CHERN-SIMONS FORMS ON ASSOCIATED BUNDLES, AND BOUNDARY TERMS

DAVID L. JOHNSON

ABSTRACT. Let E be a principle bundle over a compact manifold M with compact structural group G. For any G-invariant polynomial P, The transgressive forms $TP(\omega)$ defined by Chern and Simons in [4] are shown to extend to forms $\Phi P(\omega)$ on associated bundles B with fiber a quotient $F = G/H$ of the group. These forms satisfy a heterotic formula

$$d\Phi P(\omega) = P(\Omega) - P(\Psi),$$

relating the characteristic form $P(\Omega)$ to a fiber-curvature characteristic form. For certain natural bundles B, $P(\Psi) = 0$, giving a true transgressive form on the associated bundle, which leads to the standard obstruction properties of characteristic classes as well as natural expressions for boundary terms.

INTRODUCTION

In their groundbreaking paper [4], S-S Chern and James Simons explain that their theory of what are now known as Chern-Simons classes

> grew out of an attempt to derive a purely combinatorial formula for the first Pontryagin number of a 4-manifold. ... This process got stuck by the emergence of a boundary term which did not yield to a simple combinatorial analysis. The boundary term seemed interesting in its own right and ....

Their "boundary term" was in fact a geometric realization of the transgression co-chains which appear in the Leray-Serre spectral sequence of a principal bundle [1], and their importance has grown out of the fact that, on the base manifold, they measure finer geometric information than the primary characteristic classes of the bundle. The main result of this article is the natural extension of the Chern-Simons forms to forms on associated bundles $B \to M$ rather than on the principal bundle, whose differentials give a correspondence between the characteristic classes of the bundle and a characteristic form involving the curvature of the fibers. In certain cases, when the fiber-curvature term vanishes, these Chern-Simons forms on the associated bundle serve as true transgressions, and not only give similar secondary characteristic classes for the bundle, but also produce natural boundary terms for the traditional characteristic forms, providing simple proofs of the primary obstruction properties of these characteristic classes. Interestingly, one of the results established below re-constructs a form in the unit tangent bundle originally constructed by Chern in 1944, in conjunction with his version of the generalized Gauss-Bonnet theorem [3], which pre-dates the topological interpertation by Borel. That construction of Chern is interpreted here in terms of his and Simons' later work in a broader context.

Date: December 2, 2009.
2000 *Mathematics Subject Classification.* 53C05, 57R20, 55R25 53A55.
Key words and phrases. Characteristic classes, Chern-Simons invariants.

1. Associated bundles and Chern-Simons forms

Let M be a compact, n-dimensional manifold, and let $\pi : E \to M$ be a principal bundle over M with compact structural group G. Real characteristic classes of E can be determined by forms $P(\Omega)$, where $P \in \mathcal{I}(\mathfrak{g})$ is an adjoint-invariant polynomial on the Lie algebra $\mathfrak{g}$ of G, and Ω is the curvature form of a connection ω on E. Such forms $P(\Omega)$ are horizontal, invariant forms on E, so are naturally defined as forms on M itself.

Invariance properties of these polynomial forms, along with the Bianchi identity, traditionally are used to show that these forms are closed on M (cf. [7]). Moreover, the forms $P(\Omega)$ in the cohomology of E itself are exact, which not only verifies that $P(\Omega)$ are closed on M but also implies the existence of forms $TP(\omega)$ on E, primitives of the characteristic forms $P(\Omega)$. Given a connection ω on E, Chern and Simons derive in [4] an explicit formula for these transgressive forms.

Theorem 1. *[Chern-Simons]. Let $\pi : E \to M$ be a principal bundle over a compact n-manifold M with compact structural group G. If $P \in \mathcal{I}^k(\mathfrak{g})$ is*

$$TP(\omega) := \sum_{i=0}^{k-1} A_i P(\omega, [\omega,\omega]^i, \Omega^{k-i-1})$$

is a G-invariant form on E satisfying $dTP(\omega) = P(\Omega)$, where $A_i := (-1)^i k!(k-1)!/2^i(k+1)!(k-1-i)!$ a degree-k, adjoint-invariant polynomial on the Lie algebra $\mathfrak{g}$ of G, and if ω is a connection on E, then the $(2k-1)$-form

$$TP(\omega) := \sum_{i=0}^{k-1} A_i P(\omega, [\omega,\omega]^i, \Omega^{k-i-1})$$

is a G-invariant form on E satisfying $dTP(\omega) = P(\Omega)$, where $A_i := (-1)^i k!(k-1)!/2^i(k+i)!(k-1-i)!$ and P is realized as a symmetric, multilinear functional $P : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$ by polarization.

Let now $\pi_2 : B \to M$ be an associated bundle to the principal bundle $\pi : E \to M$ as before, with fibers F which are homogeneous spaces, quotients of the structural group G by the isotropy subgroup H of the right action of G on F. The primary example of this situation is when E is the bundle of oriented frames of an oriented Riemannian manifold M, and B the unit tangent bundle of M. In that case $G = SO(n)$ and $H = SO(n-1)$.

Note. It is not strictly necessary to restrict to bundles with homogeneous fibers, but the construction of these classes does depend upon all isotropy subgroups of the action of G on F being conjugate.

The two bundles are related, and in fact the total space of E is a principal bundle with group H over B:

$$\begin{array}{ccc} E & \xrightarrow{\pi_1} & B \\ \pi \downarrow & & \pi_2 \downarrow \\ M & = & M \end{array} \tag{1.1}$$

Decompose the connection ω in terms of the G-equivariant distribution $\overline{\mathfrak{h}} := ker((\pi_1)_*)$ on E, $\omega = \phi + \psi$, where $\psi = \omega|_{ker((\pi_1)_*)}$. ψ is the connection induced from ω on the principal bundle $\pi_1 : E \to B$ with structural group H. Since H is a reductive subgroup, if (by choice of bases) at

$p \in E$ $\omega|_{ker((\pi_1)_*)} : ker((\pi_1)_*) \to \mathfrak{h}$, then ϕ takes values in a reductive complement $\mathfrak{p}$ to $\mathfrak{h}$. Also, $[\psi, \psi]$ has values in $\mathfrak{h}$, and $[\psi, \phi]$ takes values in $\mathfrak{p}$.

If $\mathcal{H}$ represents the ψ-horizontal projection (and subspace), the curvature of the two connections are related by $\Psi = d\psi + \frac{1}{2}[\psi, \psi]$ and

$$\begin{aligned} \Omega &= d\omega + \frac{1}{2}[\omega, \omega] \\ &= d\phi + [\psi, \phi] + \frac{1}{2}[\phi, \phi] + \Psi \\ &= d_{\mathcal{H}}\phi + \Psi + \frac{1}{2}[\phi, \phi]. \end{aligned} \tag{1.2}$$

In particular, restricting to the subalgebra $\mathfrak{h}$, $\Omega_{\mathfrak{h}} = \Psi + \frac{1}{2}[\phi, \phi]_{\mathfrak{h}}$. The ψ-covariant differential $d_{\mathcal{H}}\phi := d\phi + [\psi, \phi]$ is the restriction of $d\phi$ to ψ-horizontal tangents. Similarly, by the Bianchi identity,

$$\begin{aligned} d_{\mathcal{H}}\Omega &= d\Omega + [\psi, \Omega] \\ &= [\Omega, \omega] + [\psi, \Omega] \\ &= [\Omega, \omega] - [\Omega, \psi] \\ &= [\Omega, \phi]. \end{aligned}$$

Theorem 2. *Let $\pi : E \to M$ be a principal bundle over a compact n-manifold M with compact structural group G. Let $P \in \mathcal{I}^k(\mathfrak{g})$ be a degree-k, adjoint-invariant polynomial on the Lie algebra $\mathfrak{g}$ of G, and let ω be a connection on E. For an associated bundle $\pi_2 : B \to M$ with fiber G/H as above, the form*

$$\Phi P(\omega) := \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} A_{ij} P(\phi, [\phi, \phi]^i, \Psi^j, \Omega^{k-i-j-1}),$$

where $A_{ij} := (-1)^i \frac{(i+j)!(k-j-1)!k!}{2^i(k-i-j-1)!i!(k+i)!j!}$, is a π_1-horizontal, Ad_H-invariant form on E, thus represents a form on B. In addition,

$$d\Phi P(\omega) = P(\Omega) - P(\Psi). \tag{1.3}$$

Proof. That $\Phi P(\omega)$ is π_1-horizontal and Ad_H-invariant, and so is a form on B, follows by the definitions of ϕ and Ψ, and the fact that P is invariant under Ad_G. Also, for any invariant polynomial P and equivariant, ψ-horizontal forms $\alpha_1, \ldots, \alpha_k$ of degrees $p_1, \ldots, p_k$, respectively, it is straightforward (cf. [4]) that, on B,

$$dP(\alpha_1, \ldots, \alpha_k) = \sum_i (-1)^{p_1 + \cdots + p_{i-1}} P(\alpha_1, \ldots, \alpha_{i-1}, d_{\mathcal{H}}\alpha_i, \alpha_{i+1}, \ldots, \alpha_k).$$

We now show that there are constants A_{ij} satisfying 1.3, and that they are well-defined. For any choices of A_{ij},

$$\begin{aligned}
d\Phi P(\omega) &= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i} A_{ij} dP(\phi,[\phi,\phi]^i,\Psi^j,\Omega^{k-i-j-1}) \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i} A_{ij} P(d_{\mathcal{H}}\phi,[\phi,\phi]^i,\Psi^j,\Omega^{k-i-j-1}) \\
&\quad -iA_{ij}P(\phi,2[d_{\mathcal{H}}\phi,\phi],[\phi,\phi]^{i-1},\Psi^j,\Omega^{k-i-j-1}) \\
&\quad -jA_{ij}P(\phi,[\phi,\phi]^i,d_{\mathcal{H}}\Psi,\Psi^{j-1},\Omega^{k-i-j-1}) \\
&\quad -(k-i-j-1)A_{ij}P(\phi,[\phi,\phi]^i,\Psi^j,d_{\mathcal{H}}\Omega,\Omega^{k-i-j-2}) \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i} A_{ij} P(\Omega-\Psi-\frac{1}{2}[\phi,\phi],[\phi,\phi]^i,\Psi^j,\Omega^{k-i-j-1}) \\
&\quad -iA_{ij}P(\phi,2[\Omega-\Psi-\frac{1}{2}[\phi,\phi],\phi],[\phi,\phi]^{i-1},\Psi^j,\Omega^{k-i-j-1}) \\
&\quad -jA_{ij}P(\phi,[\phi,\phi]^i,0,\Psi^{j-1},\Omega^{k-i-j-1}) \\
&\quad -(k-i-j-1)A_{ij}P(\phi,[\phi,\phi]^i,\Psi^j,[\Omega,\phi],\Omega^{k-i-j-2}) \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i} A_{ij} P([\phi,\phi]^i,\Psi^j,\Omega^{k-i-j}) \\
&\quad -A_{ij}P([\phi,\phi]^i,\Psi^{j+1},\Omega^{k-i-j-1}) - \frac{1}{2}A_{ij}P([\phi,\phi]^{i+1},\Psi^j,\Omega^{k-i-j-1}) \\
&\quad -2iA_{ij}P(\phi,[\Omega,\phi],[\phi,\phi]^{i-1},\Psi^j,\Omega^{k-i-j-1}) \\
&\quad +2iA_{ij}P(\phi,[\Psi,\phi],[\phi,\phi]^{i-1},\Psi^j,\Omega^{k-i-j-1}) \\
&\quad -(k-i-j-1)A_{ij}P(\phi,[\phi,\phi]^i,\Psi^j,[\Omega,\phi],\Omega^{k-i-j-2}).
\end{aligned}$$

Re-grouping by the powers of $[\phi,\phi]$ and Ψ, and using the identity from [4] which comes from invariance of the polynomial,

$$0 = (-1)^{p_1}P([\alpha_1,\phi],\alpha_2,\ldots,\alpha_k) + \cdots + (-1)^{p_1+\cdots+p_k}P(\alpha_1,\ldots,[\alpha_k,\phi]),$$

so that, in particular,

$$\begin{aligned}
P([\phi,\phi]^i,\Psi^j,\Omega^{k-i-j}) &= -jP(\phi,[\phi,\phi]^{i-1},[\Psi,\phi],\Psi^{j-1},\Omega^{k-i-j}) \\
&\quad -(k-i-j)P(\phi,[\phi,\phi]^{i-1},\Psi^j,[\Omega,\phi],\Omega^{k-i-j-1})
\end{aligned}$$

for ϕ a $\mathfrak{g}$-valued 1-form, (setting $A_{i,j}=0$ if either i or j is negative, or if $i+j>k-1$)

$$\begin{aligned}
d\Phi P(\omega) &= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i}(A_{ij}-A_{i,j-1}-\frac{1}{2}A_{i-1,j})P([\phi,\phi]^i,\Psi^j,\Omega^{k-i-j}) \\
&\quad -(2iA_{ij}+(k-i-j)A_{i-1,j})P(\phi,[\phi,\phi]^{i-1},\Psi^j,[\Omega,\phi],\Omega^{k-i-j-1}) \\
&\quad +2iA_{i,j-1}P(\phi,[\phi,\phi]^{i-1},[\Psi,\phi],\Psi^{j-1},\Omega^{k-i-j}) \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i}(A_{ij}-A_{i,j-1}-\frac{1}{2}A_{i-1,j})P([\phi,\phi]^i,\Psi^j,\Omega^{k-i-j}) \\
&\quad -(2iA_{ij}+(k-i-j)A_{i-1,j})P(\phi,[\phi,\phi]^{i-1},\Psi^j,[\Omega,\phi],\Omega^{k-i-j-1}) \\
&\quad -2iA_{i,j-1}\frac{1}{j}\Big(P([\phi,\phi]^i,\Psi^j,\Omega^{k-i-j})+(k-i-j)P(\phi,[\phi,\phi]^{i-1},\Psi^j,[\Omega,\phi],\Omega^{k-i-j-1})\Big) \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i}(A_{ij}-A_{i,j-1}-\frac{1}{2}A_{i-1,j}-2\frac{i}{j}A_{i,j-1})P([\phi,\phi]^i,\Psi^j,\Omega^{k-i-j}) \\
&\quad -(2iA_{ij}+(k-i-j)A_{i-1,j}+2\frac{i(k-i-j)}{j}A_{i,j-1})P(\phi,[\phi,\phi]^{i-1},\Psi^j,[\Omega,\phi],\Omega^{k-i-j-1}).
\end{aligned}$$

We will have $d\Phi P(\omega)=P(\Omega)-P(\Psi)$ if the coefficient of $P([\phi,\phi]^i,\Psi^j,\Omega^{k-i-j})$ is 1 for $i=j=0$ and -1 for $i=0$, $j=k$, and 0 otherwise, as well as the coefficients of $P(\phi,[\phi,\phi]^{i-1},\Psi^j,[\Omega,\phi],\Omega^{k-i-j-1})$ vanishing. That is, for $(i,j)\neq(0,0)$,

$$\begin{aligned}
0 &= A_{ij}-(\frac{j+2i}{j})A_{i,j-1}-\frac{1}{2}A_{i-1,j} \\
0 &= 2iA_{ij}+(k-i-j)A_{i-1,j}+2\frac{i(k-i-j)}{j}A_{i,j-1}
\end{aligned}$$

or

$$\begin{aligned}
A_{i,j-1} &= -\frac{j(k-j)}{2i(k+i)}A_{i-1,j} \\
A_{ij} &= \left(\frac{(i+j)(i+j-k)}{2i(k+i)}\right)A_{i-1,j}.
\end{aligned}$$

There is a necessary consistency condition, in that two recursive formulas must be consistent, that is

$$A_{i+1,j-1}=\left(\frac{(i+j)(i+j-k)}{2(i+1)(k+i+1)}\right)A_{i,j-1}=\left(\frac{(i+j)(i+j-k)}{2(i+1)(k+i+1)}\right)\left(-\frac{j(k-j)}{2i(k+i)}\right)A_{i-1,j}$$

versus

$$A_{i+1,j-1}=-\frac{j(k-j)}{2(i+1)(k+i+1)}A_{i,j}=-\frac{j(k-j)}{2(i+1)(k+i+1)}\left(\frac{(i+j)(i+j-k)}{2i(k+i)}\right)A_{i-1,j},$$

which indeed do give the same expression, so that the double recursion defining A_{ij} is consistent.

From the second recursion, setting $A_{0,0}=1$, we obtain

$$\begin{aligned}
A_{i,0} &= \left(\frac{(i-k)}{2(k+i)}\right)A_{i-1,0} \\
&= \left(\frac{(i-k)}{2(k+i)}\right)\left(\frac{(i-1-k)}{2(k+i-1)}\right)A_{i-2,0} \\
&= \frac{(-1)^i k!(k-1)!}{2^i(k-i-1)!(k+i)!},
\end{aligned}$$

exactly agreeing with the terms A_i of [4], as expected. Now, using the first recursion,

$$\begin{aligned}
A_{i,j} &= -\frac{2(i+1)(k+i+1)}{j(k-j)}A_{i+1,j-1} \\
&= (-1)^2\left(\frac{2(i+1)(k+i+1)}{j(k-j)}\right)\left(\frac{2(i+2)(k+i+2)}{(j-1)(k-j+1)}\right)A_{i+2,j-2} \\
&= (-1)^2\left(\frac{2^2(i+1)(i+2)(k+i+1)(k+i+2)}{j(j-1)(k-j)(k-j+1)}\right)A_{i+2,j-2} \\
&= (-1)^j\frac{2^j(i+j)!(k+i+j)!(k-j-1)!}{i!(k+i)!j!(k-1)!}A_{i+j,0} \\
&= (-1)^j\frac{2^j(i+j)!(k+i+j)!(k-j-1)!}{i!(k+i)!j!(k-1)!}\frac{(-1)^{i+j}k!(k-1)!}{2^{i+j}(k-i-j-1)!(k+i+j)!} \\
&= (-1)^i\frac{(i+j)!(k-j-1)!k!}{2^i(k-i-j-1)!i!(k+i)!j!},
\end{aligned}$$

which is of course the general term.

The nature of the recursion will guarantee that the coefficients of $P([\phi,\phi]^i,\Psi^j,\Omega^{k-i-j})$ will be 0 except when $i=j=0$, or $j=k$, and that the coefficient of $P(\Omega^k)$ will be 1, because $A_{ij}=0$ if either i or j is negative. Also, the coefficient of $P(\Psi^k)$ will be $-A_{0,k-1}$ (recalling that $A_{ij}=0$ if $i+j>k-1$). Now,

$$-A_{0,k-1} = -\frac{(k-1)!k!}{(k)!(k-1)!} = -1,$$

as claimed. □

The right-hand side of (1.3) is not, unfortunately, exactly the characteristic form $P(\Omega)$ that one might hope for. Fortunately, though, in certain circumstances it can be shown that $P(\Psi)=0$, for which bundles B the form $\Phi P(\omega)$ will represent a secondary characteristic form of $P(\Omega)$ on the associated bundle. This occurs in particular for the Gauss-Bonnet integrand on the unit tangent bundle, which gives the connection between the Chern-Simons class $Te(\omega)$ of the Riemannian connection of an even-dimensional Riemannian manifold M and the formulas for the boundary term described by Chern in [3].

It is not the case that $\Phi P(\omega)$ is the ψ-horizontal part of $TP(\omega)$, which instead is only the terms in $\Phi P(\omega)$ with $j=0$. The additional terms, those involving the curvature Ψ of ψ, can be expressed in terms of Ω, ϕ, as explained below.

Remark 3. The formula (1.3) of Theorem (2)

$$d\Phi P(\omega) = P(\Omega) - P(\Psi)$$

is a general version of the heterotic formula $dH = Tr(F\wedge F) - Tr(R\wedge R)$ of [10], in that, in the case of a tensorial bundle, the curvature term Ω is related to the curvature of the base manifold, and the curvature Ψ is a curvature of the fibers.

2. Obstructions

For specific bundles, the characteristic classes $P(\Omega)$ are obstructions to the existence of global sections. Using the forms $\Phi P(\omega)$, following Chern's original construction, the characteristic classes $P(\Omega)$ can be explicitly computed as obstructions. The same proof, applied to integration over chains rather than cycles, gives relative versions of each of these classes. The boundary term in general will depend upon a choice of section on the boundary. In the case of the Euler class of the tangent bundle, however, the normal field of the boundary gives a canonical section of the tangent bundle over the boundary.

2.1. **The Euler class.** This first result, in the case of a cycle, is of course classical, and the method of proof is essentially that of [3]. In the general case, the result does follow from Chern's result, but was not stated as such by him. Several authors have presented proofs of the general result, usually just for the tangent bundle of a manifold-with-boundary, such as [5, 6, 9]. The formulations differ from case to case, but each basically recovers Chern's transgression form, as does the current version.

Proposition 4. *Let ξ be a rank-$2k$, oriented vector bundle over a compact manifold M, with a smooth metric. Let ω be a metric-compatible connection on ξ. Let α be a smooth singular $2k$-chain in M, and let σ be a generic section of ξ. Then*

$$\int_\alpha e(\Omega) \quad = \quad \sum_{j=1}^m a_j + \int_{\partial\alpha} s^*(\Phi e(\omega)),$$

where $\{p_1, \ldots, p_m\}$ are the zeros of $\sigma|_\alpha$, with a_j the index of the zero of σ at p_j, and $s := \sigma/\|\sigma\|$.

Proof. For an oriented, rank-$2k$ vector bundle ξ over M, and for ω a metric-compatible connection on ξ, the naturally-associated intermediate bundle B is of course the unit $(2k-1)$-sphere bundle $S(\xi)$ of ξ. Within $S(\xi)$, $d\Phi e(\omega) = e(\Omega)$, since $e(\Psi) = 0$, Ψ lying in $so(2k-1)$.

Since σ is generic, we can assume that the zero-section of σ will have intersection with α a finite collection of points $\{p_1, \ldots, p_m\}$ in the interiors of $2k$-simplices of α, with nonzero, finite-degree singularities. Then, for any $\epsilon > 0$ sufficiently small, $s := \sigma/\|\sigma\|$ defines a section over $Supp(\alpha)\backslash\{B(p_1,\epsilon)\cup\cdots\cup B(p_m,\epsilon)\}$, where $B(p_1,\epsilon)$ is the ϵ-ball within the appropriate $2k$-simplex, and

$$\begin{aligned}\int_\alpha e(\Omega) \quad &= \quad \lim_{\epsilon\downarrow 0}\int_{\alpha\backslash B(p_1,\epsilon)\cup\cdots\cup B(p_m,\epsilon)} s^*(d\Phi e(\omega)) \\ &= \quad \lim_{\epsilon\downarrow 0}\sum_{j=1}^m\int_{s_*(\partial B(p_j,\epsilon))}\Phi e(\omega) + \int_{\partial\alpha} s^*(\Phi e(\omega)).\end{aligned}$$

Since each singularity of σ is of finite, nonzero degree a_j, $\lim_{\epsilon\downarrow 0} s_*(\partial B(p_j,\epsilon)) \sim a_j\pi^{-1}(p_j)$ (homologous) where $\pi : S(\xi) \to M$ is the bundle projection. Since Ω is $\pi : S(\xi) \to M$ horizontal, using the form of Ψ above in equation (1.2) and the fact that $[\phi, \phi]$ has image contained in $\mathfrak{h}$ in this case,

$$\begin{aligned}
\lim_{\epsilon\downarrow 0}\sum_{j=1}^{m}\int_{s_*(\partial B(p_j,\epsilon))}\Phi e(\omega) &= \sum_{j=1}^{m} a_j \int_{\pi^{-1}(p_j)} \Phi e(\omega) \\
&= \sum_{j=1}^{m} a_j \int_{\pi^{-1}(p_j)} \sum_{i=0}^{k-1} A_{i,k-1-i} e(\phi, [\phi,\phi]^i, \Psi^{k-1-i}) \\
&= \sum_{j=1}^{m} a_j \int_{\pi^{-1}(p_j)} \sum_{i=0}^{k-1} A_{i,k-1-i} \frac{(-1)^{k-1-i}}{2^{k-1-i}} e(\phi, [\phi,\phi]^{k-1}) \\
&= \sum_{j=1}^{m} a_j \int_{\pi^{-1}(p_j)} \sum_{i=0}^{k-1} (-1)^{k-1} \frac{(k-1)!k!}{2^{k-1}(k+i)!(k-1-i)!} e(\phi, [\phi,\phi]^{k-1}) \\
&= \sum_{j=1}^{m} a_j \int_{\pi^{-1}(p_j)} \frac{(-1)^{k-1}k}{(2k-1)2^{k-1}} e(\phi, [\phi,\phi]^{k-1}) \\
&= \sum_{j=1}^{m} a_j,
\end{aligned}$$

since the integral, being restricted to a fiber on which the behavior of ϕ is independent of M, can be normalized by applying it to the singularities of the longitudinal flow on the sphere S^{2k}. □

If $\xi = T_*(M)$ and M is a $2k$-manifold with boundary ∂M, then the usual Gauss-Bonnet-Chern theorem, with boundary, can be recovered by taking σ to be the unit normal field to $\partial M \subset M$, and of course the Poincaré-Hopf theorem.

2.2. **Chern Classes.** Since Chern classes are defined, by the splitting principle, from the Euler class [7], the situation is quite similar for Chern classes as for the Euler class. For a complex rank-k vector bundle $\xi \to M$, the transgression of the the j^{th} Chern class $c_j(\xi)$ will be naturally-defined on the Stiefel bundle $B := V_{k-j+1}(\xi)$ of $(k-j+1)$-frames on ξ, with fiber $U(k)/U(j-1)$. Within $V_{k-j+1}(\xi)$, $d\Phi c_j(\omega) = c_j(\Omega)$, since $c_j(\Psi) = 0$ for Ψ lying in $u(j-1)$.

Proposition 5. *Let ξ be a rank-k, complex vector bundle over a compact manifold M, with a smooth hermitian metric. Let ω be a metric-compatible connection on ξ. Let α be a smooth singular $2j$-chain in M, and let $(\sigma_1, \ldots, \sigma_{k-j})$ be a unitary $(k-j)$-frame of $\xi|_{Supp(\alpha)}$. Let σ be a generic section of $\xi/Span_{\mathbb{C}}\{\sigma_1, \ldots, \sigma_{k-j}\} \cong \xi^\perp$ with no zeros on $\partial\alpha$. Then*

$$\int_\alpha c_j(\Omega) = \sum_{l=1}^{m} a_l + \int_{\partial\alpha} s^*\Phi c_j(\omega),$$

where $\{p_1, \ldots, p_m\}$ are the singularities of $s := (\sigma_1, \ldots, \sigma_{k-j}, \sigma/\|\sigma\|)$ as a section of the Stiefel bundle $V_{k-j+1}(\xi)|_\alpha$, with a_l the index of the singularity of σ at p_l.

Proof. Let α be a smooth singular $2j$-cycle in M. Since $rank_{\mathbb{R}}(\xi) > 2j$, there is a unitary $(k-j)$-frame $(\sigma_1, \ldots, \sigma_{k-j})$ of $\xi|_{Supp(\alpha)}$. Let σ be a generic section of the orthogonal complement $\xi^\perp \cong \xi/Span\{\sigma_1, \ldots, \sigma_{k-j}\}$. Since σ is generic, we can assume that the zero-section of σ will have intersection with $Supp(\alpha)$ a finite collection of points $\{p_1, \ldots, p_m\}$ in the interiors of $2j$-simplices of α, with nonzero, finite-degree singularities. Then, for any $\epsilon > 0$ sufficiently small, $s := (\sigma_1, \ldots, \sigma_{k-j+1})$, with $\sigma_{k-j+1} := \sigma/\|\sigma\|$, defines a section of $V_{k-j+1}(\xi)$ over $Supp(\alpha)\backslash\{B(p_1,\epsilon)\cup\cdots\cup$

$B(p_m,\epsilon)\}$, where $B(p_1,\epsilon)$ is the ϵ-ball within the appropriate $2j$-simplex, and, since $d\Phi c_j(\omega)=c_j(\Omega)$ on $B=V_{k-j+1}(\xi)$,

$$\begin{aligned}\int_\alpha c_j(\Omega) &= \lim_{\epsilon\downarrow 0}\int_{\alpha\setminus B(p_1,\epsilon)\cup\cdots\cup B(p_m,\epsilon)} s^*(d\Phi c_j(\omega))\\ &= \lim_{\epsilon\downarrow 0}\sum_{l=1}^m\int_{s_*(\partial B(p_l,\epsilon))}\Phi c_j(\omega)+\int_{\partial\alpha}s^*\Phi c_j(\omega).\end{aligned}$$

Since each singularity of σ_{k-j+1} is of finite, nonzero degree a_l as a section of the unit sphere bundle in $\xi/Span\{\sigma_1,\ldots,\sigma_{k-j}\}$, $\lim_{\epsilon\downarrow 0}s_*(\partial B(p_l,\epsilon))=a_lS(p_l)$ where $S(p_l)$ is the $(2j-1)$-sphere in $V_{k-j+1}(\xi)|_{p_l}$ defined by fixing $\sigma_1,\ldots,\sigma_{k-j}$ at p_l, and varying $\sigma_{k-j+1}(p_l)$ among all unit vectors orthogonal to the span of $\{\sigma_1,\ldots,\sigma_{k-j}\}$. Since Ω is $\pi:V_{k-j+1}(\xi)\to M$ horizontal, using the form of Ψ above in equation (1.2),

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\sum_{l=1}^m\int_{s_*(\partial B(p_l,\epsilon))}\Phi c_j(\omega) &= \sum_{l=1}^m a_l\int_{\pi^{-1}(p_l)}\Phi c_j(\omega)\\ &= \sum_{l=1}^m a_l\int_{\pi^{-1}(p_l)}\sum_{i=0}^{j-1}A_{i,j-1-i}c_j(\phi,[\phi,\phi]^i,\Psi^{j-1-i})\\ &= \sum_{l=1}^m a_l\int_{\pi^{-1}(p_l)}\sum_{i=0}^{k-1}A_{i,j-1-i}\frac{(-1)^{j-i-1}}{2^{j-i-1}}c_j(\phi,[\phi,\phi]^i,[\phi,\phi]_{\mathfrak{h}}^{j-i-1})\\ &= \sum_{l=1}^m a_l,\end{aligned}$$

again by normalizing the integral on a test case, such as the sum $\mathcal{O}(1)\oplus\cdots\oplus\mathcal{O}(1)$ of j copies of the line bundle with $c_1=1$ on $\mathbb{CP}^n$, which has $c_j=1$, the standard generator of $H^{2j}(\mathbb{CP}^n)$. □

2.3. **Pontryagin classes.** Since the j^{th} Pontryagin class P_j of a real, rank-k vector bundle ξ is just the $2j^{th}$ Chern class of $\xi\otimes\mathbb{C}$ [7], the form $\Phi P_j(\omega)$ will be defined on the Stiefel bundle $B=V_{k-2j+1}(\xi\otimes\mathbb{C})$ of complex $(k-2j+1)$-frames of $\xi\otimes\mathbb{C}$ as the form $\Phi c_{2j}(\omega_{\mathbb{C}})$, using the natural extension of the connection ω to $F(\xi\otimes\mathbb{C})$. However, there is an interpretation of the Pontryagin classes in at least one case, P_1, which is independent of a complexification of ξ.

Let ξ be a real, oriented, rank-4 vector bundle over M. Then, ξ is a rank-one, quaternionic vector bundle over M, that is, the bundle $F(\xi)$ of oriented frames of ξ defines a bundle of (non-integrable) quaternionic structures on ξ. Each frame, that is, each quaternionic structure of ξ_x, determines 3 complex structure tensors on ξ_x, I, J, and K, with $IK=-J$. I is defined by $I(e_1)=e_2$, $I(e_3)=e_4$, and of course $I(e_2)=-e_1$and $I(e_4)=-e_3$. Similarly, K is defined at the same frame by $K(e_1)=e_3$ and $K(e_2)=e_4$, and J is defined by $J=KI$. This bundle of quaternionic structures produces 2 complementary bundles of complex structures (corresponding to I and K, to be specific). If H_1 and H_2 are the two subgroups of $SO(4)$ corresponding to the complex-linear automorphisms with complex determinant 1 with respect to I and K, respectively, each being a representation of $SU(2)$ in $SO(4)$, then the associated bundles $B_1:=F(\xi)\times_{SO(4)}SO(4)/H_1$ and $B_2:=F(\xi)\times_{SO(4)}SO(4)/H_2$ are each $\mathbb{RP}^3$-bundles over M. In general, of course, the complex-structure tensors I and K will not be well-defined on all of M, but if so, such as for the tangent bundle of a hyperkähler manifold of real dimension 4, then they would give two dual complex structures on ξ. Such complex structures would correspond to global sections of B_1 and B_2.

Now, it will not be the case that, for either bundle, the term $P_1(\Psi, \Psi) = 0$ as was the case in the previous situations. However, since

$$\begin{aligned} P_1(\Omega, \Omega) &= \frac{-1}{8\pi^2} Tr(\Omega \wedge \Omega) \\ &= \frac{1}{8\pi^2}(\Omega_{12}\Omega_{12} + \cdots + \Omega_{34}\Omega_{34}) \\ &= \frac{1}{8\pi^2} \sum_{i<j} \Omega_{ij} \wedge \Omega_{ij} \end{aligned}$$

for $\Omega \in so(4)$, and since, in this situation, the decomposition $so(4) = \mathfrak{h}_1 \oplus \mathfrak{p}_1 = \mathfrak{p}_2 \oplus \mathfrak{h}_2 = su(2) \oplus su(2)$ splits as Lie algebras rather than just as a reductive complement, then, for either B_1 or B_2 the decomposition $\omega = \phi_i + \psi_i$ of the connection satisfies $[\phi_i, \phi_i] \subset \mathfrak{p}_i$ (that is, the form takes values in $\mathfrak{p}_i$) and $[\psi_i, \psi_i] \subset \mathfrak{h}_i$. Thus, by the decomposition of $\Psi_i = \Omega_{\mathfrak{h}_i} - \frac{1}{2}[\phi_i, \phi_i]_{\mathfrak{h}_i} = \Omega_{\mathfrak{h}_i}$, so $P_1(\Psi_i, \Psi_i) = P_1(\Omega_{\mathfrak{h}_i}, \Omega_{\mathfrak{h}_i})$, denoting by Ψ_i, $i = 1, 2$, the corresponding curvature forms, for $B_1$and B_2 (both of which can be viewed as forms in $F(\xi)$ having values in $so(4)$), then

$$P_1(\Psi_1, \Psi_1) + P_1(\Psi_2, \Psi_2) = P_1(\Omega, \Omega),$$

so that while neither one of the transgressive forms has differential the Pontryagin class, their sum does,

$$d\Phi P_1(\omega_1) + d\Phi P_1(\omega_2) = P_1(\Omega),$$

where of course ω_1 and ω_2 refer to the two distinct decompositions of the bundle of frames into associated bundles (even though ω is the same in both cases).

Proposition 6. *Let ξ be a real, oriented, rank-4 vector bundle over a compact manifold M. Let B_1 and B_2 be given by $B_1 := F(\xi) \times_{SO(4)} SO(4)/H_1$ and $B_2 := F(\xi) \times_{SO(4)} SO(4)/H_2$ as above. Let α be a smooth singular 4-chain in M. Choose generic sections σ_1 of B_1 and σ_2 of B_2 with a finite set of singular points $\{p_j\}$, singular for either σ_1 or σ_2, or both, interior to 4-simplices in α, with nondegenerate singularities of indices a_{1l} and a_{2l} at p_l. Then,*

$$\int_\alpha P_1(\Omega) = \sum_{l=1}^{m} (a_{1l} + a_{2l}) + \int_{\partial\alpha} \sigma_1^*(\Phi P_1(\omega_1)) + \sigma_2^*(\Phi P_1(\omega_2)).$$

Proof. Let α be a smooth singular 4-cycle in M. B_1 and B_2 give rise to two $\mathbb{R}^4$-bundles on M, which as above will have generic sections with a discrete set of nondegenerate zeros on α, corresponding to sections σ_1 of B_1 and σ_2 of B_2 with a finite set of singular points $\{p_j\}$, singular for either σ_1 or σ_2, or both, interior to 4-simplices in α, which are limits of maps $\sigma_i : \partial B(p_j, \epsilon) \to \mathbb{RP}^3$ of finite degree when lifted to $\widetilde{\sigma}_i : \widetilde{\partial B(p_j, \epsilon)} \to S^3$, $i = 1, 2$, that is, for $\epsilon > 0$ sufficiently small, $\left(\widetilde{\sigma}_i|_{\partial B(p_j,\epsilon)}\right)_* : H_3(\partial B(p_j, \epsilon)) \to H_3(\widetilde{\pi^{-1}(p_j)})$ given by $[\partial B(p_j, \epsilon)] \mapsto a_{i,j}[\widetilde{\pi^{-1}(p_j)}]$, $a_{ij} \in \mathbb{Z}$. As above, with the projections $\pi_1 : B_1 \to M$ and $\pi_2 : B_2 \to M$,

$$\begin{aligned}
\int_\alpha P_1(\Omega) &= \lim_{\epsilon\downarrow 0} \int_{\alpha\setminus B(p_1,\epsilon)\cup\cdots\cup B(p_m,\epsilon)} \sigma_1^*(d\Phi P_1(\omega_1)) + \sigma_2^*(d\Phi P_1(\omega_2)) \\
&= \lim_{\epsilon\downarrow 0} \sum_{l=1}^m \int_{(\sigma_1)_*(\partial B(p_l,\epsilon))} \Phi P_1(\omega_1) + \int_{(\sigma_2)_*(\partial B(p_l,\epsilon))} \Phi P_1(\omega_2) \\
&\quad + \int_{\partial\alpha} \sigma_1^*(\Phi P_1(\omega_1)) + \sigma_2^*(\Phi P_1(\omega_2)) \\
&= \sum_{l=1}^m a_{1l} \int_{\pi_1^{-1}(p_l)} \Phi P_1(\omega_1) + a_{2l} \int_{\pi_2^{-1}(p_l)} \Phi P_1(\omega_2) \\
&\quad + \int_{\partial\alpha} \sigma_1^*(\Phi P_1(\omega_1)) + \sigma_2^*(\Phi P_1(\omega_2)) \\
&= \sum_{l=1}^m a_{1l} \int_{\pi_1^{-1}(p_l)} \sum_{i=0}^1 A_{i,1-i} P_1(\phi_1, [\phi_1,\phi_1]^i, \Psi_1^{1-i}) + \\
&\quad + a_{2l} \int_{\pi_2^{-1}(p_l)} \sum_{i=0}^1 A_{i,1-i} P_1(\phi_2, [\phi_2,\phi_2]^i, \Psi_2^{1-i}) \\
&\quad + \int_{\partial\alpha} \sigma_1^*(\Phi P_1(\omega_1)) + \sigma_2^*(\Phi P_1(\omega_2)) \\
&= \sum_{l=1}^m a_{1l} \int_{\pi_1^{-1}(p_l)} A_{0,1} P_1(\phi_1, [\phi_1,\phi_1]) + a_{2l} \int_{\pi_2^{-1}(p_l)} A_{0,1} P_1(\phi_2, [\phi_2,\phi_2]) \\
&\quad + \sum_{l=1}^m a_{1l} \int_{\pi_1^{-1}(p_l)} A_{1,0} P_1(\phi_1, \Omega_{\mathfrak{h}_1}) + a_{2l} \int_{\pi_2^{-1}(p_l)} A_{1,0} P_1(\phi_1, \Omega_{\mathfrak{h}_2}) \\
&\quad + \int_{\partial\alpha} \sigma_1^*(\Phi P_1(\omega_1)) + \sigma_2^*(\Phi P_1(\omega_2)) \\
&= \sum_{l=1}^m a_{1l} \int_{\pi_1^{-1}(p_l)} P_1(\phi_1, [\phi_1,\phi_1]) + a_{2l} \int_{\pi_2^{-1}(p_l)} P_1(\phi_2, [\phi_2,\phi_2]) \\
&\quad + 0 \\
&\quad + \int_{\partial\alpha} \sigma_1^*(\Phi P_1(\omega_1)) + \sigma_2^*(\Phi P_1(\omega_2)),
\end{aligned}$$

since the integration is over $\pi_i^{-1}(p_l)$, and Ω is π_i-horizontal.

Since the form $P_1(\omega, [\omega,\omega])$ has integral periods and generates the transgressive first Pontryagin form of $H^3(SO(4),\mathbb{R}) = H^3(SU(2),\mathbb{R}) \oplus H^3(SU(2),\mathbb{R})$, the projection $SO(4) \to SO(4)/H_1$ pulls the generator of $H^3(SO(4)/H_1,\mathbb{R})$, which is $P_1(\phi_1,[\phi_1,\phi_1])$, back to $P_1(\omega,[\omega,\omega])$, and so

$$\int_{\pi_1^{-1}(p_l)} P_1(\phi_1, [\phi_1,\phi_1]) = 1;$$

similarly with the other projection as well. Thus

$$\int_\alpha P_1(\Omega) = \sum_{l=1}^m a_{1l} + a_{2l} + \int_{\partial\alpha} \sigma_1^*(\Phi P_1(\omega_1)) + \sigma_2^*(\Phi P_1(\omega_2)).$$

$\square$

Example 7. As an example of this decomposition, let ξ be the tangent bundle to S^4. Since $S^4 = \mathbb{HP}^1$, S^4 admits a global quaternionic structure, though it admits no global almost-complex structures. However, if p is the South pole, $p = (-1, 0, 0, 0, 0)$, on $S^4 \backslash \{p\}$, there are certainly global almost-complex structures. The standard complex structures can be described by parallel transport of a given pair of complex structures at the North pole $n := -p$, corresponding to the canonical frame, along longitudes. For these sections σ_1 of B_1 and σ_2 of B_2, there is only one singular point, for both sections, at p. The section $\sigma_1|_{x_1=-\sqrt{1-\epsilon^2}} : S^3_\epsilon \to B_1|_{S^3_\epsilon} \cong S^3_\epsilon \times \mathbb{RP}^3$, as a map $\sigma_1 S^3 \to \mathbb{RP}^3$, lifts to a map $\widetilde{\sigma_1} : S^3_\epsilon \to S^3$ of degree 2, and σ_2 similarly lifts to a map of degree -2.

Remark 8. An eventual goal of these relative classes would be to construct a combinatorial procedure to determine the Pontryagin classes of a closed manifold, since it is well-known that they are topological invariants [8]. It is possible to begin the procedure for an oriented 4-manifold M based on this result, but the details are not apparent. Given such a manifold M^4, each 4-simplex of a fixed triangulation of M admits a standard hyperkähler structure, with prescribed behavior at the boundary (given as that of the boundary of the 4-ball in quaternionic 1-space), so that the computation reduces to the 3-skeleton. The boundary terms cancel on the interiors of the 3-simplices due to reversal of orientation of the sections, so the remaining calculations should lie on the 2-skeleton.

3. Secondary characteristic classes

One of the most extensive uses which has been made with the construction of [4] has been the construction of secondary characteristic classes. If P is a polynomial of degree k so that $P(\Omega)$ is integral for all Ω, that is, if it has integral periods, then when ω is a connection for which $P(\Omega) = 0$ (as a form, not just as a cohomology class), the Chern-Simons transgression $TP(\omega)$ will be closed, generating a cohomology class in $H^{2k-1}(E, \mathbb{R})$. Of more interest is the construction, from that class, of a cohomology class in the base M modulo integral classes. In the case of a principal bundle the existence of such a class follows by passing to the universal bundle, where every cocycle is a coboundary on the total space, and so the mod-$\mathbb{Z}$ reduction of $TP(\omega)$ will be a lift of a cocycle on the base.

The forms $\Phi P(\omega)$ can, in some cases, be more directly seen to be lifts, using the obstruction information determined by the characteristic class $P(\Omega)$. Note that the method of proof used by Chern and Simons will not work in this situation, and that the forms $\Phi P(\omega)$, and so the secondary characteristic classes determined by them, are not always the same as the Chern-Simons classes.

Theorem 9. *If ξ is either a rank-$2k$ real oriented vector bundle, or a rank-k complex vector bundle, over a compact manifold M, and if, respectively, the form $e(\Omega) = 0$ (resp., $c_j(\Omega) = 0$ for some j), then the corresponding form $\Phi e(\omega)$ (resp., $\Phi c_j(\omega)$) is well-defined as an element of $H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$ (resp., $H^{2j-1}(M, \mathbb{R}/\mathbb{Z})$).*

Proof. For the Euler class of an oriented rank-$2k$ vector bundle, where B is the sphere bundle: if $e(\Omega) = 0$ as a form, of course $\Phi e(\omega)$ will be closed, and for any section σ over the $2k$-skeleton, $\sigma^*(\Phi e(\omega))$ will be a closed form on M which lifts to $\Phi e(\omega)$, defining a secondary characteristic class on M modulo the choice of section σ.

The Gysin sequence of the $(2k-1)$-sphere bundle $\pi : B \to M$,

$$\cdots \xrightarrow{\pi_*} H^r(M, R) \xrightarrow{e\wedge} H^{r+2k}(M, R) \xrightarrow{\pi^*} H^{r+2k}(B, R) \xrightarrow{\pi_*} H^{r+1}(M, R) \xrightarrow{e\wedge}\ ,$$

for $r = -1$ yields the split short exact sequence (for any section σ)

$$0 \to H^{2k-1}(M,R) \underset{\sigma^*}{\overset{\pi^*}{\rightleftarrows}} H^{2k-1}(B,R) \xrightarrow{\pi_*} R.$$

The map π_* is integration over the fiber [2, p. 178], so in the case that $R = \mathbb{R}$ the image $\sigma^*(\Phi e(\omega))$ is well-defined modulo $\pi_*(\Phi e(\omega)) = [f]$, where $f(x) = \int_{\pi^{-1}(x)} \Phi e(\omega)$ is an integer-valued (hence constant, since it is continuous) function by the fact that the integrand over each fiber is integral, since $\Phi e(\omega)|_{ker(\pi_*)}$ is the (normalized) volume form on the fibers. Then, with the coefficient ring $R = \mathbb{R}/\mathbb{Z}$, $\Phi e(\omega) \in ker(\pi_*)$, thus there is a unique $U \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$ so that $\pi^*(U) = \Phi e(\omega)$, and since $\pi\sigma = 1$, $U = \sigma^*\pi^*(U) = \sigma^*(\Phi e(\omega))$, independent of choice of σ.

In the case of a rank-k complex vector bundle, if the form $c_j(\Omega) = 0$ then, as for the Euler class, there will be a section $\sigma : M^{(2j)} \to V_{k-j+1}(\xi)$ of the Stiefel bundle of unitary $(k-j+1)$-frames of ξ over the $2j$-skeleton of M. The Stiefel bundle splits as a tower of sphere bundles

$$V_{k-j+1}(\xi) \to \cdots \to V_2(\xi) \cong S(\pi^*(v^\perp)) \to S(\xi) \to M,$$

where $V_2(\xi)$ is the sphere bundle of the orthogonal complement bundle $v^\perp \to S(\xi)$ with fiber over $v \in S(\xi)$ the orthogonal complement of $\{v, iv\}$ in the fiber $\xi_{\pi(v)}$. The fiber at each stage is $S^{2k-2l-1}$, $l = 0, \dots, k-j$. Applying the Gysin sequence at each stage, with $r = 2(j-k)-1$ in the first stage through $r = -1$ at the last, gives

$$H^{2j-1}(M,R) \cong H^{2j-1}(S(\xi),R) \cong \cdots \cong H^{2j-1}(V_{k-j}(\xi),R),$$

and a split exact sequence (using $H^{2j-1}(V_{k-j}(\xi),R) \cong H^{2j-1}(M,R)$) for any section σ over the $2j$-skeleton of M:

$$0 \to H^{2j-1}(M,R) \underset{\sigma^*}{\overset{\pi^*}{\rightleftarrows}} H^{2j-1}(V_{k-j+1}(\xi),R) \xrightarrow{\pi_*} R.$$

The proof then proceeds as in the first case, noting that π_* can still be viewed as integration over the fiber, but over the fiber of the S^{2j-1}-bundle $V_{k-j+1}(\xi) \to V_{k-j}(\xi)$. □

3.1. **Invariants of odd-dimensional manifolds.** For a compact 3-manifold M, one of the more intriguing results in [4] is the equivalence of the existence of a critical connection (or metric) for the functional $\omega \mapsto \int_M \frac{1}{2} TP_1(\omega)$ as an $\mathbb{R}/\mathbb{Z}$-valued form on the space of Riemannian metrics on M, to the Poincaré conjecture. While we do not draw such connections for the forms $\Phi P_1(\omega)$ (which, in particular, may not be conformal invariants, a key property used by Chern and Simons), we can show that, as a secondary characteristic class $\Phi P_k(\omega)$ on the sphere bundle of the tangent bundle of a compact $(4k-1)$-manifold M will define a cohomology class in $H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$.

For a compact, oriented $(4k-1)$-manifold M, since the Euler class vanishes there will always be a section $\sigma : M \to T_1(M)$ of the tangent sphere bundle. By simple dimension, rather than any assumptions on the characteristic forms or Theorem (2), the pullback $\sigma^*(\Phi P_k(\omega))$ from the sphere bundle will satisfy $d\sigma^*(\Phi P_k(\omega)) = 0$, so will define a cohomology class depending upon the section σ. Since $d\Phi P_k(\omega) = -P_k(\Psi)$ in the sphere bundle, then the mod-$\mathbb{Z}$ reduction $\delta\widetilde{\Phi P_k}(\omega) = 0$ because the differential has integral periods. Apply the Gysin sequence again with $\mathbb{R}/\mathbb{Z}$ coefficients:

$$\cdots \xrightarrow{\pi_*} H^r(M,\mathbb{R}/\mathbb{Z}) \xrightarrow{e\wedge} H^{r+4k-1}(M,\mathbb{R}/\mathbb{Z}) \underset{\sigma^*}{\overset{\pi^*}{\rightleftarrows}} H^{r+4k-1}(B,\mathbb{R}/\mathbb{Z}) \xrightarrow{\pi_*} H^{r+1}(M,\mathbb{R}/\mathbb{Z}) \xrightarrow{e\wedge} .$$

In this case the Euler class will be 0, leading to, for $r = 0$, the split short exact sequence

$$0 \to H^{4k-1}(M, \mathbb{R}/\mathbb{Z}) \underset{\sigma^*}{\overset{\pi^*}{\substack{\rightarrow \\ \leftarrow}}} H^{4k-1}(B, \mathbb{R}/\mathbb{Z}) \overset{\pi_*}{\rightarrow} H^1(M, \mathbb{R}/\mathbb{Z}) \to 0 \ ,$$

indicating that, in particular, the class $\sigma^*(\widetilde{\Phi P_k(\omega)})$ will be independent of the section σ for a simply-connected manifold M, so that, *a priori*, for these manifolds there is a $H^{4k-1}(M, \mathbb{R}/\mathbb{Z})$-secondary class. More generally, again since the form $\Phi P_k(\omega)|_{ker(\pi_*)}$ restricts to (an integral multiple of) the volume form on the fibers $\pi^{-1}(x)$ for $x \in M$, then $\widetilde{\Phi P_k(\omega)}$ will lie in the kernel of the map π_*, which is integration over the fiber (in $\mathbb{R}/\mathbb{Z}$ coefficients). By the short exact sequence, that implies that $\widetilde{\Phi P_k(\omega)}$ is in the image of the pullback map π^*. However, any σ^* will be a left inverse of π^*, so on this subgroup (the image of π^*), σ^* will indeed be the inverse of π^*, so that $\sigma^*(\widetilde{\Phi P_k(\omega)})$ will be independent of the choice of σ. This establishes the following fact.

Proposition 10. *If M^{4k-1} is a compact, oriented manifold, with $B \to M$ the unit tangent bundle of M, and if ω is the Riemannian connection of a given Riemannian metric on M, then the forms $\sigma^*(\widetilde{\Phi P_k(\omega)})$ are well-defined as secondary characteristic classes in $H^{4k-1}(M, \mathbb{R}/\mathbb{Z})$, depending only upon the metric.*

3.2. **Examples.** As a simple example of the kind of information that can be measured by these secondary characteristic classes, and how they differ from the original classes of Chern and Simons, consider a compact 3-manifold M. Since the tangent bundle is trivial, both the Chern-Simons classes $TP_1(\omega)$ and the forms $\Phi P_1(\omega)$ determine secondary characteristic classes in $H^3(M, \mathbb{R}/\mathbb{Z})$. Start with the trivial (flat) connection ω_0 on $M \times SO(3)$, which is simply the Maurer-Cartan form on the fibers, $\omega_0|_{(x,g)} = g^{-1}dg$, where $g : SO(3) \to M(3 \times 3, \mathbb{R})$ is essentially the identity map recognizing $g \in SO(3)$ as a matrix. In this notation, any connection ω on $M \times SO(3)$ can be given as $\omega = \omega_0 + g^{-1}\pi^*(\alpha)g$, where $\alpha \in E^1(M, o(3))$ is arbitrary. It is a simple computation to show that the curvature Ω is given by $\Omega = g^{-1}\pi^*(d\alpha + \alpha \wedge \alpha)\, g$. If $o(3) = \mathfrak{p} \oplus \mathfrak{h}$ is the standard decomposition of $o(3)$ corresponding to the projection $SO(3) \to S^2$, with $\mathfrak{h} = I \oplus SO(2) \subset SO(3)$, then the connection decomposes as above into $\omega = \phi + \psi$ where $\phi = g^{-1}\pi_{\mathfrak{p}}(\pi^*(\alpha))\, g$ and $\psi = g^{-1}\pi_{\mathfrak{h}}(\pi^*(\alpha))\, g$. Since the polarized first Pontryagin polynomial is $P_1(A, B) = \frac{-1}{8\pi^2}Tr(AB)$, $P_1(\mathfrak{p}, \mathfrak{h}) = 0$, and so, for $\sigma : M \to M \times SO(3)$ the obvious map $x \mapsto (x, e)$,

$$\begin{aligned}
\sigma^*(\Phi P_1(\omega)) &= \sigma^*\left(A_{00}P_1(\phi, \Omega) + A_{01}P_1(\phi, \Psi) + A_{10}P_1(\phi, [\phi, \phi])\right) \\
&= \frac{-1}{8\pi^2}\left(Tr(\pi_{\mathfrak{p}}\alpha \wedge (d\alpha + \alpha \wedge \alpha)) + Tr(\pi_{\mathfrak{p}}\alpha \wedge \Psi) - \frac{1}{6}Tr(\pi_{\mathfrak{p}}\alpha \wedge [\pi_{\mathfrak{p}}\alpha, \pi_{\mathfrak{p}}\alpha])\right) \\
&= \frac{-1}{8\pi^2}\left(Tr(\pi_{\mathfrak{p}}\alpha \wedge (d\alpha + \alpha \wedge \alpha)) + 0 - \frac{1}{6}Tr(\pi_{\mathfrak{p}}\alpha \wedge [\pi_{\mathfrak{p}}\alpha, \pi_{\mathfrak{p}}\alpha])\right).
\end{aligned}$$

In addition, since $\pi_{\mathfrak{p}}(\alpha)$ is a 1-form with values in an $\mathbb{R}^2$, the last term will vanish. Writing

$$\alpha = \begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & b \\ -a_2 & -b & 0 \end{bmatrix},$$

we have

$$\pi_{\mathfrak{p}}\alpha = \begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & 0 \\ -a_2 & 0 & 0 \end{bmatrix},$$

and so a direct computation yields that

$$\sigma^*(\Phi P_1(\omega)) = \frac{1}{4\pi^2}\left(a_1 \wedge da_1 + a_2 \wedge da_2 - 2a_1 \wedge a_2 \wedge b\right).$$

On the other hand, the Chern-Simons form of this same connection will be

$$\begin{aligned}
\sigma^*(TP_1(\omega)) &= \sigma^*\left(A_0 P_1(\omega,\Omega) + A_1 P_1(\omega,[\omega,\omega])\right)\\
&= \frac{-1}{8\pi^2}\left(Tr(\alpha\wedge(d\alpha+\alpha\wedge\alpha)) - \frac{1}{6}Tr(\alpha\wedge[\alpha,\alpha])\right)\\
&= \frac{-1}{8\pi^2}\left(Tr(\alpha\wedge d\alpha) + \frac{2}{3}Tr(\alpha\wedge(\alpha\wedge\alpha))\right)\\
&= \frac{1}{4\pi^2}\left(a_1\wedge da_1 + a_2\wedge da_2 + b\wedge db - 2a_1\wedge a_2\wedge b\right).
\end{aligned}$$

These can clearly be seen to differ modulo forms of integral periods, if, for example, b is non-closed with $\widetilde{b \wedge db/4\pi^2} \neq 0$ in $H^3(M,\mathbb{R}/\mathbb{Z})$.

3.3. **Conformal metrics.** If $g_t = e^{2t\lambda}g_0$ is a conformal family of metrics on an arbitrary n-manifold M, for some $\lambda : M \to \mathbb{R}$, then the Riemannian connections ω_t of the family of metrics, as a connection form on the orthonormal frame bundle $F(M)$, varies according to the following Lemma. Note that this differs somewhat from Chern and Simons' formula [4], but their version is as a form on the full bundle of bases, relative to a fixed local basis, whereas this version uses a family of moving frames $\{\mathbf{e}_i(t)\} = \{e^{-t\lambda}\mathbf{e}_i\}$.

Lemma 11. *Let M be an arbitrary Riemannian manifold, with metric $g = <,>$, and let $g_t := e^{2t\lambda}g$ be any conformal family of metrics. Then, if $\omega_i^j(t)$ is the Riemannian connection one-form of the family of metrics g_t, relative to the moving frame $\{\mathbf{e}_i(t)\} = \{e^{-t\lambda}\mathbf{e}_i\}$, we have*

$$\omega_i^j(t) = \omega_i^j + t\mathbf{e}_i(\pi^*\lambda)\theta^j - t\mathbf{e}_j(\pi^*\lambda)\theta^i,$$

where $\pi : F(M) \to M$ is the bundle projection and $\{\theta^j\}$ are the solder forms.

Proof. By a direct computation, identifying λ on M with the pull-back $\pi^*(\lambda)$ on $F(M)$, and X with an arbitrary lift of X under π_*,

$$\begin{aligned}
\omega_i^j(t)(X) &= \langle\nabla(t)_X\mathbf{e}_i(t),\mathbf{e}_j(t)\rangle_t\\
&= \frac{1}{2}\big(X\langle\mathbf{e}_i(t),\mathbf{e}_j(t)\rangle_t + \mathbf{e}_i(t)\langle X,\mathbf{e}_j(t)\rangle_t - \mathbf{e}_j(t)\langle X,\mathbf{e}_i(t)\rangle_t\\
&\quad + \langle[X,\mathbf{e}_i(t)],\mathbf{e}_j(t)\rangle_t - \langle[X,\mathbf{e}_j(t)],\mathbf{e}_i(t)\rangle_t - \langle[\mathbf{e}_i(t),\mathbf{e}_j(t)],X\rangle_t\big)\\
&= \frac{1}{2}\left(X\langle\mathbf{e}_i,\mathbf{e}_j\rangle + t\mathbf{e}_i(\lambda)\langle X,\mathbf{e}_j\rangle + \mathbf{e}_i(\langle X,\mathbf{e}_j\rangle) - t\mathbf{e}_j(\lambda)\langle X,\mathbf{e}_i\rangle - \mathbf{e}_j(\langle X,\mathbf{e}_i\rangle)\right.\\
&\quad -tX(\lambda)(\langle\mathbf{e}_i,\mathbf{e}_j\rangle) + \langle[X,\mathbf{e}_i],\mathbf{e}_j\rangle + tX(\lambda)\langle[X,\mathbf{e}_j],\mathbf{e}_i\rangle - \langle[X,\mathbf{e}_j],\mathbf{e}_i\rangle\\
&\quad \left.+t\mathbf{e}_i(\lambda)\langle\mathbf{e}_j,X\rangle - t\mathbf{e}_j(\lambda)\langle\mathbf{e}_i,X\rangle - \langle[\mathbf{e}_i,\mathbf{e}_j],X\rangle\right)\\
&= \omega_i^j(X) + t\mathbf{e}_i(\lambda)\theta^j(X) - t\mathbf{e}_j(\lambda)\theta^i(X).
\end{aligned}$$

The result follows. □

Now consider a more specific situation, where M is the 3-manifold $S^1\times S^1\times S^1$ with g being the flat metric. If now $\Phi P_1(\omega(t))$ is the corresponding family of transgressive forms on the unit sphere bundle, from the previous subsection and Lemma (11) we have that $\omega(t) = \omega_0 + g^{-1}\pi^*(\alpha)g$ where, with respect to the standard frame $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ being the unit tangent fields to the factors, with coordinates $\{x, y, z\}$ and $\lambda = \lambda(x, y, z)$ being any triply-periodic function with periods 2π in each variable,

$$\alpha = \begin{bmatrix} 0 & t\frac{\partial\lambda}{\partial x}dy - t\frac{\partial\lambda}{\partial y}dx & t\frac{\partial\lambda}{\partial x}dz - t\frac{\partial\lambda}{\partial z}dx \\ -t\frac{\partial\lambda}{\partial x}dy + t\frac{\partial\lambda}{\partial y}dx & 0 & t\frac{\partial\lambda}{\partial y}dz - t\frac{\partial\lambda}{\partial z}dy \\ -t\frac{\partial\lambda}{\partial x}dz + t\frac{\partial\lambda}{\partial z}dx & -t\frac{\partial\lambda}{\partial y}dz + t\frac{\partial\lambda}{\partial z}dy & 0 \end{bmatrix},$$

so that

$$\begin{aligned}
\sigma^*(\Phi P_1(\omega)) &= \frac{1}{4\pi^2}\left(t^2\left(\frac{\partial\lambda}{\partial x}dy - \frac{\partial\lambda}{\partial y}dx\right)\wedge d\left(\frac{\partial\lambda}{\partial x}dy - \frac{\partial\lambda}{\partial y}dx\right)\right. \\
&\quad + t^2\left(\frac{\partial\lambda}{\partial x}dz - \frac{\partial\lambda}{\partial z}dx\right)\wedge d\left(\frac{\partial\lambda}{\partial x}dz - \frac{\partial\lambda}{\partial z}dx\right) \\
&\quad \left. -2t^3\left(\frac{\partial\lambda}{\partial x}dy - \frac{\partial\lambda}{\partial y}dx\right)\wedge\left(\frac{\partial\lambda}{\partial x}dz - \frac{\partial\lambda}{\partial z}dx\right)\wedge\left(\frac{\partial\lambda}{\partial y}dz - \frac{\partial\lambda}{\partial z}dy\right)\right) \\
&= \frac{1}{4\pi^2}\left(t^2\left(\frac{\partial\lambda}{\partial x}dy - \frac{\partial\lambda}{\partial y}dx\right)\wedge\left(\frac{\partial^2\lambda}{\partial x^2}dx\wedge dy - \frac{\partial^2\lambda}{\partial x\partial z}dy\wedge dz + \frac{\partial^2\lambda}{\partial y^2}dx\wedge dy + \frac{\partial^2\lambda}{\partial y\partial z}dx\wedge dz\right)\right. \\
&\quad + t^2\left(\frac{\partial\lambda}{\partial x}dz - \frac{\partial\lambda}{\partial z}dx\right)\wedge\left(\frac{\partial^2\lambda}{\partial x^2}dx\wedge dz + \frac{\partial^2\lambda}{\partial x\partial y}dy\wedge dz + \frac{\partial^2\lambda}{\partial z^2}dx\wedge dz + \frac{\partial^2\lambda}{\partial z\partial y}dx\wedge dy\right) \\
&\quad \left. -2t^3\left(\frac{\partial\lambda}{\partial x}dy - \frac{\partial\lambda}{\partial y}dx\right)\wedge\left(\frac{\partial\lambda}{\partial x}dz - \frac{\partial\lambda}{\partial z}dx\right)\wedge\left(\frac{\partial\lambda}{\partial y}dz - \frac{\partial\lambda}{\partial z}dy\right)\right) \\
&= \frac{t^2}{4\pi^2}\left(\frac{\partial\lambda}{\partial y}\frac{\partial^2\lambda}{\partial x\partial z} - \frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial y\partial z} - \frac{\partial\lambda}{\partial z}\frac{\partial^2\lambda}{\partial x\partial y} + \frac{\partial\lambda}{\partial x}\frac{\partial^2\lambda}{\partial z\partial y} - 2t\left(\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial z}\frac{\partial\lambda}{\partial y} - \frac{\partial\lambda}{\partial y}\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial z}\right)\right)dx\wedge dy\wedge \\
&= \frac{t^2}{4\pi^2}\left(\frac{\partial\lambda}{\partial y}\frac{\partial^2\lambda}{\partial x\partial z} - \frac{\partial\lambda}{\partial z}\frac{\partial^2\lambda}{\partial x\partial y}\right)dx\wedge dy\wedge dz,
\end{aligned}$$

and

$$\begin{aligned}
\int_M \sigma^*(\Phi P_1(\omega)) &= \int_M \frac{t^2}{4\pi^2}\left(\frac{\partial\lambda}{\partial y}\frac{\partial^2\lambda}{\partial x\partial z} - \frac{\partial\lambda}{\partial z}\frac{\partial^2\lambda}{\partial x\partial y}\right)dxdydz \\
&= \int_M \frac{t^2}{2\pi^2}\left(\frac{\partial\lambda}{\partial y}\frac{\partial^2\lambda}{\partial x\partial z}\right)dxdydz \\
&= 0,
\end{aligned}$$

which follows from writing λ in terms of its Fourier series, $\lambda = \sum_{n,m,p} a_{n,m,p}e^{i(mx+ny+pz)}$.

This leads to the following conclusion:

Proposition 12. *If g is a conformally-flat metric on a compact 3-manifold M, then $\sigma^*(\widetilde{\Phi P_1(\omega)}) = 0$.*

Remark 13. It can similarly be shown, by a direct calculation, that $\frac{d}{dt}\big|_0 \sigma^*(\Phi\widetilde{P_1(\omega(t))})$ will be exact, if $\omega(0)$ is the standard Riemannian connection on the round 3-sphere, so that at least infinitesimally $\sigma^*(\widetilde{\Phi P_1}(\omega))$ will be conformally invariant there as well.

References

[1] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Annals of Mathematics, **57** (1953), 115-207.

[2] Raul Bott and Loring W. Tu, *Differential forms in algebraic topology*, Springer GTM # 82, 1982.

[3] Shiing-Shen Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Annals of Mathematics, **45** (1944), 747-752.

[4] Shiing-Shen Chern and James Simons, *Characteristic forms and geometric invariants*, Annals of Mathematics, **99** (1974), 48-69.

[5] Michael Hutchings, The Gauss-Bonnet-Chern theorem, undergraduate thesis, Harvard University, 1993.
[6] Sheng Li and Yishi Duan, Decomposition theory of spin connection and topological structure of Gauss-Bonnet-Chern theorem on manifold with boundary, math-ph/9903020, 1999.
[7] J. Milnor and J. Stasheff, *Characteristic classes*, Annals of Math Studies #76, Princeton, 1974.
[8] S. Novikov, Topological invariance of rational Pontryagin classes, Dokl. Akad. Nauk SSSR, **163**, N2 (1965), 298-300.
[9] M. Spivak, *A comprehensive introduction to differential geometry*, v. 5, Publish or Perish, 1975.
[10] C. Vafa and E. Witten, *A one-loop test of string duality*, hep-th/9512078, 1995.

Current address: Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015-3174

E-mail address: `david.johnson@lehigh.edu`