Algebraic Topology:

There's an App for That

Donald M. Davis

In the spectrum of pure versus applied mathematics, algebraic topology has throughout most of its lifetime sat resolutely on the pure side of the scale. Pioneered in the early 20th century, it has seen continuous development of its abstract theories and methods. Every serious university mathematics department has a topologist—or a team of such—in its ranks, yet most laymen don't have any idea what these people do. What questions do they ask? What results do they produce? And to what end—does their work have applications in the real world, or is it an exercise in pure abstraction? This is the story of one small strand of topological research and the recent surprising discovery of its application to the field of robotics.

While topology is a branch of geometry, algebraic topology often considers questions about high-dimensional objects, or *spaces*, as they are affectionately known. The most familiar spaces are the Euclidean ones: R^n denotes the space of all real n-tuples, so R^2 is the Cartesian plane, R^3 is three-dimensional space, and so on. The superscript indicates the dimension of the space, and this notational convention is used for a large class of topological spaces known as *manifolds*.

For example, the n-dimensional sphere S^n is the set of all points $(x_1, \ldots, x_{n+1})$ in R^{n+1} for which $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$. Think about the case $n = 2$, which is like a balloon in three-space. It is called "two-dimensional" because little parts of it can be flattened out into a two-dimensional disk. In other words, a small patch of the two-sphere looks very much like a small patch of R^2.

If the curviness of the spherical surface seems fundamentally different than the flat plane, keep in mind that in the field of topology, such things do not matter. Two spaces are topologically equivalent if each can be continuously deformed into the other. To a topologist, a circle and a square are indistinguishable! It is important to understand that dimension is a topological property: topologically equivalent manifolds have the same dimension.

Note that the n-sphere sits naturally inside Euclidean $n+1$ space. For more abstract spaces, the following question is fundamental: What is the smallest dimensional Euclidean space in which a given manifold can be embedded? By an *embedding* we mean a one-to-one differentiable function, so no two distinct points on the manifold can occupy the same point in space. The identity map $f(x) = x$ is an embedding of S^n in R^{n+1}.

To get a better sense of this question, consider another family of spaces: the real projective spaces. The space RP^n is formed from S^n by abstractly gluing each point to its antipodal point; i.e., the point $-x$ directly opposite x. For $n > 1$, this gluing cannot be accomplished in the R^{n+1} in which S^n sits. More dimensions are required. For example, it can be shown that RP^2 can be embedded in R^4 but not in R^3. Finding the smallest embedding dimension for RP^n is a challenging question in topology.

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In algebraic topology, replacing embeddings with immersions turns out to be a more tractable type of problem. An immersion is similar to an embedding except that it allows well-behaved self-intersections (technically, the derivative must be injective everywhere). The easiest example of an immersion is that of the Klein bottle, which can be defined as the space obtained from a rectangle by gluing points on its edges according to the prescription in the left side of figure 1. When the long sides are glued, you get a cylinder.

One way of gluing the ends of the cylinder would yield a torus, but that is not the way they are to be glued to make a Klein bottle. The Klein bottle cannot be embedded in R^3, but it can be immersed in R^3, as seen on the right side of figure 1. Nice glass models of immersed Klein bottles can be seen at *www.kleinbottle.com*.

Similarly, RP^2 can be immersed in R^3, but it is not so easy to visualize. Figure 2 depicts an immersion of RP^2 called Boy's surface. Werner Boy discovered this immersion in 1901. It is not easy to see how this picture relates to the definition of RP^2, but it can be shown that Boy's surface is topologically equivalent to two-dimensional projective space.

In 1970, the only unknown immersion question for any RP^n with $n \leq 27$ was whether RP^{24} could be immersed in R^{38}. This question remains unresolved. An immersion question that was completely resolved in the 1980s was for RP^{28}; it can be immersed in R^{47} but not in R^{46}. Embeddings are much trickier. For example, it is not known whether RP^6 can be embedded in R^9 or in R^{10}. Although it can be embedded in R^{11}, mathematicians have proved that it cannot be embedded in R^8.

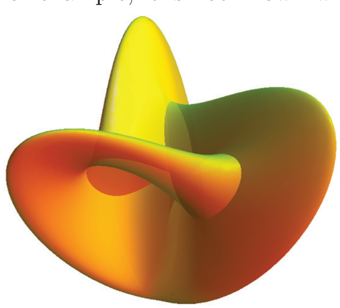

Figure 2: *An immersion of Boy's surface in R^3.*

Algebraic topology has been used to prove both positive and negative immersion results. To give a sense of what these results look like, here is a sample of two theorems. Let $\alpha(n)$ denote the number of ones in the binary expansion of a positive integer n.

> *Theorem* (Davis, 1983). For any positive integer $n \equiv 6 \pmod 8$ with $\alpha(n) = 4$, RP^n can be immersed in R^{2n-9}.

> *Theorem* (Davis, 1984). For any positive integer n, $RP^{2(n+\alpha(n)-1)}$ cannot be immersed in $R^{4n-2\alpha(n)}$.

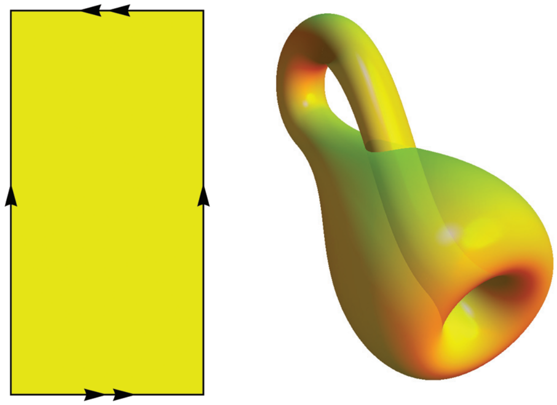

Figure 1: *How to glue the edges of a rectangle to create a Klein bottle, far left. Immersion of a Klein bottle, left.*

While the second theorem does not always give the best-known result, it is always close to it. For example, for $n = 5$, 6, and 8, we obtain that RP^{12}, RP^{14}, and RP^{16} cannot be immersed in R^{16}, R^{20}, and R^{30}, respectively. The first two are within one of best possible, and the third is optimal.

New Dimensions in Robotics

I remember that in one of my earliest talks on immersions of RP^n someone asked, "Does this have any applications?" and my response was that I considered this to be an application—of methods of algebraic topology. However, in recent years a relationship, and potential application, has been found, to robotics.

Michael Farber, a mathematician at the University of Durham, England, has been the leader in developing the field of topological robotics during the past decade. He introduced the notion of topological complexity of a space X. This is the minimum number of motion-planning rules required to tell how to move between pairs of points in X.

These rules are used to tell a robotic arm how to move from one place to another. A "rule" must be continuous, in the sense that if x is close to x', and y is close to y', then the path it takes in going from x to y is close to the path it takes in going from x' to y'. Every pair of points must be governed by exactly one rule.

Here is a simple example. Let $X = S^1$ be the circle. You can't just say, "Take the shortest arc from x to y," because that has a problem when x and y are antipodal. And you can't say, "Move counterclockwise from x to y," because then you would follow a constant path from x to x, but to move from x to a point just a tiny bit clockwise from x, you would move almost all the way around the circle, so this would not be a continuous rule.

But if you used the first rule whenever the points are not antipodal and the second rule when they are, then you have two continuous rules that cover all cases, and so the topological complexity of the circle is two.

Here's where robotics comes in. The space X might consist of all configurations of some robot. For example, consider a robot arm in the plane with two independent joints at separate spots along the arm. See the top of figure 3. Each joint can rotate through any angle, so two angles describe all the arm's possible positions. Since the points on a torus (see the bottom of figure 3) are completely described by two angles, the torus is the configuration space for this robot arm. It can be shown that the topological complexity of the torus is three, and so three rules are required to tell this robot arm how to move between any two positions.

The relationship with immersions of projective spaces came in 2003, when Farber and two colleagues proved that, if $n \neq 1$, 3, or 7, then the topological complexity of RP^n equals the smallest k such that RP^n can be immersed in R^{k-1}. Thus, immersion results from algebraic topology give information about how many motion-planning rules are required for RP^n.

Conversely, nonimmersion results would tell you that it is impossible to specify how to move between any two points in RP^n in fewer than a certain number of (continuous) rules. The proof of the theorem just mentioned describes a precise way of going from an immersion to a set of motion-planning rules.

For this to apply to robotics, we need to have an interpretation of RP^n as the space of configurations of a robot. One, which is simple but not very practical, uses the model of RP^n as the set of all lines through the origin in R^{n+1}. The relationship of this with our previous definition of RP^n as the set of antipodal pairs on S^n is that each line through the origin can be identified with the pair of antipodal points where it intersects the sphere.

Now think of a robot that consists of one arm in R^{n+1}, which pivots around the origin. Then RP^n is the configuration space for this robot, and so immersion results from algebraic topology give information on how many rules are required for this arm to move between any two positions.

It would be more practical to have a model of RP^n as the configuration space of a robot in R^3, even for high values of n. This can be done, at least in theory. There is a known method of associating to any space defined by equations of the sort that can be used to describe RP^n a mechanical linkage, that is, a set of connected physical rods of specified lengths (as in the top of figure 3), for which the given space is the space of configurations. However, this "known method" is quite complicated.

There is also a notion of symmetric motion-planning rules, in which the path from y to x must be the reverse of the path from x to y, and the path from a point to itself must be the constant path. The symmetric topological complexity of a space is the minimum number of symmetric motion-planning rules required to go between any two points. It was proved recently that, with several low-dimensional exceptions, the symmetric topological complexity of RP^n is the smallest k such that RP^n can be embedded in R^{k-1}. So the embedding question is related to robotics too.

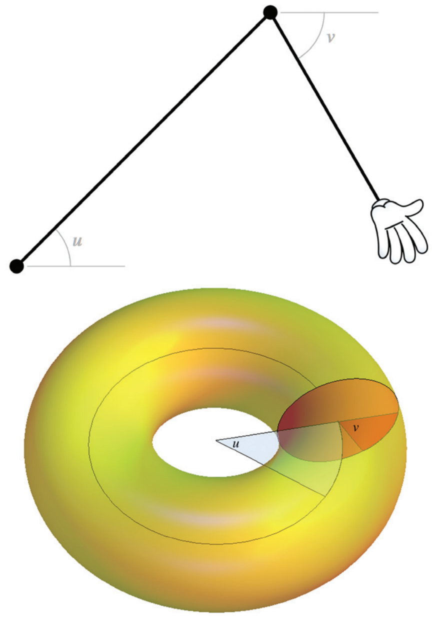

Figure 3: *At top, a robot arm in the plane with two joints. Below, its configuration space is a torus.*

In the end, there is a potential path from the exceedingly abstract realm of algebraic topology to the practical realm of robotic design. This path was completely unknown to the algebraic topologists as they conducted their research, but its existence shows yet again how patterns discovered in the lofty towers of pure mathematics can ultimately reflect some concrete aspect of our physical world. ■

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Further Reading

Michael Farber has a nice book, *Invitation to Topological Robotics* (American Mathematical Society, 2008), that carefully explores the aspects of topology relevant to engineering problems in robotics.

Since the early days of the Internet, I have maintained a table of known immersion and embedding results for real projective spaces. It can be found at *www.lehigh.edu/~dmd1/imms.html*.

An interactive illustration of Boy's surface can be found at *http://demonstrations.wolfram.com/BoySurfaceAndVariations*.

Regarding the two theorems, the first is from my article "Some New Immersions and Nonimmersions of Real Projective Spaces," *AMS Contemporary Mathematics* 19 (1983): 51–64.

The second is from another article of mine, "A Strong Nonimmersion Theorem for Real Projective Spaces," *Annals of Mathematics* 120 (1984): 517–528.

—**Donald M. Davis**