n questions on the first exam, then the student answered correctly $.75n + .81 \cdot 2n + .85 \cdot 3n$ out of $6n$. The answer is $(.75 + 1.62 + 2.55)/6 = 4.92/6 = .82$.
5. 7. [64] If $\frac{a}{7} = k\frac{2}{5}$, then $14k = 5a$, so a must be a multiple of 14. There are 7 such numbers less than 100.
6. $\sqrt{2}$. [63] The polynomial factors as $(x + 1)(x^2 - 2)$.
7. $(3, 0)$ and $(7, 5)$. [70] The number is divisible by 5 and by 9, so B must be 0 or 5, and the sum of the digits must be divisible by 9.
8. 2. [64] One easily checks the first eight values to be 3, 5, 2, -3, -5, -2, 3, 5. It becomes clear that these are repeating with period 6. Since 2013 is 3 more than a multiple of 6, $a_{2013} = a_3 = 2$.
9. $(6, 1, 5)$. [70] Since the only common divisors of 77 and 42 are 1 and 7, and $a + b > 1$, we must have $a + b = 7$. Then $a + c = 11$ and $b + c = 6$. Subtracting the first two of these equations yields $c - b = 4$. Adding this to the third gives $2c = 10$, from which the rest follows easily.
10. $1 + \sqrt{5}$. [56] The lines are $y = \pm 2x$ for $x > 0$, and $y = 0$ for $x < 0$. Inside the box, we have segments from $(0, 0)$ to $(-1, 0)$, and to $(\frac{1}{2}, \pm 1)$, of lengths 1, $\frac{1}{2}\sqrt{5}$, and $\frac{1}{2}\sqrt{5}$.
11. 120. [35] Since 120 equals $2^3 \cdot 3 \cdot 5$, it has $4 \cdot 2 \cdot 2 = 16$ divisors. Smaller candidates with lots of divisors are 60 and 96, but each of them only has 12 divisors.

12. 794. [39] There are 9 such 1-digit numbers, $9 \cdot 9 = 81$ such 2-digit numbers (since the first digit can be 1 to 9 and the second digit any other number from 0 to 9). Similarly the number of such 3-digit numbers is $9 \cdot 9 \cdot 8 = 648$, since the third digit cannot equal either of the first two. For 4-digit numbers beginning 10, there are $8 \cdot 7$ and there are none beginning 11.

13. 3. [69] We have $\frac{r^{10}-1}{r-1} = 244\frac{r^5-1}{r-1}$. Thus $r^{10} - 244r^5 + 243 = 0$. Hence $r^5 = 1$ or 243, but $r = 1$ doesn't make sense. So the answer is $\sqrt[5]{243} = 3$.

14. $\frac{1}{2}(\sqrt{2} - 1)$. [61] If one circle is at the center of the square and the others are in the corners, then we have $1 = 2r + 4\frac{r}{\sqrt{2}}$. To see this, let A (resp. B) be the center of the upper right (resp. lower left) circle. Then the vertical displacement from A to B is $4r/\sqrt{2}$, and the vertical displacement from each of these to the nearer horizontal edge is r. Thus $r(2 + \frac{4}{\sqrt{2}}) = 1$, and multiplying both sides by $\sqrt{2} - 1$ yields the claim. It is not difficult to prove that any way of putting five equal circles of radius r in the square must have two centers in a square of side-length $\frac{1}{2} - r$, and hence must satisfy $2r \le (\frac{1}{2} - r)\sqrt{2}$, which implies that the above result is optimal.

15. 5/16. [69] There are 2^6 sequences of F (forward) and B (back), each equally likely. The number with 3 F's and 3 B's is $\binom{6}{3} = 20$.

16. $3\sqrt{6}/2$. [66] By similar triangles, $AC = \frac{1}{2}DI$, $AB = \frac{1}{2}GH$, and $CB = \frac{1}{2}FE$. Thus the area of triangle ABC is $1/4$ times the area of a triangle with sides 5, 6, and 7, which, by Heron's formula, is $\sqrt{9 \cdot 2 \cdot 3 \cdot 4}$.

17. 27/128. [56] There are three ways to have four grandchildren. One is to have two children, each of whom have two children. The probability of this is $\frac{1}{2}^3 = \frac{1}{8}$. Another is to have two children, with one having one child and the other three. The probability of this is $2 \cdot \frac{1}{2} \cdot \frac{1}{4}^2 = \frac{1}{16}$. Note that the factor 2 occurs

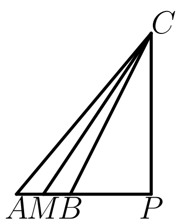
because it can be either the first child or the second child who has the three children. Similarly the probability that they have three children, one of whom has two children (and the other two have one) is $3 \cdot \frac{1}{4} \cdot \frac{1}{32} = \frac{3}{128}$. Adding these three probabilities yields the desired result.

18. 194. [57] $x + \frac{1}{x} = 2 + \sqrt{3} + \frac{1}{2+\sqrt{3}} \frac{2-\sqrt{3}}{2-\sqrt{3}} = 4$. Then $x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2 = 14$, and $x^4 + \frac{1}{x^4} = (x^2 + \frac{1}{x^2})^2 - 2 = 14^2 - 2$.

19. 51. [63] The number is clearly divisible by 4 but not by 8, and by 5. By the sum-of-digits test, it is divisible by 3 but not by 9. By the alternating-sum-of-digits test, it is divisible by 11. Thus of the numbers in S, it is divisible by 3, 4, 5, 6, 10, 11, and 12.

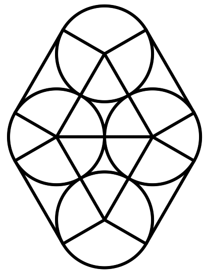
20. 80. [59] Numbers with 4-digit expansions are 27 to 80 in base 3, and 64 to 255 in base 4. For 5-digit expansions, the ranges are already nonoverlapping.

21. 12. [62] Let P be the point where the altitude from C meets the extension of AB. Then, by the Pythagorean Theorem, $AP = 5$, and the distance of M from P is $\sqrt{160 - 144} = 4$. Since M lies between A and P, $AM = 5 - 4 = 1$, so $AB = 2$. Thus the desired area is $\frac{1}{2} 2 \cdot 12$.

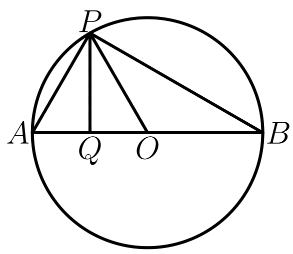


22. $8 + 2\pi$. [47] The diagram below makes it quite clear that the band is composed of four segments of length 2 plus four circular arcs which altogether compose 360 degrees of arc, 120 for the

top and bottom and 60 for the two sides.

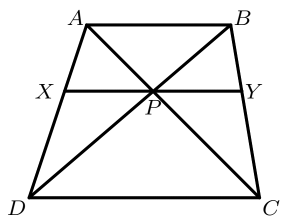


23. $\frac{1}{2}(\sqrt{3}-1)$. [54] Triangle APO is equilateral, so angle A equals 60 degrees and $AP = 1$. Thus $PB = \sqrt{3}$. Thus triangle APB has area $A = \sqrt{3}/2$ and perimeter $p = 3+\sqrt{3}$. Its inradius r satisfies $A = \frac{1}{2}rp$, so $r = \frac{\sqrt{3}}{3+\sqrt{3}} = \frac{1}{2}(\sqrt{3}-1)$.



24. $2\sqrt{2}$. [38] The largest value of $x+y$ on the circle occurs at the point $(\sqrt{2}, \sqrt{2})$. We check to see whether it satisfies the inequality $\sqrt{2}(1+\sqrt{5}) \geq 2\sqrt{5}$. Squaring both sides, this reduces to $\sqrt{5} \geq 2$, which is true.

25. 40/3. [54] In the diagram below, by similar triangles we have $\frac{PY}{AB} = \frac{YC}{BC}$ and $\frac{PY}{CD} = \frac{BY}{BC}$. Adding these yields $PY(\frac{1}{12} + \frac{1}{15}) = 1$ and so $PY = 60/9$. A similar argument shows that XP also equals 60/9.



26. 20. [46] The equation may be rewritten as $(x-1)(y-1) = -48$. Each of the five factorizations of 48 as a product of two positive integers, $1 \cdot 48$, $2 \cdot 24$, $3 \cdot 16$, $4 \cdot 12$, and $6 \cdot 8$ leads to four ordered pairs, by reversing order and choosing which one to negate.

27. 286. [51] There is a standard trick for this. Put 14 x's in a row, and then put vertical lines at three distinct positions separating adjacent x's. This divides the x's up into 4 groups. The number in each group corresponds to the 4-tuple. Since there are 13 spaces where the lines can go, the number of such sequences is $\binom{13}{3} = \frac{13 \cdot 12 \cdot 11}{6} = 286$.

28. 1000/1001. [22] By the quadratic formula, the points of intersection have $x = \frac{1}{16} \pm \frac{1}{4}\sqrt{2n+2}$. The difference between these x-values is $\frac{1}{2}\sqrt{2n+2}$. Since the lines have slope 1, the length of the line connecting the points is $d_n = \sqrt{n+1}$. The associated summand is $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. The sum of these as n goes from 1 to 1000 is $1 - \frac{1}{1001}$.

29. 110. [16] There exist numbers a, b, c, and d such that

$$\begin{aligned} & (x+c)(x^2-5x+4) + ax + b = P(x) \\ = \quad & (x+d)(x^2-5x+6) + 2ax + 2b. \end{aligned}$$

Then $4c + b = 100 = 6d + 2b$. Now $P(5) = 4(5+c) + 5a + b = 120 + 5a$ and $P(5) = 6(5+d) + 10a + 2b = 130 + 10a$. Since these are equal, $a = -2$ and $P(5) = 110$.

30. $(3, \infty)$. [8] By the quadratic formula, we have two distinct positive roots if and only if $a > 0$ and $0 < b < a^2/4$. The desired expression can equal any value strictly between $\frac{3}{4}a + \frac{3}{a}$ to $a + \frac{3}{a}$. This can be arbitrarily large. To find its minimum value without using calculus, we note that the minimum value of a sum of two positive numbers whose product is fixed occurs when they are equal. Thus the minimum value of $\frac{3}{4}a + \frac{3}{a}$ occurs when $a = 2$, and the value equals 3. Our expression must be strictly greater than this.

31. 6. [26] If x is the desired expected value, then $x = \frac{1}{3}\cdot 2+\frac{2}{3}(x+2)$, since after two steps the bug will either be at the center or back at a corner vertex. Solving yields $x = 6$.

32. 28. [28] The only numbers with exactly 6 divisors are p^5 and p^2q, with p and q prime, $q \neq p$. The sum of the reciprocals of the divisors of p^5 is less than 2, so we want $(1+\frac{1}{p}+\frac{1}{p^2})(1+\frac{1}{q}) = 2$, hence $(p^2+p+1)(q+1) = 2p^2q$. This implies $p^2+p+1 = 2q$ and $q+1 = p^2$, or $p^2+p+1 = q$ and $q+1 = 2p^2$. These simplify to $p^2-p-3 = 0$ or $p^2-p-2 = 0$, of which only the second has a positive integer solution, $p = 2$, yielding $q = 7$.

33. 16. [23] Since all fourth powers are either 0 or 1 mod 5, either x or y must be a multiple of 5. Since $5^4 = 625$, and $15266-625 = 14641 = 11^4$, $(x, y) = (5, 11)$ works. Since $8^4 < 5266 < 9^4$, we cannot get a pair using 10^4, and 15^4 is too large.

34. 17, 25, 39, 113. [18] Since ABC is a right triangle, we must have $30^2+x^2 = 4r^2$, where x and r are positive integers. Note that x must be even, so let $x = 2y$, and we have $225+y^2 = r^2$, for integers r and y. Thus $225 = (r-y)(r+y)$, and we are led to consider the ways of writing 225 as a product $a\cdot b$ of positive integers. Then $r = \frac{1}{2}(a+b)$. The factorizations are $225\cdot 1$, $75\cdot 3$, $45\cdot 5$, $25\cdot 9$, and $15\cdot 15$. The last one would yield $y = 0$, so it is invalid.

35. 14365. [38] Note that $2107 = 7^2\cdot 43$. Thus n can be of the form $p^6q^6r^{42}$ or $p^{48}q^{42}$ or p^6q^{300} or p^{2106}, where p, q, and r are distinct primes. The number of divisors of n^2 in these three cases are $13\cdot 13\cdot 85$, $97\cdot 85$, and $13\cdot 601$, resp. The first of these is the largest.

36. 359. [13] Since $(n+1)\cdot n! = (n+1)!$, there would be no point in using more than n copies of $n!$. Every positive integer can be written uniquely as $\sum a_i\cdot i!$ with $0 \le a_i \le i$. This can be proved by induction on the largest value of i. [If all sums with $i < n$ give all numbers less than $n!$, then adjoining all

multiples of $n!$ gives all numbers less than $(n+1)!$.] Thus using 10 ($= 1+2+3+4$) or fewer factorials gets us up to $5!-1 = 119$. Adding $5!$ to these gets us up to $2 \cdot 5! - 1 = 239$ using 11 or fewer. Using $2 \cdot 5!$ plus lower combinations gets to everything less than $2 \cdot 5! + 4 \cdot 4! + 3 \cdot 3! + 2 \cdot 2! + 1!$ with 11 or fewer, but this value, which equals $3 \cdot 5! - 1$, cannot be done with 11 factorials.

37. -3. [11] Let k be the common value of the three fractions. Then a, b, and c are roots of the polynomial $x^3 - kx + 1$, and since they are distinct, it factors as $(x-a)(x-b)(x-c)$. Hence $x^3 - kx + 1 = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc$. Thus $a+b+c = 0$, $abc = -1$, and

$$\begin{aligned} & a^3+b^3+c^3 \\ = \; & (a+b+c)(a^2+b^2+c^2-ab-ac-bc)+3abc = -3. \end{aligned}$$

38. $24\sqrt{5}/97$. [3] Let $AB = 7$, $BC = 8$, and $CA = 9$, and $AP = x$, $BP = y$, and $CP = z$. Then

$$\begin{aligned} y^2 &= 7^2 + x^2 - 14x\cos\alpha \\ z^2 &= 8^2 + y^2 - 16y\cos\alpha \\ x^2 &= 9^2 + z^2 - 18z\cos\alpha \end{aligned}$$

Adding these gives $(14x+16y+18z)\cos\alpha = 194$. Computing the area of triangle ABC both by Heron's formula and by summing the areas of the three subtriangles yields $\sqrt{12 \cdot 3 \cdot 4 \cdot 5} = \frac{1}{2}(7x+8y+9z)\sin\alpha$. Dividing the two equations gives $\frac{1}{4}\tan\alpha = \frac{12\sqrt{5}}{194}$.

39. 41. [18] Suppose that removing the last two digits of n^2 leaves the perfect square s^2. Then $0 < n^2 - (10s)^2 < 100$. Then $n^2 - (10s)^2 = (n+10s)(n-10s) \geq n+10s > 10s+10s = 20s$, so $20s < 100$. Therefore $s \leq 4$ and hence $n^2 < 100+(10s)^2 \leq 1700$. Therefore $n \leq 41$, and $41^2 = 1681$, so $n = 41$ works.

40. $\frac{1023}{512}$. [1] Let $Q(x) = P(2x) - P(x) - 1$. Then $Q(2^i) = 0$ for $0 \leq i \leq 9$, and hence $Q(x) = \alpha(x-2^0)(x-2^1)\cdots(x-2^9)$

for some number α. The coefficient of x in $P(x)$ equals the coefficient of x in $Q(x)$, and this equals $-Q(0)\left(\frac{1}{2^0}+\frac{1}{2^1}+\cdots+\frac{1}{2^9}\right)=\frac{1}{2^0}+\frac{1}{2^1}+\cdots+\frac{1}{2^9}=2-\frac{1}{2^9}$.