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2012 contest. Questions are annotated with the number of people answering them correctly. The first is the number out of the 44 people who scored at least 21, and the second is out of the other 316 people.

1. [44,314] Express $\frac{1}{3} + \frac{2}{5}$ as a reduced fraction.
2. [44,280] List all values of x that satisfy $\frac{(x+2)(2x-1)}{x-3} = 0$.
3. [43,271] What is the area of a triangle whose sides are 13, 13, and 10?
4. [43,238] How many of the 2-digit numbers from 10 to 99 have the property that both digits are perfect squares? For example, 10 is the smallest such number and 99 is the largest.
5. [42,134] How many digits are in the base 10 number $4^{22} \cdot 5^{40}$? For example, 9001 has four digits.
6. [43,239] Find A if there is a polynomial identity

$$(x^3+2x^2-3x-2)(x^4+3x^3+Ax^2-6x+1) = x^7+5x^6+5x^5-13x^4-23x^3+16x^2+9x-2.$$

7. [38,98] If $1.0000042376^2 = 1.00000xyz521795725376$, what is the value of $x+y+z$?
8. [42,249] The ratio of children to adults at a party is 2:3. A busload of 30 more children arrives at the party, and now the ratio of children to adults is 3:2. How many people were at the party before the bus arrived?
9. [41,109] Find the area of the region consisting of all points (x, y) satisfying $1 \leq |x| + |y| \leq 2$.
10. [38,113] A sequence is defined by $a_1 = 1$ and, for $n \geq 1$,

$$a_{n+1} = \begin{cases} 0 & \text{if } a_n = 0 \text{ and } n \text{ is odd} \\ 2 & \text{if } a_n = 0 \text{ and } n \text{ is even} \\ 1 & \text{if } a_n = 1 \text{ and } n \text{ is odd} \\ 0 & \text{if } a_n = 1 \text{ and } n \text{ is even} \\ 1 & \text{if } a_n = 2. \end{cases}$$

How many of the numbers $a_1, a_2, \ldots, a_{2012}$ are equal to 2?

11. [35,130] You are trying to guess an integer between 1 and 1000, inclusive. Each time you make a guess, which must be an integer, you are told whether your number is too high, too low, or correct. What is the smallest number of guesses required to guarantee that you will have guessed the number?
12. [27,30] A number system based on 26 uses the letters of the alphabet as its digits, with $A = 0$, $B = 1$, $C = 2, \ldots, Y = 24$, and $Z = 25$. Express $ONE + ONE$ in this system.

13. [38,65] A circle is inscribed in quadrilateral $ABCD$. If $AB = 4$, $BC = 5$, and $CD = 8$, what is DA?

14. [41,195] What is the 4-tuple (w, x, y, z) which satisfies all four of the following equations?

$$\begin{aligned} w + x + y &= 4 \\ x + y + z &= -5 \\ y + z + w &= 0 \\ z + w + x &= -8 \end{aligned}$$

15. [41,195] The outside of a cube is painted black, and then it is cut up into 64 smaller congruent cubes. How many of the smaller cubes have at least one face painted black?

16. [34,43] This problem deals with 5-digit numbers, and for the purposes of this problem, we allow 0's as the initial digits. Thus the number of 5-digit numbers is 10^5 under this interpretation (which differs from the usual convention). Write as a decimal the probability that a randomly selected 5-digit number contains exactly 4 distinct digits (under the interpretation of this problem).

17. [37,47] How many ordered pairs (x, y) of integers satisfy $x^2 + 6x + y^2 = 16$?

18. [33,72] Circles of radius 10 and 17 intersect at two points. The segment connecting these points of intersection has length 16. List all possible values for the distance between the centers of the circles?

19. [40,80] Of all the nonempty subsets S of $\{1, 2, 3, 4, 5, 6, 7\}$, how many do not contain the number $|S|$, where $|S|$ denotes the number of elements in S? For example, $\{3, 4\}$ is one such subset, since it does not contain the number 2.

20. [32,40] Let $(x_1, y_1), \ldots, (x_6, y_6)$ denote the vertices of a regular hexagon whose center is at $(2, 0)$ and which has one vertex at $(3, 0)$. Let $z_1 = x_1 + iy_1, \ldots, z_6 = x_6 + iy_6$ denote the corresponding complex numbers. What is the product $z_1 \cdots z_6$ of these six complex numbers?

21. [33,47] What is the value of the continued fraction

$$3 + \cfrac{1}{4 + \frac{1}{3 + \frac{1}{4 + \cdots}}}?$$

The 3's and 4's alternate indefinitely.

22. [37,48] List all ordered triples (a, b, c) of positive integers which satisfy $a + \frac{b}{c} = 11$ and $b + \frac{a}{c} = 14$.

23. [23,6] Let F_n denote the Fibonacci numbers, defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. Express the infinite sum

$$S = \frac{F_1}{5} + \frac{F_2}{5^2} + \frac{F_3}{5^3} + \frac{F_4}{5^4} + \cdots$$

as a reduced fraction.

24. [16,8] Let $A_1, \ldots, A_6$ denote the vertices of a regular hexagon inscribed in the circle $x^2 + y^2 = 3$. Circles of radius 1 are drawn with centers at each of these six points, and a seventh circle of radius 1 is drawn, centered at the origin. What is the area of the region common to at least two of the 7 circles of radius 1; i.e., the union of the intersections?

25. [26,17] Let $A = \sin x + \cos x$. Write $\sin^4 x + \cos^4 x$ as a polynomial in A.

26. [33,67] What is the largest positive integer n such that n is divisible by every positive integer m which satisfies $m^2 + 4 \leq n$?

27. [27,31] Points A, B, C, and D lie in a plane, with A, B, and C in order in a straight line. The angles between some of these points, measured in degrees, satisfy $BDC = 78$, $DBC > DCB$, $ABD = 4x + y$ and $DCB = x + y$, for some numbers x and y. How many positive integer values can y take on, satisfying all these conditions?

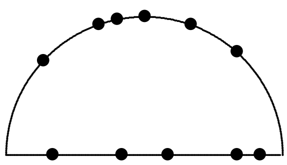
28. [27,37] Alice and Bill are walking in opposite directions along the same route between A and B. Alice is going from A to B, and Bill from B to A. They start at the same time. They pass each other 3 hours later. Alice arrives at B 2.5 hours before Bill arrives at A. How many hours are required for Bill's trip from B to A?

29. [25,29] What is the smallest positive integer that can be written as the difference of two positive perfect squares in at least three distinct ways?

30. [30,18] What is the ratio V^2/A^3, where V is the volume of a regular octahedron and A is its surface area?

31. [21,9] In triangle ABC, the median from A is perpendicular to the median from B. If $AC = 6$ and $BC = 7$, then what is AB?

32. [5,1] Eleven points are arranged on a semicircle with five on the straight line segment and six on the arc. See the diagram below for a possible configuration. Every pair of these points is joined by a straight line segment, and it turns out that no three of the line segments intersect at a common point in the interior of the semicircle. How many points are there in the interior of the semicircle where two of the line segments intersect?



33. [3,0] A trapezoid has ratio of its bases 2:1, ratio of its legs (nonparallel sides) 2:1, and ratio of its diagonals 2:1. What is the ratio of its shorter base to shorter leg?

34. [13,8] If x and y are randomly selected real numbers between 0 and 1, what is the probability that the integer closest to $\frac{y-x}{y+x}$ is odd?

35. [27,14] In triangle ABC, $AB = 3$, $BC = 4$, and $AC = 6$. If BC is extended through C to D so that $CD = BC$, what is AD?

36. [15,1] Alice, Bill, Chris, Don, and Emily take a test. In how many different orders can they finish if ties are allowed? (For example, if there were three people, it would be 13, since there are 6 orders without ties, 1 with a 3-way tie, 3 with a 2-way tie for first, and 3 with a 2-way tie for last.)

37. [9,0] List all positive integers b for which there is a positive integer $a < b$ such that exactly $1/100$ of the consecutive integers $a^2, a^2 + 1, \ldots, b^2$ are perfect squares.

38. [18,4] Let a_n denote the nth smallest positive integer for which the sum of its decimal digits equals 3. For example, $a_1 = 3$, $a_2 = 12$, $a_3 = 21$, and $a_4 = 30$. How many digits are there in a_{2012}?

39. [1,1] You flip a fair coin repeatedly until either four consecutive Heads (H) or six consecutive tails (T) occur. What is the probability that the sequence HHHH occurs before the sequence TTTTTT?

40. [7,1] Eight points on the circumference of a circle are chosen and all $\binom{8}{2} = 28$ chords connecting them are drawn. It turns that no three chords intersect in the same point in the interior of the circle. Into how many regions do these chords divide the interior of the circle?

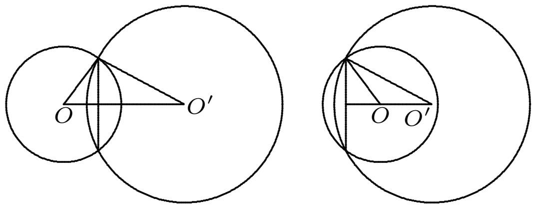
Solutions to 2012 contest.

1. 11/15. It is $\frac{5}{15}+\frac{6}{15}$.
2. -2 and 1/2 (must have both). Must make the numerator equal to zero while keeping the denominator nonzero.
3. 60. This is an isosceles triangle composed of two 5-12-13 right triangles, each of which has area 30.
4. 12. The first digit can be any of 1, 4, or 9, while the second digit can be any of 0, 1, 4, or 9.
5. 42. The number is $4^2 \cdot (4 \cdot 5^2)^{20} = 16 \cdot 10^{40}$.
6. 2. The coefficient of x^2 in the product, 16, must equal $-2A + 18 + 2$. Solve to get $A = 2$.
7. 19. $1 + t^2 = 1 + 2t + t^2$. Here $t = .0000042376$. The t^2 is much too small to affect the first three nonzero digits of $2t$. We obtain $xyz = 847$.
8. 60. $C = \frac{2}{3}A$, while $C + 30 = \frac{3}{2}A$. Solving yields $\frac{5}{6}A = 30$, so $A = 36$ and $C = 24$.
9. 6. The region consists of the points lying inside a square whose vertices are at $(\pm 2, 0)$ and $(0, \pm 2)$ and outside a square whose vertices are at $(\pm 1, 0)$ and $(0, \pm 1)$. The outer square has area 8 and the inner square area 2.
10. 502. Starting with a_1, the sequence goes 1, 1, 0, 0, 2, 1, 0, 0, 2, and keeps repeating with period 4. Thus $a_n = 2$ if and only if $n = 5$, 9, etc., 1 greater than a positive multiple of 4. a_{2009} will be the 502nd 2.
11. 10. This is because $2^{10} = 1024 > 1000$. The best that you can guarantee after 1 guess is to limit it to 500 numbers, then 250, 125, 62, 31, 15, 7, 3, and 1. After 9 guesses, you know what the answer is, but you still must make the tenth guess.
12. $BDAI$. The rightmost position is $4 + 4 = 8 = I$. The next position is $13 + 13$, for which we enter $0 = A$ and carry 1. The next position is $1 + 14 + 14$, for which we enter $3 = D$ and carry $1 = B$ to the leftmost position.
13. 7. Let a, b, c, and d denote the length of a tangent from A, B, C, and D to the inscribed circle. Then $a + b = 4$, $b + c = 5$, and $c + d = 8$. Subtract the second equation from the sum of the others to obtain $a + d = 7$.
14. $(2, -3, 5, -7)$. Add the four equations, obtaining $w + x + y + z = -3$. Subtract each equation from this to obtain the value of one of the variables.
15. 56. The cube is cut into fourths in each direction. You can enumerate the number of pieces from the center of a face (24), an edge (24), and a corner (8), or you can subtract the $2 \times 2 \times 2$ cube in the center of the big cube, which are the only subcubes without a black face.

16. .504 or $\frac{63}{125}$. There are 10 ways to choose which digit is repeated, and $\binom{5}{2} = 10$ ways to choose which two positions of the number will have the repeated digit. Then there are $9 \cdot 8 \cdot 7$ ways to fill the other three positions with distinct remaining digits. The answer is $10 \cdot 10 \cdot 9 \cdot 8 \cdot 7/10^5$.

17. 12. The equation can be written as $(x+3)^2 + y^2 = 25$. This implies that the set $\{x+3, y\}$ is $\{0, \pm 5\}$ or $\{(\pm 3, \pm 4\}$. There are four ordered pairs of the first type and eight of the second.

18. 9 and 21. (Must have both.) The line connecting the centers is the perpendicular bisector of this chord, intersecting it at point P. If O and O' are the centers of the circles, then $OO' = PO' \pm OP$, and OP and PO' are bases of right triangles with height 8 and hypotenuses 10 and 17, respectively. By the Pythagorean Theorem, OP and PO' are 6 and 15.



19. 63. There are $2^7 - 1 = 127$ nonempty subsets of $\{1, \ldots, 7\}$. The number which have k elements and do contain the number k is $\binom{6}{k-1}$. Summing this as k goes from 1 to 7 yields $2^6 = 64$ subsets S which contain the number $|S|$. The number which do not is $127 - 64 = 63$.

20. 63. The numbers z_j are 3, 1, $\frac{5}{2} \pm i\frac{\sqrt{3}}{2}$, and $\frac{3}{2} \pm i\frac{\sqrt{3}}{2}$. Their product is $3\frac{25+3}{4}\frac{9+3}{4} = 63$. Another method notes that the numbers z_j are the complex roots of $(z-2)^6 - 1 = 0$ and so their product is $2^6 - 1$.

21. $\frac{3}{2} + \sqrt{3}$. If x is the answer, then $3 + 1/(4 + \frac{1}{x}) = x$. This reduces to $x = (x-3)(4x+1)$, hence $4x^2 - 12x - 3 = 0$, which is easily solved by the quadratic formula. Since the number is positive, we choose only the positive square root.

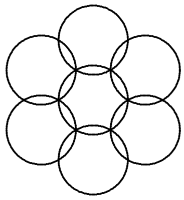
22. $(8, 12, 4)$. Let $a = \alpha c$ and $b = \beta c$, with α and β integers. Subtract equations and obtain $(\beta - \alpha)(c-1) = 3$. If $c = 2$, then $\beta - \alpha = 3$ and $2\alpha + \beta = 11$, implying $\alpha = 8/3$, not an integer. If $c = 4$, then $\beta - \alpha = 1$ and $4\alpha + \beta = 11$, yielding $\alpha = 2$, $\beta = 3$.

23. 5/19. $5S = 1 + \frac{F_2}{5} + \frac{F_3}{5^2} + \cdots$. Subtract the original equation and obtain

$$4S = 1 + \frac{F_2 - F_1}{5} + \frac{F_3 - F_2}{5^2} + \frac{F_4 - F_3}{5^3} + \cdots = 1 + \frac{F_1}{5^2} + \frac{F_2}{5^3} + \cdots = 1 + \frac{1}{5}S.$$

Thus $\frac{19}{5}S = 1$.

24. $4\pi - 6\sqrt{3}$. There are 12 regions, each of which is twice as large as the difference between a 60 degree wedge in a unit circle and an equilateral triangle of side length 1. Thus the answer is $24(\frac{\pi}{6} - \frac{\sqrt{3}}{4})$. Note that the centers of any two adjacent circles are $\sqrt{3}$ apart, which implies that the radius to an intersection point makes a 30 degree angle with the line connecting the centers.



25. $\frac{1}{2}+A^2-\frac{1}{2}A^4$. We have $A^4 = \sin^4 x+\cos^4 x+4\sin x\cos x(\cos^2 x+\sin^2 x)+6\sin^2 x\cos^2 x$. Also $A^2 = \sin^2 x + \cos^2 x + 2\sin x\cos x$. Thus $\sin x\cos x = \frac{1}{2}(A^2-1)$, and $\sin^4 x + \cos^4 x = A^4 - 4(\frac{1}{2}(A^2-1)) - 6(\frac{1}{2}(A^2-1))^2 = A^4 - 2A^2 + 2 - \frac{3}{2}(A^4 - 2A^2 + 1)$.

26. 24. 24 works, since it is divisible by 1, 2, 3, and 4, while 25, 26, 27, and 28 do not, since they are not divisible by both 3 and 4. To see that no larger n works, let $k^2+4 \le n < (k+1)^2+4$, with $k \ge 5$. Then n must be divisible by both k and $k-1$, and hence by $k(k-1)$, since k and $k-1$ are relatively prime. However, since $k \ge 5$, we have $2k(k-1) \ge (k+1)^2+4$, so n is less than $2k(k-1)$ and hence must equal $k(k-1)$, and thus cannot be $\ge k^2+4$.

27. 24. Let $\alpha = DBC$ and $\beta = DCB$. We have $\alpha+\beta = 102$, $180-\alpha = 4x+y$, and $x+y=\beta$. These imply $3x = 78$, so $x = 26$. Then $76-y = \alpha > \beta = 26+y$, so $y < 25$. We easily check that each value of y from 1 to 24 works.

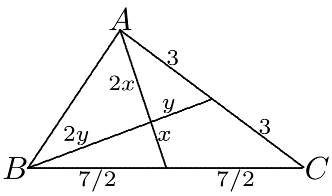
28. 7.5. Let s_A and s_B denote their speeds, and t the desired time when Bill arrives at A. Then $s_A(t-2.5)$, s_Bt, and $3(s_A+s_B)$ all equal the distance between the two points. Let $r = \frac{s_A}{s_B}$. Then $r(t-2.5) = t = 3(r+1)$, which reduces to $t^2 - 8.5t + 7.5 = 0$, so $t = 7.5$, since $t > 1$.

29. 45. It can be written as $(7+2)(7-2)$, $(9+6)(9-6)$, and $(23+22)(23-22)$. This shows that what we are really looking for is factorizations of n as a product of two positive integers whose difference is even (since $(a+b)-(a-b) = 2b$). If n is odd, we require three divisors of n (including 1) which are less than $\sqrt{n}$, while if n is even, we require three even divisors d of n, each less than $\sqrt{n}$, such that n/d is also even. It is easy to see that 45 is the smallest n for which this can be done.

30. $\sqrt{3}/324$. The octahedron is formed from eight equilateral triangles. Let s denote the sidelength. Then $A = 8s^2\sqrt{3}/4 = 2\sqrt{3}s^2$. The altitude from the middle of the center square to a peak is $s\sqrt{\frac{3}{4}-\frac{1}{4}} = s\sqrt{2}/2$, so the volume is $s^3\sqrt{2}/3$. The desired ratio is $\frac{2/9}{24\sqrt{3}}$.

31. $\sqrt{17}$. Let x and y be as in the diagram below, in which the two medians are perpendicular. Then $x^2 + 4y^2 = 49/4$ and $4x^2 + y^2 = 9$. Hence

$$AB^2 = 4x^2 + 4y^2 = \tfrac{4}{5}(\tfrac{49}{4} + 9) = 17.$$

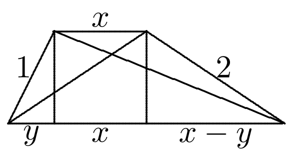


32. 265. Any four of the eleven points which include at least two on the semicircle will determine exactly one point of intersection. The total number of such groups is $\binom{6}{4} + \binom{6}{3}\binom{5}{1} + \binom{6}{2}\binom{5}{2} = 15 + 100 + 150$.

33. $\frac{1}{2}\sqrt{10} : 1$ or just $\frac{1}{2}\sqrt{10}$. We let the shorter leg equal 1, and wish to find x in the diagram below, in which the height is h. We have $1 - y^2 = h^2 = 4 - (x - y)^2$, and then comparing diagonals, $4(h^2 + (x + y)^2) = h^2 + (2x - y)^2$. We substitute the two expressions for h^2 into this, and obtain

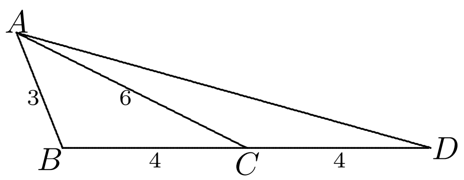
$$4(4 - (x - y)^2 + (x + y)^2) = 1 - y^2 + (2x - y)^2,$$

which simplifies to $20xy + 15 = 4x^2$. Equating the two equations for h^2 yields $2xy + 3 = x^2$. Multiply this by 10 and subtract the earlier equation to obtain $6x^2 = 15$.



34. 1/3. Let $r = \frac{y}{x}$. The specified ratio is $\frac{r-1}{r+1} = 1 - \frac{2}{r+1}$. This is an increasing function of r for $0 \le r < \infty$, and assumes values -1, $-\frac{1}{2}$, $\frac{1}{2}$, 1, for $r = 0$, $\frac{1}{3}$, 3, and ∞. Thus for (x, y) in the unit square (and, in fact, in the first quadrant), the integer closest to the specified ratio is -1 for (x, y) below the line $y = \frac{1}{3}x$, 1 above the line $y = 3x$, and 0 in between. For (x, y) in the unit square, the area for which this value is odd is 1/3, the union of two right triangles with legs 1/3 and 1.

35. $\sqrt{95}$. By the Law of Cosines $6^2 = 3^2 + 4^2 - 24\cos B$, so $\cos B = -11/24$. Now $AD^2 = 3^2 + 8^2 - 48\cos B = 73 + 22$.



36. 541. Let $f(n)$ denote the number of outcomes with n people, with the convention that $f(0) = 1$. Then $f(n) = \sum_{i=0}^{n-1} \binom{n}{i} f(i)$, where the i-summand is where there are i people who were not first or tied for first. Iteratively compute $f(1) = 1$, $f(2) = 3$, $f(3) = 13$, $f(4) = 75$, and $f(5) = 541$.

37. 60, 68, 100 (must have all). We require $(b - a + 1)/(b^2 - a^2 + 1) = 1/100$. This can be manipulated to $(b + a - 100)(b - a) = 99$. Then each $2b - 100$ equals the sum of the two factors of a factorization of 99 as a product of two integers, i.e. (11,9), (33,3), and (99,1). Therefore $2b$ equals 100 plus one of 20, 36, or 100. (The corresponding values of a are 49, 65, and 99, respectively.)

38. 22. There are $\binom{d+2}{3}$ numbers with $\leq d$ digits of the desired form. This can be seen by filling in $d + 2$ positions with exactly three O's and the rest X's, with the correspondence being that the ith digit of the number is the number of O's between the $(i - 1)$st and ith X's. For example, with $d = 5$, $XXOOXXO$ corresponds to 00201. Since $\binom{24}{3} = 2024$, while $\binom{23}{3}$ is much smaller, the result follows.

39. 21/26. Let E denote the event that HHHH occurs before TTTTTT. If $\mathbf{s}$ a sequence of tosses, let $P(E|\mathbf{s})$ denote the probability that E occurs, given that $\mathbf{s}$ occurs at the start. Let $x = P(E|H)$ and $y = P(E|T)$. Then

$$x = \tfrac{1}{2}P(E|HT) + \tfrac{1}{4}P(E|HHT) + \tfrac{1}{8}P(E|HHHT) + \tfrac{1}{8} = \tfrac{7}{8}y + \tfrac{1}{8},$$

while

$$\begin{aligned} y = \tfrac{1}{2}(&P(E|TH) + \tfrac{1}{4}P(E|TTH) + \tfrac{1}{8}P(E|TTTH) + \tfrac{1}{16}P(E|TTTTH) \\ &+ \tfrac{1}{32}P(E|TTTTTH) = \tfrac{31}{32}x. \end{aligned}$$

Solving these equations yields $x = \frac{32}{39}$ and $y = \frac{31}{39}$, so the desired probability is $\frac{1}{2}(x + y) = \frac{63}{78}$.

40. 99. The eight points on the circumference and points of intersection of the chords together with the portions of chords and arcs between vertices form a graph (in the sense of graph theory). It has $\binom{8}{4} = 70$ vertices in the interior, and so 78 vertices altogether. There are $\frac{1}{2}(70 \cdot 4 + 8 \cdot 9) = 176$ edges, obtained as $\frac{1}{2}$ times the number of vertex-edge intersections. Here we have used that each of the eight vertices on the circumference meets 7 chords plus two arcs. By Euler's formula, $78 - 176 + R = 1$.