LEARNING AND TEACHING ABOUT FRACTALS

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I have been a professor since 1972, and have published many papers on my research in algebraic topology. During the past ten years, teaching has become increasingly important to me, and fractals have had a lot to do with that. In this note, I will discuss how I learned about fractals, and how I passed this learning on to a wide variety of people.

My introduction to fractals came in May 1984 at the Cornell Topology Festival, a large annual research conference at which I was one of the invited speakers. During the lunch break, John Hubbard entertained many of us with some of his blowups of the Mandelbrot set. This was the first time that I had heard of this concept. I believe that Hubbard told us the definition of the M set, and perhaps a bit more. I remember being very struck by the strange beauty of these photographs, and within a year my living room wall was adorned with a framed print of part of the M set with lightning shooting out into a red background, purchased from Art Matrix, the Cornell group marketing these photos. I enjoyed explaining to dinner guests a little bit of the mathematics behind this picture, and in particular that the baby bug out in the lightning was really attached to the main body by a thin filament, even though we couldn't see the filament. During this period, 1984-1986, my early years of fractal appreciation, I read the Scientific American article by Dewdney, and wrote BASIC programs to create the M set in multihour runs on my PC.

Although Lehigh University, where I have taught since 1974, is generally thought of as an engineering school, there are about 1500 undergraduates who major in humanities or social science, and they have a distribution requirement of at least one semester of mathematics. For a few years, their requirement was two math courses. In the early 1980's, I introduced two courses, Introduction to Finite Mathematics and Introduction to Mathematical Thought, to service many of these students who preferred not to take calculus. In the latter course, I could teach whatever topics I thought might catch their fancy. In Spring 1988, I tried fractals for the first time, devoting four lectures, including one which introduced complex numbers and one in a computer lab, to try to get them to understand the definition of the M set and how a computer can magnify a small portion of the set. I was disappointed with the results when an exam showed that most of them didn't understand. But I did receive a note after the course was over from an art major in the course saying how much she had enjoyed the course, and that she had learned things that might be useful to her.

During Summer 1988, I saw Peitgen and Richter's book "The Beauty of Fractals" at the home of a mathematician friend, and was taken by the beautiful blowup sequence of the M set and also by the somewhat obscure mathematical discussions. I bought a copy for myself and showed it to some of my colleagues. One of them told Lehigh's Director of Galleries about it, and he looked at it and asked whether I could obtain such pictures for an exhibit. Obtaining prints from Art Matrix and Media Magic, I became Curator of an exhibit of fractal art in one of Lehigh's galleries during spring of 1989. This exhibit obtained extensive publicity, with full color pictures of fractals (and a small black and white of me) occupying most of the front page of the D section of the Morning Call, the newspaper that virtually everyone in Allentown and Bethlehem reads. There was also a nice interview on the local television station, featuring the exhibit.

An eighth grade student named Paul Martino in a city about 40 miles south of Lehigh was told about the Morning Call article by one of his teachers. Paul had a science project on fractals, and called me to see if he could meet with me in the evening to see my exhibit and talk about fractals. We did this, and it seemed as if Paul knew almost as much about fractals as I did. During each of his three high school years, Paul had a math-related science project, and would always come up to Lehigh with his father to tell me about his project and to look for materials in Lehigh's library. They were impressed by all the studying going on in the library and also by this professor who would spend an evening with them, and so Paul ended up applying only to Lehigh, Harvard, and Princeton. A Trustees' Fellowship which covered all his expenses led him to choose Lehigh, and he has been a phenom here. His 3.96 GPA ranks him first in the College of Arts and Sciences, where he is a Computer Science major with a minor in mathematics. He has been in several Scholars programs at Lehigh which have enabled him to spend a significant amount of time with people such as Lee Iacocca and Dan Quayle. He will graduate from Lehigh in three years in Spring 1995, and attend one of the top graduate programs in the country in computer science. His life would have been different had it not been for the article about my fractal exhibit.

Spring of 1989 was when fractals started taking over a large part of my life. In addition to the exhibit, I gave two courses and several demonstrations for the general public. One of the courses was the same Intro to Math Thought course that had included three lectures on the M set in 1988. This time I devoted six lectures to fractals, which seemed to help the students' understanding of the M set. By this time, I was starting to think about writing a book for my course. Each year, I would try a different text, but would always add so much of my own material that the students would complain "If you weren't going to use the book, why did you make us buy it?" I started writing detailed lecture notes and using them instead of a text. The three main topics that I had found most effective were non-Euclidean geometry, Number Theory and Cryptography, and Fractals. In 1989, I tried covering all of them, but was still disappointed with the amount that the students learned, and so in subsequent years, I would just cover one or two of the topics. My lecture notes kept growing and were finally published in 1993 by Princeton University Press as "The Nature and Power of Mathematics." This year (Spring 1995) will be the second year I have used the actual book for the course, plus three years when it was in almost complete form as lecture notes. I will be doing mostly fractals this spring, but will diverge somewhat from the text, primarily to use Fractint to replace BASIC programs.

My book contains five chapter, and these are divided into sections as follows:

1. Some Greek Mathematics
    - 1.1 π and Irrational Numbers
    - 1.2 Euclidean Geometry
    - 1.3 Greek Mathematics and Kepler
2. Non-Euclidean Geometry
    - 2.1 Formal Axiom Systems
    - 2.2 Precursors to Non-Euclidean Geometry
    - 2.3 Hyperbolic Geometry
    - 2.4 Spherical Geometry
    - 2.5 Models of Hyperbolic Geometry
    - 2.6 The Geometry of the Universe
3. Number Theory
    - 3.1 Prime Numbers
    - 3.2 The Euclidean Algorithm
    - 3.3 Congruence Arithmetic
    - 3.4 The Little Fermat Theorem
4. Cryptography
    - 4.1 Some Basic Methods of Cryptography
    - 4.2 Public Key Cryptography
5. Fractals
    - 5.1 Fractal Dimension
    - 5.2 Iteration and Computers
    - 5.3 Mandelbrot and Julia Sets

It is written for good liberal arts students. It could also work for a course for good high school students, or for college freshmen thinking about a possible math major, or for the interested non-student. For a one-semester course I recommend one of the following

i. 1.1, 1.2, and Chapter 2
ii. 1.1, 1.3, and Chapter 5
iii. Chapters 3 and 4

If I become frustrated when students don't learn the concepts or complain about the computer assignments, I can take solace from the occasional student who lets me know that he or she appreciates it. I received a letter in December 1994 from a student who had graduated in May 1994, and had been in my Intro to Math Thought course in 1992. He explained in his letter that he was driving his mother somewhere, and she asked him what he had gained by passing a certain car. He told her that other cars might come in between them, and that by getting through some traffic lights that the other car misses, he might arrive at his destination many minutes ahead of the other car. He compared this to what I had taught them about the butterfly flapping its wings and causing a change in the weather far away. He then mentioned Jurassic Park as another application of chaos theory, and closed by writing "Aside from all this, I just wanted to tell you that your Math 5 class was the most interesting class I had at Lehigh." Teachers need an occasional letter like that.

The other fractals course which I gave in Spring 1989 was a once-a-week seminar on the mathematics of fractals, which was attended by about 15 people, including undergraduates, graduate students, and professors from departments of mathematics, physics, computer science, and engineering. During the first half of the semester, I focused on Barnsley's book "Fractals Everywhere." I felt that the application of the Contraction Mapping Theorem for metric spaces to guarantee that there is a unique attractor of an iterated function system (IFS) was an excellent example of abstract mathematics getting applied in an interesting way. The second half of the semester was devoted to Julia sets and the M set, only scratching the surface of the theoretical treatment of the various criteria for Julia sets. I wrote some programs in Pascal for IFS's and Julia sets, which I shared with the seminar. I think they found IFS's most interesting.

One student in the seminar was my Ph.D. student, Ken Monks. His thesis work was in algebraic topology, but he was also an excellent computer programmer, having worked in industry for several years between undergraduate and graduate school. His C programs for IFS's and Julia sets were much more effective than mine, and added much to the class. Ken has maintained an interest in using fractals in teaching during his years of teaching at University of Scranton. We have done much exchanging of materials for courses on fractals. Ken introduced me to Fractint, the remarkably fast and versatile fractal software.

Ken was also invaluable in the preparation of my book, "The Nature and Power of Mathematics." One of his contributions was to think of the name of the book, after my editor and Benoit Mandelbrot both disapproved of my original title, "Selecta Mathematica: Selected topics in mathematics for liberal arts students." Another was to do a more detailed job of proofreading than anyone else. But the biggest was his production of the sequence of blowups of the Mandelbrot set which appear in the book. He did everything: finding an interesting region on which to focus, letting his 386 machine generate extraordinarily detailed pictures on overnight runs, and experimenting with different camera settings while photographing the screen. We were both disappointed with the quality of the photos which ultimately appeared in the book. Much of the brilliance of Ken's colors was lost. But still his sequence of blowups has a number of features that I haven't seen anywhere else. He focuses on the anus of the M set, and shows that it is an extremely sharp crevice. Inside it, he shows some highly asymmetrical baby bugs, and then focuses on a region that seems to be far away from the M set, but after lots of blowing up, we finally see a baby bug at a magnification of about 10^{14}, surrounded by a spectacular array of colors.

It was also in the spring of 1989 that I realized how effective the topic of fractals is for general lectures. I gave a slide show, with slides from Art Matrix, associated with the art exhibit, which attracted about 100 people, mostly from Lehigh's art department. Also, I gave the first of three annual lectures to prospective students and their parents at Lehigh's Candidates' Day. I used a computer projection system to show these people the Koch curve, a fractal fern, and the M set, and explain a little bit about their properties. This was a very popular talk, as I learned from some of these students after they matriculated at Lehigh.

Two years later, I gave a demo and talk to Math Conn, a science symposium for middle school girls at nearby Cedar Crest College. I let the girls try some hands-on manipulation with Fractint, and then gave my slide show to a group of some 200 students. I received a good deal of newspaper publicity for this. I can't imagine another topic in mathematics being so effective for such a presentation. I would be surprised if that did not nudge some girls toward a career in mathematics.

I also gave a similar talk at my church one summer about that time. Our Unitarian Church changes its format during the summer, to lay-led services of a not-necessarily-religious theme. One member of the congregation who was present teaches at a high school about 25 miles away. He told a math teacher there about my talk, and she contacted me to see if I could give a presentation to her gifted class, which I did. By that time, I was using Fractint for such presentations, and they really enjoyed it. She showed her appreciation by giving me a Fractals calendar as a present that year, and the book "Einstein's Dreams" a year later.

My only meeting to date with Benoit Mandelbrot ultimately led to a similar relationship with another high school teacher. In Spring 1991, I drove to Princeton to hear Mandelbrot give two talks. I gave him a copy of the fractals part of a draft of my book between talks. His hosts there, Fred Almgren and Jean Taylor, were able to incorporate me into the dinner party they were having at their house for Mandelbrot and some of their students. This gave me an excellent opportunity to talk to the man. He didn't forget me, for some months later I received a telephone call from a high school teacher about 60 miles away who had been told about me by Mandelbrot. It seems that this teacher had decided to introduce his students to fractals and had the gumption to call Mandelbrot to ask how to do it. Mandelbrot told him that I lived near him and had written a book about fractals, and suggested that he contact me. This teacher and I arranged for him to visit me at Lehigh to see what materials were available, and we stayed in contact for some time after that. [1]

I have given two other types of fractals courses at Lehigh, which have disappointed me slightly in my inability to get through to most of the students. One was a course in Fractal Geometry for math majors which I gave in Spring 1994. The main text was Barnsley, which we used for IFS's and fractal dimension. Then I used a small portion of Falconer's "Fractal Geometry: Mathematical foundations and applications" for the theory of Julia sets and the M set. I find Barnsley's treatment of this topic a little bit quirky. This was a real math course, with minimal computer illustrations. I expected the students to prove theorems; they were math majors, after all. There were nine students in the course. It was clear that two of them were much better than the others, but it wasn't until the final exam that I realized the extent to which this was the case. I thought that I was gearing the course toward the lower echelon, but all except the two good students did very poorly on the final. I will probably teach this course again in 1996. I will have to adjust my sights a little, to realize that most of our math majors are not as adept at proofs as I would like.

The other course is a freshman seminar called "Fractals: unexpected beauty in mathematics." I gave it in the spring and fall semesters of 1993. Lehigh's College of Arts and Sciences has a recent requirement that all freshmen must take a freshman seminar, which is an ordinary 3-credit course, but generally in a nonstandard topic and in a class of no more than 16 students. One purpose of the seminar is to try to encourage discussion rather than lecturing. It is in this regard that my seminar

[1] This teacher, Bob Swaim of Souderton High School, received national publicity, including an appearance on "Good Morning, America," in January, 1995, for pointing out that a Boston Chicken advertisement was confusing permutations and combinations.

failed. I had it in a room in which every student was seated at a computer, networked together. The computers were used in virtually every class meeting. Fractint was the primary software, although many of the BASIC programs from my book were also used. The problem with the computers was that the students could hide behind them to avoid any discussions. Seminars in other subjects were held in a roundtable format, so that students could hardly avoid talking. Many students complained in their evaluations of the lack of discussion, but I am not sure that they really wanted it. The math professor who taught a freshman seminar prior to me said that the students only seemed happy when he would lecture to them. He is the most popular teacher in the department, and was giving a seminar on "The Art of Mathematics." If he couldn't engage the students in conversation, I begin to think that the problem is the nature of the subject.

One of the problems I had with the seminars was hardware-related. The students had to do a lot of creating of figures using Fractint, and to print out the result of their labors. The problem was that the network on which Fractint was installed was often overloaded with users, which made it hard to print your pictures. It seems that Fractint's printing option, at least as we had it installed, was somewhat ornery, even if there were not other users. The third difficulty with the seminars was that, as usual, I overestimated the abilities of the lower half of the class. Even though this was not a high-powered math course, students would still get lost and be afraid to ask for help. Subsequent topics would then be wasted on these students. The course really had four almost-independent topics: Fractal dimension and L-systems, IFS's, chaos, and the M set. On each topic, students could almost start afresh, but within a topic, they could get really lost.

There were good things about the seminars. I think the best was having the students create their own L-systems and IFS's using Fractint. This was something that gave them a lot of room to be creative, more than any other math course that I have taught. It was fun for them to share their creations with the class. I intend to do this in my Intro to Math Thought class this spring (1995), although we won't have individual computers, and if I get to revise my book, I would certainly include much more of these creative activities. The IFS's were particularly successful. The first time I gave the course, one student who had access to a scanner made a fractal map of Africa, using an IFS with eight transformations. It came out beautifully. I showed this to the class the second time I taught the seminar, and I think it inspired many in the class to become really involved with this project. Two of the best were a fractal Donald Duck and a cow's head. Several students, including the maker of the cow's head, worked extremely hard and had all the concepts right, but just couldn't get their picture to come out the way they thought it ought to look. They came to me for help, and after much scrutiny, I was able to find a careless mistake which had messed up their picture. The gratitude that they showed when their picture came out the way they had envisioned it convinced me that this was more than just a homework problem to them; it was something they really wanted to accomplish.

In order to create her fractal cow's head, my student Sarah Goode, who has cows on her family farm in Ohio, began by sketching on graph paper her desired cow's head. Then, using freehand drawing, she tried to fill it in with shrunken versions of the same thing. Sarah used 19 mini cow's heads. For each of these, she had to find the coordinates of the tips of the ears and bottom of the mouth, and use a computer program to find the coefficients of the affine transformation which would send the points of the big head to the corresponding points on each little head. Finally, she put all of these coefficients into a file for Fractint, and it plots a sequence of thousands of points, each obtained by applying one of the 19 affine transformations to the previous point. The sequence of points fills up the desired cow's head in a fractal fashion. Sarah's original cow drawing and its fractal version appear at the end of this paper.

One of the best things for me in teaching about fractals is that, more than with other courses, I speculate and learn while I am teaching. A lot of this has to do with its being an emerging subject, without a complete text. One of these topics about which I have shared my wonder with my classes the last two years deals with the relationship of the Feigenbaum bifurcation diagram and the Mandelbrot set. Wherever there is a baby bug in the proboscis of the M set, there corresponds a region of periodic behavior in the bifurcation diagram, usually of small period for the easily visible bugs. But these baby bugs are dense along the proboscis of the M set, and that means that periodic behavior must occur at a dense open set of values along the bifurcation diagram. All the dark area of the bifurcation diagram is usually described as chaotic behavior, but it seems from this that much of it is periodic behavior of high period. I don't know that my students appreciate this difference or these speculations, but I find it interesting to try to sort out things like this that I can't find in books.

In conclusion, teaching about fractals has been very important to me. I feel that I reach a broader spectrum of people through it than through other mathematical topics. There are only a few people about whom I can definitely say "My teaching about fractals influenced this person's opinion about mathematics," but there are a lot of people for whom I think it is probably true.

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