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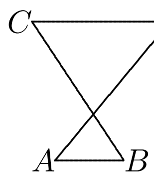
1. What simple reduced fraction equals $\frac{1}{3}+\frac{1}{4}+\frac{1}{6}$?
2. The legs of a right triangle have ratio 5:12. The hypotenuse has length 39. What is the length of the shortest leg of the triangle?
3. What values of x satisfy $\frac{(4x-1)(x-2)}{x+1}=0$?
4. The population increases by 20% one year and then decreases by 20% the next year. What is its total change, in percent, over the 2-year period?
5. How many 2-digit numbers have the sum of their digits divisible by 6? (Note, 6 is not considered a 2-digit number.)
6. What is the sum of the areas of the two triangles in the diagram below, in which $AB=3$, $CD=6$, and the distance between the parallel lines AB and CD is 6?



7. If $f(x)=x^2+3$ and $g(x)=2x-1$, express $f(g(x))$ as an explicit function of x, using no parentheses in your answer.
8. Determine the number of pairs (m,n) of positive integers for which

$$m^3+n=1000000008=10^9+8.$$

9. The streets in a city form a grid, with some running north-south, and others east-west. Alice, Bob, Carol, and Don live at four street intersections which are vertices of a rectangle, with Alice and Don at opposite vertices. The school that they all attend is at an intersection inside the rectangle, and they all walk to it by their shortest possible route. Alice walks 10 blocks, Bob walks 12 blocks, and Carol walks 15 blocks. How many blocks does Don walk to get to school?
10. What is the number of noncongruent rectangles with integer sides and area 315?
11. A rectangular sheet of paper is folded in half, with the crease parallel to the shorter edge. It happens that the ratio of longer side to shorter side remains unchanged. What is this ratio?
12. How many lines in the plane are at distance 2 from the point $(0,0)$ and also at distance 3 from the point $(6,0)$?
13. What is the remainder when $x^{135}+x^{125}-x^{115}+x^5+1$ is divided by x^3-x?
14. A "turn" is a flip of a pair of fair coins. What is the probability that, in 4 turns, you obtain a pair of heads at least once?

15. For positive integers a, b, and c, what is the smallest possible value of $a+b+c$ for which $abc = 7(a+b+c)$?

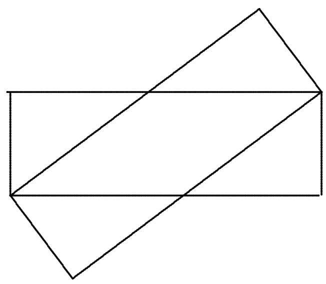
16. The exterior angles of a triangle are in the ratio $3:5:6$. What is the ratio, similarly expressed and reduced to lowest terms, of the interior angles adjacent to them, in the same order?

17. All positive integers which can be written as a sum of one or more distinct powers of 3 are written in increasing order. The first four of them are 1, 3, 4, and 9. What is the 81st?

18. Triangle ABC has $AB = 25$, $AC = 24$, and $BC = 23$. Let D be the point on AC for which $BD \perp AC$. What is the difference $AD - DC$?

19. List all solutions in the domain $0 \le x \le 2$ of the equation $\sqrt{2-x} = 1 + \sqrt{x}$.

20. Two 3-by-9 rectangles are arranged as below, so that an opposite pair of vertices of each coincide. What is the area of the parallelogram which forms their intersection?



21. Three boys, A, B, and C, can work separately or together on a job. If working together, their efforts combine efficiently. It takes A working alone twice as long as it takes B working alone. It takes A four times as long as B and C working together to do the job. Also, it takes A two hours longer to do the job than it takes all three boys working together. How many hours does it take B to do the job?

22. We call a number *ascending* if each digit is greater than the digit that precedes it. For example, 457 is ascending, but 447 is not. How many ascending numbers are there between 400 and 5000?

23. Out of all polynomials with integer coefficients which have both $\frac{1}{2}\sqrt{2}$ and $\frac{1}{2}\sqrt{3}$ as roots, consider those of smallest possible degree. Among all of these, what is the smallest positive coefficient which occurs in any of them?

24. A triangle has vertices at $(0,0)$, $(4,2)$, and $(5,1)$. What is the tangent of its angle at the vertex $(4,2)$?

25. What is the smallest positive integer n such that the product $19999 \cdot n$ ends in the four digits 2010?

26. A parallelogram has area 36 and diagonals of length 10 and 12. What is the length of its longest side?

27. The numbers 1 to 9 are arranged in tables with 3 rows and 3 columns in all the 9! possible ways. For each table, we take the smallest row-sum and the largest row-sum. We do this for all the tables, and add all the $2 \cdot 9!$ numbers obtained in this way. What is the value of this sum? You may use factorials in your answer.

28. There are six doors in front of you. Behind each of five of them is $120, and behind the sixth is a goat. You may open as many doors as you like. If all the doors you open contain money, then you keep all the money. But if the goat is behind one of the doors that you open, it eats all the money, and you get nothing. How many doors should you open to maximize the expected value of your winnings, and what is the expected value? Your answer should be the ordered pair (n, m), where n is the number of doors you open, and m the expected number of dollars that you win.

29. List all integer values of x for which $x^2 - 5x - 1$ is a perfect square. (Note that x may be positive or negative.)

30. For what positive integer n does the decimal expansion of $\frac{n}{810}$ equal a repeating decimal $.9d59d5\cdots$ with 3-digit repetend $9d5$, where d is some digit from 0 to 9?

31. For a positive integer n, let $p(n)$ denote the (base 10) product of the digits of the base-5 expansion of n. For example, $p(24) = 4 \cdot 4 = 16$ since the base-5 expansion of 24 is 44. Evaluate $\sum_{n=1}^{124} p(n)$.

32. Out of four integers, which are not necessarily distinct, we form the sums of each of the six pairs, and find that the values of these sums are 27, 41, 44, 47, and 61. Thus one of these sums is achieved twice. What is the largest of the four integers?

33. How many distinct integers occur in the list

$$\left[\frac{1^2}{2010}\right], \left[\frac{2^2}{2010}\right], \left[\frac{3^2}{2010}\right], \ldots, \left[\frac{2010^2}{2010}\right],$$

where $[x]$ denotes the greatest integer less than or equal to x?

34. 2^{29} is a 9-digit integer with distinct digits. What digit (from 0 to 9) does it not contain?

35. What rational number equals $\sqrt[3]{9+4\sqrt{5}} + \sqrt[3]{9-4\sqrt{5}}$?

36. An army marches at a steady pace. When the front of the army passes a suspected bomb, a runner runs at a steady pace from the front of the army to the rear, and then turns around and runs back to the front at the same steady pace, arriving back at the front just as the rear passes the suspected bomb. What is the ratio of the speed of the runner to the speed of the army?

37. Let ABC be an isosceles right triangle with right angle at C. Let P be a point inside the triangle such that $AP = 3$, $BP = 5$, and $CP = 2\sqrt{2}$. What is the area of the triangle ABC?

38. What reduced fraction $\frac{p}{q}$ with $8 \le q \le 99$ is closest to $\frac{3}{7}$?

39. Write all positive integers which occur as $\max\{a, b, c, x, y, z\}$, where a, b, c, x, y, and z are positive integers satisfying $abc = x+y+z$ and $xyz = a+b+c$. A correct answer to this question will consist of four integers. Half credit will be granted for an answer consisting exactly of three of the four.

40. Let EF be the diameter of a semicircle, A a point on this diameter, and C, D, and B points on the semicircle such that $AD \perp EF$, AD bisects angle CAB, BE bisects angle ABC, and CF bisects angle ACB. If $AB = 6$ and $AC = 3$, what is AD?

SOLUTIONS: In brackets is the number of people who answered it correctly out of the 54 people who got at least 20 right.

1. 3/4. [53] It is $\frac{4+3+2}{12}$.

2. 15. [54] If the legs are $5a$ and $12a$, then the hypotenuse is $\sqrt{(5a)^2 + (12a)^2} = 13a$. Hence $a = 3$.

3. 1/4 and 2. [53] The fraction is not defined when $x = -1$.

4. 4% decrease or -4% or just -4. (We eventually gave credit for an answer of 4 or 4%, too, because so many high-scoring students wrote that as their answer. My feeling is that "change" is a signed quantity, but since so many smart people interpreted it differently, credit was granted.) [51] If the initial population is P, then after 1 year it is $1.2P$, and after 2 years it is $.8 \cdot 1.2P = .96P$, 4 percent less than the initial.

5. 14. [44] The sum of the digits can be 6, 12, or 18. The numbers are 15, 24, 33, 42, 51, and 60; then 39, 48, 57, 66, 75, 84, and 93; finally 99.

6. 15. [53] The triangles are similar. Thus their altitudes are 2 and 4, and the sum of the areas is $\frac{1}{2}(3 \cdot 2 + 6 \cdot 4)$.

7. $4x^2 - 4x + 4$. [52] It is $(2x - 1)^2 + 3$.

8. 1000. [45] m can be any integer from 1 to 1000. Then $n = 10^9 + 8 - m^3$.

9. 17. [51] The total distance walked by Alice and Don equals that total walked by Bob and Carol, and so Don walks $12 + 15 - 10$ blocks.

10. 6. [43] It is the number of factors of $315 = 3^2 \cdot 5 \cdot 7$ which are less than $\sqrt{315}$, and this is between 17 and 18. These factors are 1, 3, 5, 7, 3^2, and $3 \cdot 5$, which give the shorter side of the rectangle.

11. $\sqrt{2}$. [51] Letting ℓ and s refer to the length of the longer and shorter sides of the initial rectangle, we have $\frac{\ell}{s} = \frac{s}{\ell/2}$. Hence $2s^2 = \ell^2$.

12. 4. [36] The circles of radius 2 centered at (0,0) and of radius 3 centered at (6,0) do not intersect. The desired lines must be tangent to each. There are two common internal tangents and two common external tangents.

13. $2x + 1$. [30] Write $x^{135} + x^{125} - x^{115} + x^5 + 1 = q(x)(x^3 - x) + ax^2 + bx + c$. Letting $x = 0$, 1, and -1 yields $1 = c$, $3 = a + b + c$, and $-1 = a - b + c$. These are easily solved to give $c = 1$, $b = 2$, and $a = 0$.

14. $\frac{175}{256}$. [43] The desired answer is 1 minus the probability of never getting a HH on four turns. This latter probability is $(3/4)^4 = 81/256$.

15. 15. [43] One of the integers must be divisible by 7. Call this one a. For the smallest possible value, we should try $a = 7$. Then we have $bc = 7 + b + c$, which yields $8 = (b - 1)(c - 1)$. Then $\{b, c\} = \{3, 5\}$ or $\{2, 9\}$. Thus $\{a, b, c\} = \{7, 3, 5\}$ yields the smallest sum.

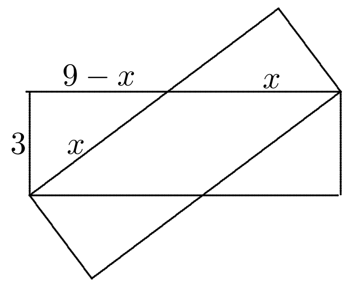
16. $4:2:1$. [45] The sum of the exterior angles is 360. Hence the three exterior angles are $\frac{540}{7}$, $\frac{900}{7}$, and $\frac{1080}{7}$. The interior angles are $\frac{1260}{7}$ minus these, and hence are $\frac{720}{7}$, $\frac{360}{7}$, and $\frac{180}{7}$, which are clearly in the ratio $4:2:1$.

17. 811. [34] These are the positive numbers whose base-3 expansion consists entirely of 0's and 1's. The order of these numbers will be the same as that of the corresponding base-2 numbers. Since $81 = 2^6+2^4+2^0$, our desired number is $3^6+3^4+3^0 = 729+81+1$.

18. 4. [43] Let $x = AD$ and $h = BD$. There are right triangles with $25^2 = x^2 + h^2$ and $23^2 = (24-x)^2+h^2$. Subtracting yields $25^2-23^2 = 48x-24^2$. Thus $(25-23)(25+23) = 48(x-12)$ and so $x = 14$. The desired difference is $14 - 10$.

19. $1-\frac{1}{2}\sqrt{3}$. [20] Sketching the graphs suggests that there should be exactly one solution. Squaring yields $0 = 2x + 2\sqrt{x} - 1$. The quadratic formula gives $\sqrt{x} = \frac{1}{2}(-1 + \sqrt{3})$, hence $x = 1 - \frac{1}{2}\sqrt{3}$.

20. 15. [45] In the diagram below, the Pythagorean theorem implies $x^2 = 3^2 + (9 - x)^2$, so $x = 5$. The area of the parallelogram is $3 \cdot 5$.



21. 5/4 or 1.25. [41] Let f_A, f_B, and f_C denote the fraction of the job that each of the boys can do in one hour. Then $f_B = 2f_A$, $f_B+f_C = 4f_A$, and $1/(f_A+f_B+f_C)+2 = 1/f_A$. We obtain $2 = \frac{4}{5} \cdot \frac{1}{f_A}$. Hence $f_A = \frac{2}{5}$ so $f_B = \frac{4}{5}$. The desired answer is the reciprocal of 4/5.

22. 141. [40] For 3-digit numbers, the digits can be any three of 4, 5, 6, 7, 8, and 9, thus $\binom{6}{3} = 20$ such numbers. There are $\binom{9}{4} = 126$ 4-digit ascending numbers altogether. From these, we delete the $\binom{5}{4} = 5$ which start with a digit ≥ 5.

23. 3. [38] Such polynomials must also have $-\frac{1}{2}\sqrt{2}$ and $-\frac{1}{2}\sqrt{3}$ as roots, and hence be divisible by $(x^2 - \frac{1}{2})(x^2 - \frac{3}{4})$. The desired polynomials are all integer multiples of $8(x^4 - \frac{5}{4}x^2 + \frac{3}{8})$, and so the smallest positive coefficient is 3.

24. -3. [33] The Law of Cosines says that $26 = 20 + 2 - 2\sqrt{40}\cos\alpha$, hence $\cos\alpha = -\frac{1}{\sqrt{10}}$ and so $\tan\alpha = -3$. Alternatively, the sides going out from the vertex in question make angles with the horizontal having tangents equal to 1/2 and 1. Thus the sum of these two angles has tangent $(\frac{1}{2} + 1)/(1 - \frac{1}{2}) = 3$, and our desired angle, being 180 minus this sum, has tangent the negative of 3.

25. 7990. [42] The last four digits of a number are its residue mod 10000. Since $19999 \equiv -1 \mod 10000$, our n will satisfy $2010 \equiv 19999n \equiv -n \mod 10000$, and hence $n = 10000 - 2010 = 7990$.

26. $\sqrt{109}$. [23] Let BD be the longer diagonal, A another vertex, and let $AM \perp BD$. Since the area of ABD is 18, $AM = 3$. Since the point O where the diagonals meet divides each in half, $AO = 5$ so $OM = \sqrt{5^2 - 3^2} = 4$ and $AD = \sqrt{3^2 + (4+6)^2}$.

27. $30 \cdot 9!$ (or $3 \cdot 10!$). [17] To each table, associate the complementary table obtained by replacing each entry i by $10 - i$. The largest row-sum of a table plus the smallest row-sum of its complementary table is $3 \cdot 10 = 30$. Similarly for the smallest of the first plus the largest of the complementary table. Thus each of the $\frac{9!}{2}$ pairs of tables contributes 60. An easier, but less thorough, method uses that $30 = \frac{2}{3}(1 + \cdots + 9)$ is the average of the sum of the largest and smallest rows of each of the 9! tables.

28. $(3, 180)$. [40] If you open n doors, your expected winning is $120n(1 - \frac{n}{6})$, since the probability that the goat is behind one of the selected doors is $n/6$. For $n = 1, 2, 3, 4, 5$, this is, respectively, $\frac{5}{6}120 = 100$, $\frac{2}{3}240 = 160$, $\frac{1}{2}360 = 180$, $\frac{1}{3}480 = 160$, and $\frac{1}{6}600 = 100$.

29. -5 and 10. (Must have both.) [28] From $x^2 - 5x - 1 = n^2$, we obtain that $x = \frac{1}{2}(5 \pm \sqrt{29 + 4n^2})$ and hence $29 + 4n^2 = t^2$ for some integer t. Thus $29 = (t - 2n)(t + 2n)$ and so $t - 2n = 1$ and $t + 2n = 29$. (Note that negating n does not change x, and so reversing the 1 and 29 gives no new information.) We obtain $t = 15$ and $n = 7$. Thus $x = \frac{1}{2}(5 \pm 15)$.

30. 750. [40] We have $\frac{n}{810} = \frac{9d5}{999}$, hence $37n = 30 \cdot 9d5$. Thus $9d5$ is divisible by 37, and trying the division shows it must be that the quotient is 25 and $d = 2$. Thus $n = 30 \cdot 25$.

31. 1110. [32] Note that $p(5) + \cdots + p(24)$, the sum over all numbers whose base-5 expansion has two digits, is $(0 + 1 + 2 + 3 + 4)^2 = 100$, since both of these give all products of two of these digits. Similarly the sums over all numbers whose base-5 expansion has 1 (resp. 3) digits is 10 (resp. 1000). Our desired sum is then $10 + 100 + 1000$.

32. 32. [40] Let the integers be $a \le b \le c \le d$. Then $a + b = 27$ and $c + d = 61$. Thus $a + b + c + d = 88$, and we can see that $a + d = 44$ and $b + c = 44$. Now $b + d = 47$, the second largest sum. Adding yields $a + b + c + 3d = 152$ and so $2d = 152 - 88 = 64$.

33. 1508. [9] Let $g(n) = \left[\frac{n^2}{2010}\right]$ and $f(n) = \frac{n^2}{2010}$. Then $f(n) - f(n-1) = \frac{2n-1}{2010}$ is < 1 if $n \le 1005$, and is > 1 if $n > 1005$. Thus every nonnegative integer $\le g(1005) = 502$ is achieved as $g(n)$ for at least one integer value of n, yielding 503 such values, while for $1005 \le n \le 2010$, each $g(n)$ takes on a different value. Thus the answer is $503 + 1005$, with the 1005 being the values of $g(n)$ for $1006 \le n \le 2010$.

34. 4. [38] Using $2^6 \equiv 1 \bmod 9$, we obtain $2^{29} \equiv 1^4 \cdot 32 \equiv 5 \bmod 9$. Since $0 + 1 + \cdots + 9 = 45 \equiv 0 \bmod 9$, we deduce that the missing digit is $9 - 5$. In fact, $2^{29} = 536870912$.

35. 3. [22] Let $\alpha = \sqrt[3]{9 + 4\sqrt{5}} + \sqrt[3]{9 - 4\sqrt{5}}$. Since $(9 + 4\sqrt{5})(9 - 4\sqrt{5}) = 1$, we obtain

$$\alpha^3 = (9 + 4\sqrt{5}) + 3\sqrt[3]{9 + 4\sqrt{5}} + 3\sqrt[3]{9 - 4\sqrt{5}} + 9 - 4\sqrt{5} = 18 + 3\alpha.$$

Thus α is a root of $x^3 - 3x - 18 = 0$, of which $x = 3$ is clearly a root. In fact, $x^3 - 3x - 18 = (x-3)(x^2+3x+6)$ and so 3 is the only real root.

36. $1+\sqrt{2}$. [24] Let L denote the length of the army, f the desired ratio, and αL the distance the army had moved when the runner reached the rear. Then the runner ran $L - \alpha L$ while the army moved αL, and the runner ran $L + (L - \alpha L)$ while the army moved $L - \alpha L$. Hence $\frac{1-\alpha}{\alpha} = f = \frac{2-\alpha}{1-\alpha}$ so that $2\alpha - \alpha^2 = 1 - 2\alpha + \alpha^2$ and hence $\alpha = 1 - \frac{1}{2}\sqrt{2}$. Thus $f = \frac{\sqrt{2}}{2-\sqrt{2}} = 1+\sqrt{2}$. **Alternative solution:** Without loss of generality, assume that the army is 1 mile long and marches at 1 mph. Let r denote the speed of the runner. During the first part of his run, he is moving at rate $(1+r)$ with respect to the army and moves 1 mile with respect to the army, so the time required for this is $1/(1+r)$. During the second part of his run, he is running at rate $(r-1)$ with respect to the army and again moves 1 mile with respect to the army. Thus his total time is $\frac{1}{1+r} + \frac{1}{r-1}$ and this equals 1 since the army will have marched for an hour. Thus $r^2 - 2r - 1 = 0$ and $r = 1+\sqrt{2}$.

37. $29/2 = 14.5$. [5] Let s be the length of a leg of the triangle, so that its area is $s^2/2$. Place the triangle with right angle at the origin, and other vertices at $(s,0)$ and $(0,s)$. Let (x,y) be the coordinates of P. Then $x^2+y^2 = 8$, $(s-x)^2+y^2 = 9$ and $(s-y)^2+x^2 = 25$. We obtain $s^2 - 2sx = 1$ and $s^2 - 2sy = 17$. Solve for x and y in terms of s, and place in the first equation, obtaining $(s^2-1)^2 + (s^2-17)^2 = 4s^2\cdot 8$. This simplifies to $s^4 - 34s^2 + 145 = 0$, hence $s^2 = 5$ or 29. Only $s^2 = 29$ is consistent with the given data.

38. 41/96. [13] Since $\left|\frac{3}{7} - \frac{p}{q}\right| = \frac{|3q-7p|}{7q}$, we try to find q as large as possible but ≤ 99 for which there is an integer p with $3q - 7p = \pm 1$. We look for multiples of 7 which are slightly less than 300 and are not divisible by 3. The largest is $287 = 41\cdot 7$, which differs from $3\cdot 96$ by 1. Thus $\frac{41}{96}$ differs from $3/7$ by $1/(7\cdot 96)$, which is clearly as small as can be obtained.

39. 3, 6, 7, 8. (any order, but all must be listed). [7] We may assume $a \leq b \leq c$ and $x \leq y \leq z$. We easily see that it is impossible to have $a = b = c = 1$ or $x = y = z = 1$. Therefore $z \geq 2$. If $a = b = 1$, then $xyz = 2 + c = 2 + x + y + z \leq 4z$ which implies $xy \leq 4$. We obtain the following solutions $(x,y,z,c) = (1,2,5,8)$, $(1,3,3,7)$, and $(2,2,2,6)$. Now we may assume $b \geq 2$ and $y \geq 2$. Then

$$abc = x+y+z \leq 3z \leq \tfrac{3}{2}xyz = \tfrac{3}{2}(a+b+c) \leq \tfrac{3}{2}\cdot 3c = \tfrac{9}{2}c.$$

Thus $ab \leq 4$. Similarly $xy \leq 4$. Examination of ten possible ways of selecting a, b, x, and y with products ≤ 4 yields only the solution $(a,b,c,x,y,z) = (1,2,3,1,2,3)$.

40. $3\sqrt{2}$. [7] Angles ECF and EBF are right angles. Let α, β, and γ be the half angles of triangle ABC at A, B, and C, respectively. Then $\alpha+\beta+\gamma = 90$. $\angle CAE = 90 - \alpha = \angle BAF$. Now $\angle ACE = 90 - \gamma = \alpha + \beta = 180 - \angle BAF - \angle ABF = \angle AFB$. Thus triangles ACE and AFB are similar, and so $\frac{AE}{AC} = \frac{AB}{AF}$. Now $AD^2 = AE\cdot AF =$

$AB \cdot AC = 18.$