On the work of Don Davis

Don was born a bit over 70 years ago in Fort Knox Kentucky.

He received his PhD in 1972.

From Jim Milgram.

He did important work on vanishing lines in the Adam's spectral sequence, immersions and non immersions of RP^n, the Segal Burnside ring conjecture, v_1 and v_2 periodicity, *bo*-reolutions, v_1-periodic homotopy of Lie groups, stable geometric dimension of vector bundles over RP^n, combinatorial number thoery, and most recently topolgical complexity.

Don's ability to calculate is legendary.

I concocted and example that involves some of Don's favorite functions.

$\alpha(n)$: The number of ones in the binary expansion of n.

$\nu_p(n)$: The $p-$adic valuation of n defined by

$$n = p^{\nu_p(n)} m$$

with $(m, p) = 1$.

$S(n,k)$: the Sterling number of the second kind.

$S(n, k)$ is the number of ways to partition n objects into k non-empty subsets.

Here is its formula that defines the Sterling number:

$$(e^x - 1)^j = \sum_{k \geq j} S(k, j) \frac{x^k}{k!}$$

$$
\begin{bmatrix}
3 & 0 & 5 & 0 & 0 & 8 & 9 & 0 & 0 \\
1 & 1 & 6 & 1 & 1 & 9 & 10 & 1 & 1 \\
3 & 0 & 7 & 0 & 0 & 10 & 11 & 0 & 0 \\
7 & 2 & 1 & 2 & 2 & 11 & 12 & 2 & 2 \\
15 & 0 & 10 & 0 & 0 & 12 & 13 & 0 & 0 \\
31 & 1 & 65 & 1 & 1 & 13 & 14 & 1 & 1 \\
63 & 0 & 350 & 0 & 0 & 1 & 15 & 0 & 0 \\
127 & 3 & 1701 & 3 & 3 & 28 & 1 & 3 & 3 \\
255 & 0 & 7770 & 0 & 0 & 462 & 36 & 0 & 0 \\
511 & 1 & 34105 & 1 & 1 & 5880 & 750 & 1 & 1 \\
1023 & 0 & 145750 & 0 & 0 & 63987 & 11880 & 0 & 0 \\
2047 & 2 & 611501 & 2 & 2 & 627396 & 159027 & 2 & 2 \\
4095 & 0 & 2532530 & 0 & 0 & 5715424 & 1899612 & 0 & 0 \\
8191 & 1 & 10391745 & 1 & 1 & 49329280 & 20912320 & 1 & 1 \\
16383 & 0 & 42355950 & 0 & 0 & 408741333 & 216627840 & 0 & 0 \\
32767 & 4 & 171798901 & 4 & 4 & 3281882604 & 2141764053 & 4 & 4 \\
65535 & 0 & 694337290 & 0 & 0 & 25708104786 & 20415995028 & 0 & 0 \\
131071 & 1 & 2798806985 & 1 & 1 & 197462483400 & 189036065010 & 1 & 1 \\
262143 & 0 & 11259666950 & 0 & 0 & 1492924634839 & 1709751003480 & 0 & 0 \\
524287 & 2 & 45232115901 & 2 & 2 & 11143554045652 & 15170932662679 & 2 & 2 \\
1048575 & 0 & 181509070050 & 0 & 0 & 82310957214948 & 132511015347084 & 0 & 0
\end{bmatrix}
$$

The $(i,j)th$ entry is

$$\begin{cases} S(i,j+1) & \text{if } \alpha(j+1) \text{ is odd} \\ & \text{and } i \geq j+1 \\ i+j+1 & \text{ if } \alpha(j+1) \text{ is odd} \\ & \text{and } i < j+1 \\ \nu_2\big(\sum\limits_{i\geq 1} 80^{i-1}\big(\binom{j+1}{i} + (2^{4(j+1)}-1)\binom{i+j+1}{i}\big)\big) & if\, \alpha(j+1) \text{ is even} \end{cases}$$

As of yesterday Don wrote 120 papers.

Anderson-Davis

"A vanishing theorem in homological algebra" (1973)

They prove a vanishing line for the Adams spectral sequence.

There are the classes

$$P_t^s, \quad t > s \geq 0$$

in the mod 2 Steenrod algebra which are dual to the Milnor element

$$\xi_t^{2^s}.$$

These classes satisfy

$$(P_t^s)^2 = 0$$

So for any module, M over the Steenrod algebra we can define the *Margolis cohomology*

$$H^*(M; P_t^s)$$

Suppose you are given a module over the Steenrod algebra, M, with the property that

$$H^*(M; P^{s_0}_{t_0}) \neq 0$$

for some $P^{s_0}_{t_0}$ with $s_0 < t_0$.

The dimension of $P_{t_0}^{s_0}$ is

$$d = 2^{s_0}(2^{t_0} - 1)$$

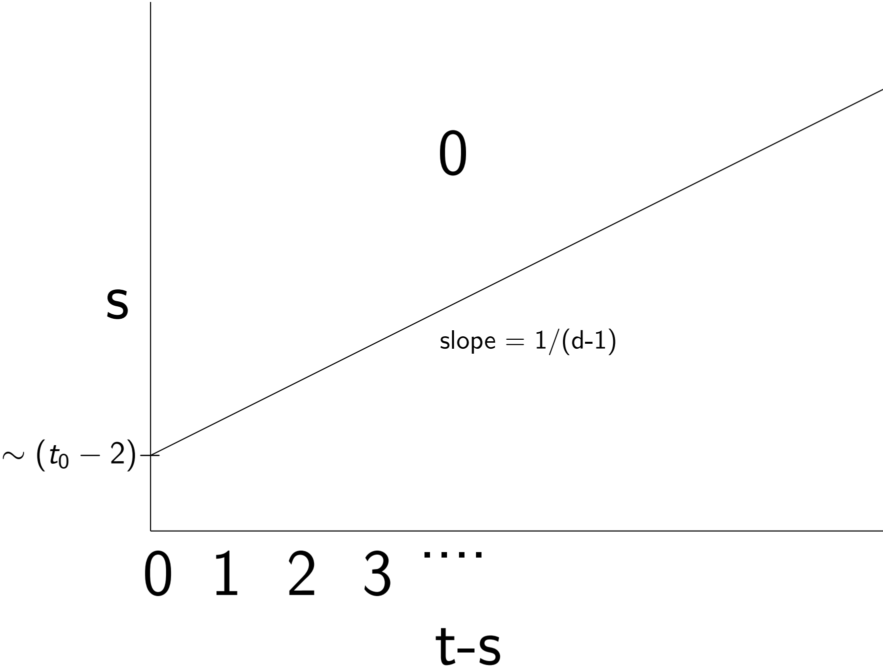
Suppose you are lucky and you find that for all P_t^s of dimension less than d

$$H^*(M; P_t^s) = 0$$

Then the picture for

$$Ext_{\mathcal{A}}(M, \mathbb{Z}_2)$$

has a vanishing line.


0
s
slope = 1/(d-1)
$\sim (t_0 - 2)$
0 1 2 3 ....
t-s

For example if Y is a $p-$local finite CW complex with

$$H^*(Y; P_t^s) = 0$$

for all

$$P_t^s, \text{ with } t + s \leq n + 1$$

and

$$P_t^0, \quad t \neq n + 1$$

Then Y has a v_n-self map.

Don's work on immersions of real projective spaces.

We start with two facts about real projective spaces:

- The stable tangent bundle of RP^n is

$$(n+1)\xi$$

- $$2^L\xi$$

is trivial for $L >> 0$.

In particular

$$(2^L - n - 1)\xi$$

is the stable normal bundle.

So the geometric dimension of

$$(2^L - n - 1)\xi \leq nk - n$$

if

$$RP^n \looparrowright R^k$$

The converse is a special case of a theorem of Hirsh.

Now the geometric dimension of

$$(2^L - n - 1)\xi \leq k - n$$

means that

$$(2^L - n - 1)\xi$$

has

$$(2^L - k - 1)$$

linearly independent sections

These sections can be used to construct a map

$$P^n \times P^{2^L-k-2} \to P^{2^L-n-2}$$

which is homotopic to the inclusion on each factor. (Such a map is called an axial map.)

So one way to prove that there cannot be an immersion is to apply your favorite cohomology theory, E^*, to an asserted axial map.

Specifically there is a class, $X \in E^2(RP^n)$ such that the axial maps sends

$$X^i \to (X_1 + X_2)^i$$

up to a unit.

The method is to pick an i so that $X^i = 0$, for dimensional reasons, but $(X_1 + X_2)^i \neq 0$.

Don used $BP2* = \mathbb{Z}_{(2)}[v_1, v_2]$ and an amazing tour de force of a calculation of most of

$$BP2^*(P^{m_1} \times P^{m_2})$$

to prove what is probably the best general non immersion theorem known

$$RP^{2(m+\alpha(m)-1)} \not\subseteq \mathbb{R}^{4m-2\alpha(m)}$$

Other theories, (*tmf*, ER(n)) give slightly stronger results. But in some sense this non immersion is within 3 of all know results.

I guess I have to explain

“in some sense".

One can change the dimension of the projective space or the Euclidean space.

For example: In 1983 using $MO[8]$ Don showed that

$$RP^{124} \not\subseteq \mathbb{R}^{231}$$

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$$RP^{124} \not\subseteq \mathbb{R}^{231}$$

His general theorem shows that

$$RP^{126} \not\subseteq \mathbb{R}^{232}$$

Positive results.

There is a fibration

$$\begin{array}{c} BO(k) \\ \big\downarrow \\ BO \end{array}$$

The stable normal bundle, ν, of RP^n fits into this picture

$$
\begin{tikzcd}
 & BO(k) \arrow[d] \\
RP^n \arrow[r, "\nu"] & BO
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & BO(k) \arrow[d] \\
RP^n \arrow[ur, "\nu_k"] \arrow[r, "\nu"] & BO
\end{tikzcd}
$$

The obstructions to such a lift live in

$$H^*(RP^n; \pi_{*-1}(F))$$

where we can take F to be a stunted projective space.

The Postnikov tower of this fibration was invented to create a framework for computing the obstructions.

It is the kind of hideous calculation that is a challenge even for Don.

Fortunately, in an example of convergent evolution, there was a 14 year older mathematician who shared Don's love of calculation.

Of course I am referring to Mark Mahowald.

Don wrote 36 papers with Mark as a coauthor.

It is hard to overestimate Mark's influence on Don and on homotopy theory.

Mark (later improved in the work of Gitler and Mahowald) modified the Postnikov tower by inserting the obstruction one Adams filtration at a time.

In particular one starts with a minimal resolution of the $\mathcal{A}$-module $H^*(F;\mathbb{Z}/2)$ to compute the $k-$invariants through a range.

The resulting tower is cleverly called a Modified Postnikov tower.

Here is a sample of the kind of immersion Don obtains using MPT's.

$$RP^n \looparrowright \mathbb{R}^{2n-d}$$

n	$\alpha(n)$		d
mod (8)			
4	3		9
6	4		9
1	5		12
0		$\alpha(n-1)=6, n\neq 64$	13
mod (16)			
14	5	$n\neq 62$	14
12	≥ 5	$\nu(n+4)<7$	12

Don derived these immersions
in a 1983 paper.

Don's work on the Segal Burnside Ring Conjecture conjecture and W.H. Lin's theorem.

The Segal conjecture was motivated by a theorem of Atiyah and Segal on the equivariant $K-$theory of a G space.

The Segal conjecture for a finite group, G, is that there is an isomorphism between

$$\varprojlim [S^n \wedge BG^{(k)}, S^n]$$

and the completion of the Burnside ring of finite G-sets.

In 1979 Lin proved a conjecture of Mahowald which implied the Segal conjecture for $G = \mathbb{Z}/2$.

His proof involved very difficult lambda algebra calculations which, according to Don, were hard to understand.

The proof was published in a joint paper by

Lin, Davis, Mahowald and Adams

Lin proved a conjecture of Mark's involving truncated real projective spectra

$$P_j^{\infty}$$

which makes sense even for $j \leq 0$.

Specifically

$$P_j^{\infty}$$

is the Thom spectrum of a virtual bundle which is j copies of the canonical line bundle over $P^{\infty}, j \in \mathbb{Z}$.

So there is a spectrum

$$P_{-\infty}^{\infty} = \varprojlim P_j^{\infty}.$$

$$H^*(P^{\infty}_{-\infty}; \mathbb{Z}/2) = \mathbb{Z}/2[x, x^{-1}].$$

There is a map

$$S^{-1} \to P_{-\infty}^{\infty}$$

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$$S^{-1} \to P_{-\infty}^{\infty}$$

which in cohomology induces

$$(\Sigma a_i x^i) \mapsto a_{-1}$$

Namely

$$\begin{tikzcd} & P^{\infty}_{-n} \arrow[d] \\ S^{-1} \arrow[r] & P^{\infty}_{-1} \end{tikzcd}$$

Namely

$$\begin{tikzcd} & P_{-n}^{\infty} \arrow[d] \\ S^{-1} \arrow[ur] \arrow[r] & P_{-1}^{\infty} \end{tikzcd}$$

Lin's theorem, proven in [LDMA] is that this map induces a homotopy equivalence after $2-$adic completion.

This is the heart of the proof of the Segal conjecture for $\mathbb{Z}/2$.

This theorem suggest an invariant of classes in the stable homotopy groups of the spheres.

$$\begin{tikzcd} & & P^{\infty}_{1-n} \arrow[d] \\ S^{t-1} \arrow[r, "\alpha"] & S^{-1} \arrow[r] & P^{\infty}_{-1} \end{tikzcd}$$

$$
\begin{tikzcd}
 & & P^{\infty}_{-n} \arrow[d] \\
 & & P^{\infty}_{1-n} \arrow[d] \\
S^{t-1} \arrow[r, "\alpha"] & S^{-1} \arrow[r] \arrow[uur] & P^{\infty}_{-1}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & S^{-n} \arrow[r] & P^{\infty}_{-n} \arrow[d] \\
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\end{tikzcd}
$$

$$
\begin{tikzcd}
 & S^{-n} \arrow[r] & P^{\infty}_{-n} \arrow[d] \\
 & & P^{\infty}_{1-n} \arrow[d] \\
S^{t-1} \arrow[r, "\alpha"] \arrow[uur, dashed] & S^{-1} \arrow[r] \arrow[uur] & P^{\infty}_{-1}
\end{tikzcd}
$$

Don's work on v_1-periodic homotopy.

For a fixed prime, p, The v_1-periodic homotopy groups, $v_1^{-1}\pi_i(X)$ of a space is often a direct summand of some actual homotopy group, $\pi_{i+L}(X)_{(p)}$.

So if you can compute the v_1-periodic homotopy groups of such a space you have computed some actual homotopy.

$v_1^{-1}\pi_*(S^n)$ have been computed by Mahowald and Thompson.

The v_1-periodic groups are completely calculable for many spaces, (e.g. Lie groups), but are complicated enough to be interesting.

If you want elements of large order, the place to look are the v_1-periodic groups.

In fact for the spheres there are v_1-periodic classes that achieve the largest order.

I will start by defining the v_1-periodic groups.

I will then tell you the answer for $SU(n)$.

The Sterling numbers will appear.

For a prime, p, there are the mod p^e homotopy groups

$$\pi_n(X; \mathbb{Z}/p^e) = [M^n(p^e), X]$$

There is the self map introduced by Adams which induces a $K-$theory isomorphism.

$$A : M^{n+s(e)}(p^e) \to M^n(p^e)$$

If p is odd $s(e)$ is

$2p^{e-1}(p-1)$.

If $p = 2$ $s(e)$ is

$\max(8, 2^{e-1})$.

So one can define

$$v_1^{-1}\pi_i(X;\mathbb{Z}/p^e)$$

by iterating A.

We can now vary e and take the direct limit of

$$v_1^{-1}(X;\mathbb{Z}/p^e) \to v_1^{-1}(X;\mathbb{Z}/p^{e+1}) \to$$

$$v_1^{-1}\pi_i(X) = \lim_e v_1^{-1}\pi_{i+1}(X;\mathbb{Z}/p^e)$$

The periodic homotopy groups of $SU(n)$ depend on the $p-$adic valuation of the Sterling numbers

$$\nu_p(S(k,j))$$

Define numbers

$$e_p(k,n) = \min_{n \le j \le k} \{\nu_p(S(k,j))\}$$

For odd primes

- $v_1^{-1}\pi_{2k}(SU(n)) \approx \mathbb{Z}/p^{e_p(k,n)}$

- $v_1^{-1}\pi_{2k-1}(SU(n))$
 is a group of the same order

You would be hard pressed to extract any information from this result.

Don was not deterred!

Don analyzed these numbers and proved that $\pi_*(SU(n))$ has an element of order greater than

$$n + \left[\frac{n-2}{p^2}\right] + \left[\frac{n+p^2-p-1}{p^3}\right]$$

The odd groups are somewhat more difficult.

The first person to figure out how to compute the odd groups was a student of Don's

Huajian Yang

From 1988 to 2003 Don and his coauthors completed the calculation of the v_1-periodic homotopy of all Lie groups at all primes.

The tools were the Unstable Novikov Spectral Sequence, The unstable $K-$theory spectral sequence, representation theory at the prime 2.

and work of Bousfield (1999) that reduced the calculation for 1-connected H spaces to the Adams operations at the odd primes

By adjointing the maps in the definition of periodic homotopy:

$$v_1^{-1}\pi_i(X) = \lim_e v_1^{-1}\pi_{i+1}(X;\mathbb{Z}/p^e)$$

Davis and Mahowald, in 1990 constructed an omega spectrum

$$\Phi(X)$$

such that

$$\pi_*^s(\Phi(X)) = v_1^{-1}\pi_*(X)$$

Bousfield computed

$$KU^*(\Phi(X))$$

at odd primes as a module over the Adams operations.

He then plugged this into the stable $K-$theory Adams spectral sequence.

The answer came out in a form Don could use to complete the computation of v_1 periodic homotopy of Lie groups at odd primes.

Even the v_1 period homotopy of the spheres has some interesting complications.

The elements of ImJ, ρ_j generate cyclic groups in the odd stems with orders related to Bernoulli numbers.

There are unstable cyclic groups in adjacent even stems with the same orders.

We know the spheres of origin of the elements of ImJ.

So it makes sense to talk about the smallest sphere where the composite

$$\rho_j \circ \rho_i$$

is defined.

Except for a few cases at the prime 2 these composites are unstable v_1 classes.

Here is the way the game works:
If you understand the multiple of the (unstable) generator that represents

$$\rho_j \circ \rho_i$$

when it is born.

Then you know when the compositions die in v_1-periodic homotopy.

The compositions were studied by Mahowald and Thompson (1988)

Don completely determined the life of the compositions of ImJ from the moment the two classes mate to their demise.

The answer (for $p = 2$) involves the number

$$\nu_2(\sum_{i\geq 1} 80^{i-1}(\binom{j+1}{i} + (2^{4(j+1)} - 1)\binom{i+j+1}{i})).$$

we saw some time ago.

I was fortunate to have worked with Don on the project

Don's work on combinatorial number theory

Motivated by trying to understand the orders of the v_1 periodic homotopy groups of $SU(n)$, Don wrote 10 papers in combinatorial number theory

They are all quite technical.

I will say a few words about his 2012 paper

"For which $p-$adic integers x can $\sum_k \binom{x}{k}^{-1}$ be defined?"

The function

$$f(n) = \sum_{k=0}^{n} \binom{n}{k}^{-1}$$

is viewed as taking values in the p-adic numbers, $\mathbb{Q}_p$.

Don studies the properties of this function.

Specifically:

For a $p-$adic integer

$$x = \sum_{i=0}^{\infty} \epsilon_i p^i$$

When does

$$\lim_n f\left(\sum_{i=0}^{n} \epsilon_i p^i\right)$$

converge in the $\mathbb{Q}_p$ topology.

The limit obviously exist if
$x \in \mathbb{N}$

Here is a theorem:

There are certain primes for which the limit only exist for

$$x \in \mathbb{N} \text{ and } x = -1.$$

An odd prime is good if

for every n such that

$$1 \le n \le p - 2$$

$$\nu_p(f(n)) \le 1$$

If an odd prime is not good it is called a Davis prime.

The good primes are the primes for which the limit only exist for the natural numbers and -1.

Here is the bizarre fact.

The only Davis prime less than

$$100,000,000$$

is

$$23$$

Don gives a separate argument for the non convergence if $p = 23$.

So the non convergence theorem is true for all primes less than 100,000,000.

Are there more Davis primes?

Noam Elkies thinks there are infinitely many.

So for Don

42
may be
23.

Don's work on Topological complexity.

The topological complexity, $TC(X)$, of a space X:

There is the fibration

$$E : PX \to X \times X$$

$$\gamma \mapsto (\gamma(0), \gamma(1))$$

Cover X by contractible open sets

$$\{U_1, \cdots, U_{r+1}\}$$

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Such that over each U_i E has a section.

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$$\{U_1, \cdots, U_{r+1}\}$$

Such that over each U_i E has a section.
The smallest r is the topological complexity of X.

It is easy to explain why Don became interested in the problem of computing the topological complexity for Lens spaces.

In a 2013 paper Gonzalez, Velasco and Wilson proved that the smallest integer, k such that there is a nice map

$$L^{2n+1}(2^e) \times L^{2n+1}(2^e) \to L^{2k+1}(2^e)$$

is related to $TC(L^{2n+1}(2^e))$

If k is the smallest such number then

If k is the smallest such number then

$$2k \leq TC(L^{2n+1}(2^e)) \leq 2k + 1$$

The condition on the map is not too dissimilar to the axial condition I mentioned for the immersion problem.

In fact the topological complexity for RP^n is one more than the best immersion dimension.

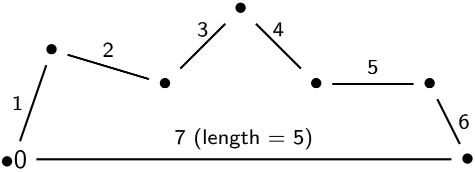
So the linear algebra technology Don developed to prove his non immersion theorem gave him the tools to extract what are probably the best lower bounds for $TC(L^{2n+1}(2^e))$ implied by ku.

Don rediscovered a cool connection between M_n the moduli space of polygons in $\mathbb{R}^2$ and real projective spaces.

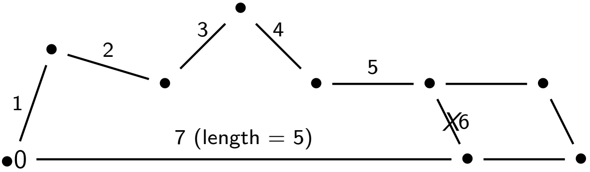
Specifically the $n-$gons he looks at have one edge on the X axis between $(0,0)$ and $(0,n-2)$. All other edges length 1.

The Moduli space of such configurations is S^{n-3} with $\mathbb{Z}/2$ action flipping about the X axis.

For example RP^4:



For example RP^4:



This is a really interesting realization of what topological complexity is telling us. One can think of the polygon as the arms of a robot.

$TC(RP^n)$ tells us something about how many rules are required to move a robot from one configuration to another.

The case of the long side having length $n - 2$ can be generalized.

On might consider the moduli space of $n - gons$ where the long side has length r. $\mathbb{Z}/2$ acts on these manifolds just as it did in the previous case.

The space of such $n-$gons modulo reflection is denoted $\overline{M}_{n,r}$

So $\overline{M}_{n,n-2} = RP^{n-3}$

Otherwise it is some $n-3$ dimensional manifold.

In Don's 120th paper Don gives bounds on $TC(\overline{M}_{n,r})$

For example he proves that

$$TC(\overline{M}_{n,n-4}) \geq 2n - 6$$

I could not cover all of Don's work.
It is too vast.
So I will end here and simply wish Don

HAPPY BIRTHDAY