DAVID L. JOHNSON AND A. M. NAVEIRA

A TOPOLOGICAL OBSTRUCTION TO THE GEODESIBILITY OF A FOLIATION OF ODD DIMENSION

ABSTRACT. Let M be a compact Riemannian manifold of dimension n, and let $\mathscr{F}$ be a smooth foliation on M. A topological obstruction is obtained, similar to results of R. Bott and J. Pasternack, to the existence of a metric on M for which $\mathscr{F}$ is totally geodesic. In this case, necessarily that portion of the Pontryagin algebra of the subbundle $\mathscr{F}$ must vanish in degree n if $\mathscr{F}$ is odd-dimensional. Using the same methods simple proofs of the theorems of Bott and Pasternack are given.

AMS(MOS) subject classifications (1970): Primary 57D30; Secondary 53B25, 53C20.

Key words and phrases: Foliations, totally geodesic foliations, integrability obstructions, geodesibility.

0. INTRODUCTION

If $\mathscr{F}$ is a codimension-k distribution on a compact smooth manifold M, there is a well-known topological obstruction, due to R. Bott, to the integrability of $\mathscr{F}$; the Pontryagin algebra of $T_*(M)/\mathscr{F}$ must vanish in degrees greater than $2k$ [1]. This result was greatly improved by J. Pasternack in his thesis under the additional assumption that the metric on M is fiberlike with respect to the foliation $\mathscr{F}$ [7]. In that case, the characteristic algebra of $T_*(M)/\mathscr{F}$ must vanish in degrees greater than k. This article gives a simple proof of these facts, using tensors similar to those introduced by B. O'Neill [6] (cf., [5]). Also, there is a similar obstruction theorem derived in the case where $\mathscr{F}$ is totally geodesic and of odd dimension. However, in this case the obstruction is in the characteristic algebra of the subbundle $\mathscr{F}$ itself; if M is n-dimensional, the characteristic algebra of $\mathscr{F}$ must vanish in degree n.

1. PRELIMINARIES

Let M be, as above, a smooth, compact Riemannian n-manifold. Let $\mathscr{F}$ be a foliation on M of codimension k. Denote also by $\mathscr{F}$ the associated distribution and the orthogonal projection onto this distribution. Similarly, if $\mathscr{H} = \mathscr{F}^\perp$ is the orthogonal distribution, denote by $\mathscr{H}$ the orthogonal projection, and, if $\mathscr{H}$ is integrable, denote the resulting foliation also by $\mathscr{H}$. Vectors in $\mathscr{H}$ (resp., $\mathscr{H}$) will be called *horizontal* (resp., *vertical*). As in [5] and [6], define tensors T and A on M by, for all vector fields $E, F \in \mathscr{X}(M)$,

$$T_E F = \mathscr{H}\nabla_{\mathscr{F}E}\mathscr{F}F + \mathscr{F}\nabla_{\mathscr{F}E}\mathscr{H}F,$$

$$A_E F = \mathscr{H}\nabla_{\mathscr{H}E}\mathscr{F}F + \mathscr{F}\nabla_{\mathscr{H}E}\mathscr{H}F.$$

As in [5], it is easily seen that $\mathscr{F}$ is totally geodesic if and only if $T = 0$,

Geometriae Dedicata **11** (1981) 347–352. 0046–5755/81/0113–0347$00.90.

and that the metric is fiberlike (i.e., locally there are Riemannian submersions defining the foliation) if and only if $A_X Y = -A_Y X$ for all $X, Y \in \Gamma(\mathscr{H})$.

The properties of the tensors A and T may equivalently be given in terms of a single tensor $\mathscr{P}$, where $\mathscr{P}$ is the automorphism $\mathscr{P}: T_*(M) \to T_*(M)$ given by $\mathscr{P} = \mathscr{F} - \mathscr{H}$. In a forthcoming article the second author will classify the various geometric almost-product and foliated structures defined naturally in terms of this automorphism, analogously to the work of A. Gray and L. M. Hervella on almost-complex structures [4]. At present there is the following partial classification.

PROPOSITION (1.1)

(a) *$\mathscr{P}$ is parallel if and only if M is locally isometric to a Riemannian product.*

(b) $\nabla_V(\mathscr{P}) = 0$ for $V \in \Gamma(\mathscr{F})$ *if and only if $\mathscr{F}$ is totally geodesic.*

(c) *For $X, Y \in \Gamma(\mathscr{H})$, $\nabla(\mathscr{P})_X Y + \nabla(\mathscr{P})_Y X = 0$ if and only if the metric is fiberlike.*

(d) For $X, Y \in \Gamma(\mathscr{H})$, $\nabla(\mathscr{P})_X Y - \nabla(\mathscr{P})_Y X = 0$ *if and only if $\mathscr{H}$ is integrable.*

Proof. A calculation verifies that

$$\nabla(\mathscr{P})_E F = -2\mathscr{F}\nabla_{\mathscr{F}E}\mathscr{H}F + 2\mathscr{H}\nabla_{\mathscr{F}E}\mathscr{F}F - 2\mathscr{F}\nabla_{\mathscr{H}E}\mathscr{H}F + 2\mathscr{H}\nabla_{\mathscr{H}E}\mathscr{F}F.$$

By taking the various cases of E and F either vertical or horizontal it is clear that $\mathscr{P}$ is parallel if and only if both A and T vanish. In that case [5] shows that M is locally isometric to a Riemannian product, verifying part (a). The remaining portions of the Proposition follow from this formula for $\nabla(\mathscr{P})$ and [5]. □

$X \in \Gamma(\mathscr{H})$ is basic if, for some local submersion $f_U : U \to \mathbb{R}^k$ defining $\mathscr{F}|_U$, X is f_U-related to a vector field $\bar{X}$ on $\mathbb{R}^k$; that is, $f_{U*}(X) = \bar{X}$.

PROPOSITION (1.2)

(a) *If X is basic, and if V is vertical, then $[X, V]$ is vertical.*

(b) *If the metric is fiberlike, it is possible to choose X and Y basic (with arbitrary horizontal values at a given point) so that $\nabla_X Y$ is also vertical.*

Proof. The first statement is trivial; since X is f_U-related to $\bar{X}$, and V is f_U-related to zero, $[X, V]$ is f_U-related to $[\bar{X}, 0] = 0$. For the second, the definition of a fiberlike metric [5] implies the existence on $\mathbb{R}^k$ of a metric for which $f_U : U \to \mathbb{R}^k$ is a Riemannian submersion. In this case, if X and Y are basic, f_U-related to $\bar{X}$ and $\bar{Y}$, respectively, then, for $\bar{\nabla}$ the Riemannian covariant derivative on $\mathbb{R}^k$, $\mathscr{H}\nabla_X Y$ is f_U-related to $\bar{\nabla}_{\bar{X}} \bar{Y}$ [6]. Choosing vector fields $\bar{X}$, $\bar{Y}$ so that $\bar{\nabla}_{\bar{X}}\bar{Y} = 0$ completes the proof. Note that, in the general case X and Y may be chosen with $[X, Y]$ vertical by a similar argument. □

2. THEOREMS OF BOTT AND PASTERNACK

On $\mathscr{H}$, define a connection $\tilde{\nabla}$ by $\tilde{\nabla}_E X = \mathscr{H}\nabla_E X - A_X \mathscr{F}E$, for $E \in \mathscr{X}(M)$ and $X \in \Gamma(\mathscr{H})$. It is evident that $\tilde{\nabla}$ is a connection; $\tilde{\nabla}$ is a geometrically natural choice of Bott's connection on $T^*(M)/\mathscr{F} \simeq \mathscr{H}$. Unfortunately $\tilde{\nabla}$ is not, in general, symmetric.

THEOREM (2.1). *If $\tilde{\Omega}$ is the curvature of $\tilde{\nabla}$, $\tilde{\Omega}(V, W) = 0$ if both V and W are vertical.*

Proof. For X basic,

$$\tilde{\nabla}_V X = \mathscr{H}(\nabla_V X - \nabla_X V) = \mathscr{H}[V, X] = 0,$$

by Proposition (1.2). Then,

$$\tilde{\Omega}(V, W)X = \tilde{\nabla}_v \tilde{\nabla}_W X - \tilde{\nabla}_W \tilde{\nabla}_V X - \tilde{\nabla}_{[V,W]} X = 0$$

due to the integrability of $\mathscr{F}$.

COROLLARY (2.2) [Bott]. Char^p $(\mathscr{H}) = 0$ *for $p > 2k$, where $Char^p(\mathscr{H})$ is that part of the real characteristic algebra (Pontryagin or Chern) of $\mathscr{H}$ in degree p.*

Remark. In general, this is the Pontryagin algebra of $\mathscr{H}$, and specifically does not include terms involving the Euler class, since the connection is not symmetric. In the case where $\mathscr{H}$ is complex, all appropriate Chern classes must vanish.

Proof. If $\mathscr{I}^{p/2}$ is the space of all $\mathfrak{gl}(k, \mathbb{R})$-invariant polynomials of degree $p/2$ (resp., $\mathfrak{gl}(k/2, \mathbb{C})$-invariant polynomials), it is well-known [2] that Char^p $(\mathscr{H})$ is generated by all $P(\tilde{\Omega})$, for $P \in \mathscr{I}^{p/2}$. As $P(\tilde{\Omega})$ is tensorial, it suffices to compute $P(\tilde{\Omega})$ $(A_1, \ldots, A_p)$ where each A_j is chosen to be either vertical or horizontal. However, if $p > 2k$ each monomial must possess a component of $\tilde{\Omega}(A_i, A_j)$ with both A_i and A_j vertical. □

In the case where the metric is fiberlike the connection $\tilde{\nabla}$ will be symmetric; an exactly analogous argument yields Pasternack's theorem.

PROPOSITION (2.3). *If the metric is fiberlike, $\tilde{\nabla}$ is symmetric.*

Proof. The condition that

$$\langle \tilde{\nabla}_E X, Y \rangle + \langle X, \tilde{\nabla}_E Y \rangle = E\langle X, Y \rangle$$

is clearly tensorial, thus it suffices to consider only the case where X and Y are basic. If E is vertical, Proposition (1.2) implies that the left-hand side vanishes. That the right-hand side is also zero may be found in [5]. If E is horizontal, taking E to be basic yields

$$\langle \tilde{\nabla}_E X, Y \rangle + \langle X, \tilde{\nabla}_E Y \rangle = \langle \bar{\nabla}_{\bar{E}} \bar{X}, \bar{Y} \rangle + \langle \bar{X}, \bar{\nabla}_{\bar{E}} \bar{Y} \rangle$$

by [6]. As $\bar{\nabla}$ is symmetric the proposition is verified, since $E\langle X, Y \rangle = \bar{E}\langle \bar{X}, \bar{Y} \rangle$ at corresponding points.

THEOREM (2.4). *If the metric is fiberlike, $\tilde{\Omega}(X, V) = 0$ for X horizontal, V vertical.*

Proof. Let Y be chosen to be basic and so that $\nabla_X Y \in \Gamma(\mathscr{F})$ by Proposition (1.2). X, as usual, will be assumed to be basic. Then, $\tilde{\nabla}_X Y = \mathscr{H}\nabla_X Y = 0$ as well as $\tilde{\nabla}_V Y = 0$. As $[X, V]$ is vertical, evidently $\tilde{\Omega}(X, V) Y = 0$. □

COROLLARY (2.5) [Pasternack]. *If the metric on M is fiberlike,* $\mathrm{Char}^p(\mathscr{H}) = 0$ for $p > k$.

Remark. Here the appropriate terms involving the Euler class may be included; that is, if $\mathscr{H}$ is orientable, consider all $so(k)$-invariant polynomials of degree $p/2$.

Proof. In this case it is necessary that each monomial in $P(\tilde{\Omega})(A_1, \ldots, A_p)$ has a component of $\tilde{\Omega}(A_i, A_j)$ where at least one of A_i and A_j is vertical. □

3. Totally geodesic foliations

A foliation $\mathscr{F}$ is *totally geodesic* if each leaf is a totally geodesic submanifold of M. In [5] the first author and L. Whitt found a strong obstruction to the existence of a totally geodesic foliation $\mathscr{F}$ of codimension one under the assumption that $\mathscr{F}$ has at least one closed leaf; in that case M must fiber over a circle. In contrast, H. Gluck has shown that there is no obstruction to the existence of a totally geodesic foliation of dimension one on a simply-connected manifold of odd dimension. It thus seems reasonable to suspect that the topological obstructions to geodesibility of a foliation $\mathscr{F}$, above the integrability obstructions, should lie in the bundle $\mathscr{F}$ rather than the normal bundle.

Define a connection $\hat{\nabla}$ on $\mathscr{F}$ by $\hat{\nabla}_E V = \mathscr{F}\nabla_E V$. $\hat{\nabla}$ is clearly a symmetric connection. Note that, since $\mathscr{F}$ is totally geodesic, $T = 0$. More generally it would be desirable, analogously to Bott's connection, to consider $\mathscr{F}\nabla_E V - T_V \mathscr{H} E$; however, the nonintegrability of $\mathscr{H}$ prevents any transparent consequences in general.

PROPOSITION (3.1). *If $X_m \in T_*(M, m)$ is horizontal, and $Y_m \in T_*(M, m)$ is vertical, there are extensions $X \in \Gamma(\mathscr{H})$ and $Y \in \Gamma(\mathscr{F})$ so that $\hat{\nabla}_X V = 0$.*

Proof. Choose X to be basic. Let $\bar{\gamma}$ be any integral curve of $\bar{X}$ on $\mathbb{R}^k$, where $f_U : U \to \mathbb{R}^k$ is a chosen local submersion with $f_{U*}(X) = \bar{X}$. Let $\Sigma = f_U^{-1}(\bar{\gamma})$. □

LEMMA (3.2). *If Σ is given the induced metric, the restriction $\mathscr{F}^\Sigma$ of $\mathscr{F}$ to Σ is totally geodesic. Also, note that the orthogonal distribution $\mathscr{H}^\Sigma$ is integrable. The metric on Σ is fiberlike with respect to the foliation $\mathscr{H}^\Sigma$.*

Proof. Since the Riemannian covariant derivative ∇^Σ on Σ is given by the orthogonal projection $\Pi_{T_*\Sigma}\nabla$, the first statement is trivial. That the metric on Σ is fiberlike with respect to $\mathscr{H}$ follows from the duality between fiberlike metrics and totally geodesic foliations described in [5]. □

Now let $g_\Sigma : \Sigma \to \mathbb{R}$ be a local submersion defining $\mathscr{H}$. As the induced metric on Σ is fiberlike, there is a metric on $\mathbb{R}$ so that g_Σ is a Riemannian submersion. Choose V to be basic with respect to g_Σ. Proposition (1.2) then implies that $[V, X] \in \Gamma(\mathscr{H}^\Sigma)$. However, as V is vertical and $\mathscr{F}$ is totally geodesic, $\nabla_V X \in \Gamma(\mathscr{H}^\Sigma)$ as well, so that $\nabla_X V \in \Gamma(\mathscr{H}^\Sigma) \subset \Gamma(\mathscr{H})$ (V may be extended to a vector field on U using a smooth family of g_Σ's for the various integral curves of $\bar{X}$). Thus, $\hat{\nabla}_V X = 0$. □

THEOREM (3.3). *If $\mathscr{F}$ is totally geodesic, and if $\hat{\Omega}$ is the curvature of $\hat{\nabla}$, then $\hat{\Omega}(V, X) = 0$ if V is vertical and X is horizontal.*

Proof. Extend X to be basic, and, as in Proposition (3.1), choose V to be $\mathscr{H}^\Sigma$-basic and so that $\hat{\nabla}_X V = 0$. Let W be another $\mathscr{H}^\Sigma$-basic vector field, for which, using Proposition (1.2), $\nabla_V W$ is in $\Gamma(\mathscr{H})$. As $\mathscr{F}$ is totally geodesic, $\nabla_V W = 0$, thus $\hat{\nabla}_V W = 0$. The proof of Proposition (3.1) implies that $\hat{\nabla}_X W = 0$ as well, since W is $\mathscr{H}^\Sigma$-basic. Also, $[X, V] = 0$ as $[X, V]$ must be both horizontal and vertical, applying Proposition (1.2) twice. Thus $\hat{\Omega}(X, V)W = 0$. □

COROLLARY (3.4). *If $\mathscr{F}$ is totally geodesic and if* $\dim(\mathscr{F})$ *is odd,* $\mathrm{Char}^n(\mathscr{F}) = 0$, *where* $n = \dim(M)$.

Proof. If P is any $o(n-k)$-invariant polynomial of degree $n/2$, consider $P(\hat{\Omega})\ (A_1, \ldots, A_n)$ where A_i is either vertical or horizontal. Each monomial must possess a component of $\hat{\Omega}(A_i, A_j)$ where one is vertical and the other is horizontal, as $\dim(\mathscr{F})$ is odd, thus each monomial must vanish. □

4. An Example

Let M be a compact 8-dimensional orientable manifold with $\chi(M) = 0$ but Hirzebruch signature nonzero. Thurston [8] has shown that there is a foliation $\mathscr{F}$ on M of codimension one. However,

PROPOSITION (4.1). *No codimension-one foliation $\mathscr{F}$ on M is geodesible.*

Proof. Let $\mathscr{H} = \mathscr{F}^\perp$. As $T_*(M) \simeq \mathscr{H} \oplus \mathscr{F}$, the total Pontryagin class $p_*(M)$ is given by $p_*(M) = p_*(\mathscr{H})p_*(\mathscr{F})$. But $\mathscr{H}$ is one-dimensional, so that $p_*(\mathscr{H}) = 1$, hence $p_2(M) = 0$ and $p_1(M) = p_1(\mathscr{F})$. By the Hirzebruch signature theorem, the signature $\sigma(M)$ of M satisfies $\sigma(M) = \frac{1}{45}(7p_2(M) - p_1(M)^2) = -\frac{1}{45}p_1(\mathscr{F})^2$, which is nonzero by assumption. Corollary (3.4) then implies that $\mathscr{F}$ cannot be totally geodesic. □

References

1. Bott, R.: *Lectures on Characteristic Classes and Foliations*, Lecture Notes in Math., vol. 279, Springer-Verlag, 1972, pp. 1–80.
2. Chern, S.-S. and Simons, J.: 'Characteristic Forms and Geometric Invariants', *Ann. Math.* **99** (1974), 48–69.
3. Gluck, H.: 'Can Space be Filled by Geodesics, and If So, How?' (to appear).
4. Gray. A. and Hervella, L. M.: 'The Sixteen Classes of Almost Hermitian Manifolds and their Linear Invariants, *Ann. Math. Pura Appl.* (to appear).
5. Johnson, D. L. and Whitt, L. B.: 'Totally Geodesic Foliations', *J. Diff. Geo.* (to appear).
6. O'Neill, B.: 'The Fundamental Equations of a Submersion', *Michigan Math. J.* **13** (1969), 459–469.

7. Pasternack, J.: 'Classifying Spaces for Riemannian Foliations, Differential Geometry', *Proc. Symposia Pure Math.* **27** (1975), 303–310.
8. Thurston, W. P.: 'Existence of Codimension-one Foliations', *Ann. Math.* **104** (1976), 249–268.

Authors' addresses

David L. Johnson,
Department of Mathematics,
Texas A & M University,
College Station
Texas 77843,
U.S.A.

A. M. Naveira,
Departamento de Geometria
y Topologia,
Facultad de Ciencias Matemáticas,
Universidad de Valencia,
Valencia,
Spain

(Received January 21, 1980)