VOLUME-MINIMIZING FOLIATIONS ON SPHERES

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ABSTRACT. The volume of a k-dimensional foliation $\mathcal{F}$ in a Riemannian manifold M^n is defined as the mass of image of the Gauss map, which is a map from M to the Grassmann bundle of k-planes in the tangent bundle. Generalizing the construction by Gluck and Ziller in [4], "singular" foliations by 3-spheres are constructed on round spheres S^{4n+3}, as well as a singular foliation by 7-spheres on S^{15}, which minimize volume within their respective relative homology classes. These singular examples provide lower bounds for volumes of regular 3-dimensional foliations of S^{4n+3} and regular 7-dimensional foliations of S^{15}.

0. INTRODUCTION

In [4], Herman Gluck and Wolfgang Ziller asked which foliations were "best-organized", in that an energy functional they called the *volume* was minimized. The *volume* of a foliation is the mass of the image of the Gauss map, which in the case of a one-dimensional foliation is the mass of the unit tangent flow field in $T_1(M)$.

They were able to show that the standard one-dimensional foliation (or *flow*, in their terminology) of S^3 by the fibers of the Hopf fibration $S^3 \to S^2$ minimized volume among all foliations of the round S^3. Their method of proof, involving calibrations, did not generalize, however.

It is not the case that even the most obvious generalization of Gluck and Ziller's example to higher dimensions, the Hopf fibration $S^5 \to \mathbb{CP}^2$, is volume-minimizing [5]. Sharon Pedersen showed in her thesis that there was a foliation of S^5 with much less volume than the Hopf fibration, although her example is singular [8]. It may well be that the volume-minimizing one-dimensional foliations on S^5 is be singular, although it is not clear whether Pedersen's example is that minimizer. Gluck and Ziller did describe a "singular foliation" on S^{2n+1} that minimizes the volume functional, but their singular minimum is of a different sort than Pedersen's. Pedersen's foliation is a smooth foliation on all but one point in S^5, and is a limit of smooth foliations, while Gluck and Ziller's example is not homologous to a foliation except on S^3.

There is, then, something peculiar about the Hopf fibration on S^3 which enables the calibration argument that Gluck and Ziller used to show the minimization of the volume of that foliation, beyond the evident geometric properties for the Hopf fibrations in general.

In this article we expand the method used by Gluck and Ziller to 3-dimensional foliations of S^{4n+3} and 7-dimensional foliations of S^{15}. What we find is that the generic situation Gluck and Ziller described for flows on S^{2n+1} holds; that is, there are singular foliations which minimize volume in these cases, but that it does not appear that the Hopf fibrations will minimize volume.

1. DEFINITIONS AND THE MINIMIZATION QUESTION

The original question considered by Gluck and Ziller in [4], extended by a number of authors, is to find the dimension-k foliation $\mathcal{F}$ on a compact Riemannian manifold M, considered as a section

$$\sigma_{\mathcal{F}} : M \to G_o(k, M)$$

of the bundle of oriented k-planes tangent to M, which is "most efficient" or "best-organized" in that its *volume* is minimized, where the *volume* is defined as the Hausdorff n-dimensional measure of the image $\sigma_{\mathcal{F}}(M) \subset G_o(k, M)$, where the Grassmann bundle has a natural Sasaki metric induced from the original metric on M. Volume-minimization should be considered within each homology class of foliations, and it is possible for one homology class to admit a smooth minimizer, but for others to have no smooth minimizer.

Remark 1.1. It may seem more appropriate to consider homotopy classes of such foliations rather than homology classes, but a simple construction shows that two homotopy classes of one-dimensional foliations on S^3 can be constructed (within one homology class, of course), one of which has a smooth volume-minimizer, but the other does not, since there is a sequence within the one homotopy class whose volume converges to the minimum of the other class. Since the only foliations achieving that minimum are within the first homotopy class (see, for example, [4]), there can be no smooth minimizer within the first.

As mentioned in the Introduction, Gluck and Ziller showed that the natural candidate, the fibers of the Hopf fibration from S^3 to S^2, is volume-minimizing among all (smooth) one-dimensional foliations on the (round) 3-sphere.

Several authors [5, 8] showed that this natural candidate volume-minimizer did not extend even to the next simplest case of the fibers of the Hopf fibration $S^5 \to \mathbb{CP}^2$. Pedersen's example, in particular, is singular in the sense that there is one point of S^5 which must be removed in order for her example to be a smooth foliation. It is the case, however, that Pedersen's example is the limit of smooth foliations (it is the limit of the sequence of geodesic flows stretching away from one pole towards the other, applied to any smooth one-dimensional foliation).

Because of Pedersen's example, it seems necessary to consider singular foliations in general.

Definition 1.2. An *oriented singular k-dimensional distribution* on a manifold M is defined as an n-dimensional rectifiable current $D \subset G_o(k, M)$ of the bundle of k-dimensional subspaces of $T_*(M)$, so that on an open dense subset $U \subset M$, $D|_{\pi^{-1}(U)}$ is a smooth, k-dimensional distribution on U, that is, a smooth cross-section of $G(k, U) \to U$ (resp., $G_o(k, U) \to U$). The distribution D is *integrable*, or is a *singular foliation*, if $D|_{\pi^{-1}(U)}$ is integrable.

As an example, any unit vector field on a manifold M with finitely many singularities, each with finite index, is an oriented singular foliation in this sense. Note that these currents need not be cycles, in general; for example in the case of a unit vector field with some point singularity of odd degree.

This notion of a singular foliation is similar to, but more general than, that studied by the second-named author and Smith in [6]. In that article, the singular sections of arbitrary vector bundles that are considered are those in the weak closure of the space of smooth sections. Many of the singular foliations considered here are not in the closure of the space of smooth sections, by topological considerations.

2. 1-DIMENSIONAL SINGULAR FOLIATIONS OF S^{2n+1}

2.1. The calibration. The bundle of oriented 1-planes tangent to S^{2n+1}, the unit tangent bundle $T_1(S^{2n+1})$, is isomorphic to the flag manifold of oriented lines in oriented 2-planes in $\mathbb{R}^{2n+2}$, which is the Stieffel manifold of 2-frames in $\mathbb{R}^{2n+2}$.

This gives rise to the following diagram:

$$
\begin{array}{ccc}
T_1(S^{2n+1}) & \longrightarrow & F_o(1,2,\mathbb{R}^{2n+2}) \\
h\uparrow\pi\downarrow & & \pi\downarrow \\
S^7 & & G_o(2,\mathbb{R}^{2n+2})
\end{array}.
$$

$G_o(2, 2n+2)$ has two universal bundles, the universal 2-plane bundle $U(2, 2n+2)$ and the dual $2n$-plane bundle $V(2n, 2n+2)$, defined by

$$
\begin{aligned}
U(2, 2n+2) & := \cup_{x \in G_o(2,2n+2)} x \\
V(2n, 2n+2) & := \cup_{x \in G_o(2,2n+2)} x^{\perp}.
\end{aligned}
$$

The respective Euler classes $E(U)$ and $E(V)$ satisfy $E(U) \cup E(V) = 0$ in $H^{2n}(G_o(2, 2n+2))$, since $U \oplus V$ is trivial. In particular, if ω is the universal connection on $U(2, 2n+2)$ defined by Narasimhan and Ramanan (cf. [7]), and ω^* is the "dual" connection on $V(2n, 2n+2)$, then the associated Euler forms, $e(\Omega)$ and $e(\Omega^*)$, satisfy $e(\Omega) \wedge e(\Omega^*) = 0$. Consider the form

$$
\Phi := C\, Te(\omega) \wedge e(\Omega^*),
$$

which is well-defined on $F_o(1, 2, \mathbb{R}^{2n+2})$ since that is the frame bundle $FU(2, 2n+2)$ of oriented orthonormal frames on $U(2, 2n+2)$, which is an $SO(2)$-principal bundle. Here, $Te(\omega)$ is the transgressive Chern-Simons form corresponding to the Euler form $e(\Omega)$ of $U(2, 2n+2)$ [3]. Because $d(Te(\omega)) = e(\Omega)$, we have that $d\Phi = 0$. The constant C is simply chosen so that the comass of Φ is one. This is the same calibration defined in [4].

2.2. Calculations. We will consider $G_o(2, 2n+2)$ as $SO(2n+2)/SO(2) \times SO(2n)$, and the principal bundle $FU(2, 2n+2)$ as $SO(2n+2)/I_2 \times SO(2n)$. The universal connection ω on $FU(2, 2n+2)$ can be defined as the truncation of the restriction of the Maurer-Cartan form on $o(2n+2)$, denoted $\mu = [\mu_{ij}]$, to the tangents to $FU(2, 2n+2)$. That is, the components of the connection ω_{ij} are defined for $i, j \in \{1, 2\}$ by $\omega_{ij}(A) = A_{ij}$, for any

$$
A \in T_*(FU(2, 2n+2), (U_0, \{e_1, e_2\})) = \left\{ A \in o(2n+2)\,|\, A = \begin{bmatrix} R & S \\ -S^t & 0 \end{bmatrix}, R \in o(2) \right\},
$$

if $U_0 = \mathbb{R}^2 \times 0 \subset \mathbb{R}^{2n+2}$, with basis $\{e_1, e_2\}$. By homogeneity, all calculations in $FU(2, 2n+2)$ can be taken to be at this point.

The curvature Ω of this connection is given by $\Omega_{ij}(X, Y) = -\omega_{ij}([X, Y])$ for left-invariant vector fields that are horizontal at U_0, that is, of the form $\begin{bmatrix} 0 & S \\ -S^t & 0 \end{bmatrix}$. In terms of the Maurer-Cartan form, $\Omega_{ij} = +\sum_{k=2}^{2n+2} \mu_{ik} \wedge \mu_{jk}$, for $i, j \in \{1, 2\}$.

Similarly, the connection ω^* on the dual principal bundle $FV(2n, 2n+2) = SO(2n+2)/SO(2) \times I$ at $U_0 = \mathbb{R}^2 \times 0 \subset \mathbb{R}^{2n+2}$ is the restriction of the same Maurer-Cartan form μ to the other block, and the curvature $\Omega^*_{kl} = \sum_{i=1}^{2} \mu_{ik} \wedge \mu_{il}$, for $k, l \in \{3, \ldots, 2n+2\}$. Either of the tangent spaces to these principal bundles can be canonically embedded into the tangent space $o(2n+2)$ of $SO(2n+2)$ at the identity.

The Euler form $e(\Omega)$ of $FU(2, 2n+2)$ is the form

$$
\begin{aligned}
e(\Omega) &:= \frac{1}{2\pi} \left(\Omega_{12} \right) \\
&= \frac{1}{2\pi^2} \left(\mu_{1k} \wedge \mu_{2k} \right),
\end{aligned}
$$

where the sum is taken over all $k \in \{3, \ldots, 2n+2\}$. Dually, the Euler form $e(\Omega^*)$ of $FV(2n, 2n+2)$ is the form

$$
\begin{aligned}
e(\Omega^*) &:= C \left(\sum_{\sigma \in S_{2n}} (-1)^\sigma \Omega_{\sigma(3)\sigma(4)} \wedge \cdots \wedge \Omega_{\sigma(2n+1)\sigma(2n+2)} \right) \\
&= C \left(\sum_{\sigma \in S_{2n}, i_1, \ldots, i_n} (-1)^\sigma \mu_{\sigma(3)i_1} \wedge \mu_{\sigma(4)i_1} \wedge \cdots \wedge \mu_{\sigma(2n+1)i_n} \wedge \mu_{\sigma(2n+2)i_n} \right),
\end{aligned}
$$

where the sum is taken over all $\sigma \in S_{2n}$ as permutations of $\{3, \ldots, 2n+2\}$, $i_1, \ldots, i_n \in \{1, 2\}$, and the constant depends just on the dimension.

Proposition 2.1. $e(\Omega) \wedge e(\Omega^*) \equiv 0$.

Proof. Each monomial in this product is of the form

$$
\mu_{1k} \wedge \mu_{2k} \wedge \mu_{3i_1} \wedge \mu_{4i_1} \wedge \cdots \wedge \mu_{(2n+1)i_n} \wedge \mu_{(2n+2)i_n}
$$

or a permutation thereof. k can be in $3, \ldots, 2n+2$. No matter what k is, since $i_1, \ldots i_n$ are either 1 or 2, then this form must be 0. $\square$

Thus, the form $\Phi := C \, Te(\omega) \wedge e(\Omega^*)$ is indeed closed.

It remains to find the maximum of $\Phi(W)$ for $2n+1$-planes W in the total space of $\pi F(1, 2, \mathbb{R}^{2n+2}) \to G(2, 2n+2)$.

Certainly the vertical direction will be a maximum for $Te(\omega)$, which is (up to scale) exactly the volume form of the fibers. Thus the maximum is achieved only when one direction of the $(2n+1)$-plane is vertical.

It is interesting to note that, since the maximum of Φ must necessarily have a vertical direction at each point, any current calibrated by Φ must be a contained in a union of fibers of the projection $\pi : F(1, 2, \mathbb{R}^{2n+2}) \to G(2, 2n+2)$, so must be of the form $\pi^{-1}(M) \cap U$ for some current $M \subset G(2, 2n+2)$. Since, for $W \in G(2, 2n+2)$, the preimage

$$
\pi^{-1}(W) = \{x \mid x \in W, |x| = 1\} = \{\{e_1, e_2\} \mid \{e_1, e_2\} \text{ is a basis of } W\}
$$

is, as a subset of $T_1(S^{2n+1})$, the unit velocity field of the great circle $S^{2n+1} \cap W$ with orientation determined by W. In terms of the foliations determined by these calibrated currents, they must then consist of arcs of great circles, and must be great circle foliations if they are regular.

To see what currents Φ calibrates, we now need only find those $2n$-plane directions maximizing $e(\Omega^*)$.

Since

$$
\begin{aligned}
e(\Omega^*) &:= C \left(\sum_{\sigma \in S_{2n}} (-1)^\sigma \Omega_{\sigma(3)\sigma(4)} \wedge \cdots \wedge \Omega_{\sigma(2n+1)\sigma(2n+2)} \right) \\
&= C \left(\sum_{\sigma \in S_{2n}, i_1, \ldots i_n} (-1)^\sigma \mu_{\sigma(3)i_1} \wedge \mu_{\sigma(4)i_1} \wedge \cdots \wedge \mu_{\sigma(2n+1)i_n} \wedge \mu_{\sigma(2n+2)i_n} \right),
\end{aligned}
$$

if E_{ij} is the basis of tangent vectors dual to μ_{ij}, for any fixed permutation $\sigma \in S_{2n}$,

$$e(\Omega^*)(E_{1\sigma(3)}, E_{1\sigma(4)}, \ldots, E_{1\sigma(2n+2)}) = (-1)^\sigma (2n)! C = e(\Omega^*)(E_{2\sigma(3)}, \ldots, E_{2\sigma(2n+2)}).$$

It is straightforward to see that, if $i_{j_1} \neq i_{j_2}$, then some permutations in the sum will evaluate to 0, so that

$$\left| e(\Omega^*)(E_{i_1\sigma(3)}, E_{i_2\sigma(4)}, \ldots, E_{i_{2n}\sigma(2n+2)}) \right| < (2n)! C.$$

Finally, if $\{i_1, \ldots, i_{2n}\}$ does not have at least n pairs of values, or if $\{k_1, \ldots, k_{2n}\}$ does not consist of some permutation of $\{3, \ldots, 2n+2\}$, then $e(\Omega^*)(E_{i_1 k_1}, \ldots, E_{i_{2n} k_{2n}}) = 0$.

For any decomposable, unit $\xi \in \Lambda_{2n}(G(2, 2n+2), W_0)$ which is tangent to the variety $G(2, 2n+2)$ at W_0,

$$\xi = \sum_{i_1, \ldots, i_{2n}, k_1 \leq \cdots \leq k_{2n}} \xi_{i_1, \ldots, i_{2n}, k_1, \ldots, k_{2n}} E_{i_1 k_1} \wedge \cdots \wedge E_{i_{2n} k_{2n}}.$$

Since ξ is decomposable, ξ satisfies the Plücker condition $\xi \wedge \xi = 0$, implying that, in particular (restricting to the case where $\{k_1, \ldots, k_{2n}\} = \{3, \ldots, 2n+2\}$ since otherwise $e(\Omega^*) = 0$), and denoting $\xi_{i_1, \ldots, i_{2n}, 3, \ldots, (2n+2)}$ by $\xi_{i_1, \ldots, i_{2n}}$,

$$\xi_{1, \ldots, 1} \xi_{2, \ldots, 2} - \xi_{2, 1, \ldots, 1} \xi_{1, 2, \ldots, 2} - \xi_{1, 2, 1, \ldots, 1} \xi_{2, 1, 2, \ldots, 2} + \cdots = 0,$$

and similarly for all other such combinations. Thus,

$$
\begin{aligned}
(\xi_{1, \ldots, 1} + \xi_{2, \ldots, 2})^2 &= \xi_{1, \ldots, 1}^2 + \xi_{2, \ldots, 2}^2 + 2\xi_{1, \ldots, 1}\xi_{2, \ldots, 2} \\
&= \xi_{1, \ldots, 1}^2 + \xi_{2, \ldots, 2}^2 + 2\xi_{2, 1, \ldots, 1}\xi_{1, 2, \ldots, 2} + 2\xi_{1, 2, 1, \ldots, 1}\xi_{2, 1, 2, \ldots, 2} \pm \cdots \\
&\leq \xi_{1,1,1,1}^2 + \xi_{2,2,2,2}^2 + \xi_{2, 1, \ldots, 1}^2 + \xi_{1, 2, \ldots, 2}^2 + \xi_{1, 2, 1, \ldots, 1}^2 + \xi_{2, 1, 2, \ldots, 2}^2 + \cdots \\
&\leq 1,
\end{aligned}
$$

since ξ is a unit. Thus, on any such ξ,

$$e(\Omega^*)(\xi) \leq (2n)! C,$$

the maximum being achieved on those ξ so that $(\xi_{1, \ldots, 1, 3, \ldots, (2n+2)} + \xi_{2, \ldots, 2, 3, \ldots, (2n+2)}) = 1$ which have the proper orientation. Those $2n$-planes are, except where $n = 1$, *not* those which are complex $2n$-planes in $T_*(G(2n, 2n+2), W_0)$ under some complex structure on that space induced from one of $\mathbb{R}^{2n+2}$ for which W_0 is complex.

Theorem 2.2. *The standard foliation H of S^3 by the fibers of the Hopf fibration $S^3 \to S^2$ for some complex structure on $\mathbb{R}^4 \supset S^3$ minimizes the volume of one-dimensional foliations of S^3. The singular foliation NS of S^{2n+1}, $n > 1$ consisting of all great circles through a pair of antipodal points with indices ± 1 minimizes volume of all singular foliations on S^{2n+1} with those singular points and indices, and provides a lower bound for the volume of all one-dimensional oriented foliations of S^{2n+1}.*

Remark 2.3. The minimization of the Hopf fibration in the case $n = 1$ is due to Gluck and Ziller in [4]. They also showed a bound on the minimum-volume flow in higher dimensions by constructing a specific cycle in twice the homology of a flow. The first-named author, along with P. Chacón and A. M. Naveira, in [1], showed that this bound is attained by the specific singular foliation NS, and is a strict lower bound for volumes of smooth foliations. The notation NS ("north-south") refers to the fact that this foliation is by longitude lines from one pole to the other.

Proof.

Case 1. $n = 1$

In the case $n = 1$ any complex 2-plane will maximize $e(\Omega^*)$, since $T_*(G(2,4), W_0)$ is $\mathbb{C}^2$ and a real 2-plane ξ in $\mathbb{C}^2$ is complex (for a given complex structure) if and only if $< \xi, \alpha > + < \xi, \beta > = 1$ for any orthogonal pair α, β of complex lines, where the inner product is the standard induced inner product on $\Lambda_2(\mathbb{C}^2)$ induced from the inner product on $\mathbb{C}^2$ itself. Using coordinate planes and the standard complex structure on $G(2,4)$ (which is as the projective variety in $\mathbb{CP}^3$ defined by $z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0$), this condition is equivalent to $(\xi_{1,1,3,4} + \xi_{2,2,3,4}) = 1$. So any complex submanifold $M \subset G(2,4)$ will be calibrated by $e(\Omega^*)$.

Not every such complex submanifold corresponds to a foliation of S^3, however, not even a singular one. For any $W \in M$, the preimage $\pi^{-1}(W) \subset F(1,2,4) = T_1(S^3)$ corresponds to the image in $T_1(S^3)$ of the intersection of S^3 with the 2-plane W via the tangent map, a great circle on S^3. So, if M (complex or not) corresponds to a smooth or singular foliation of S^3, it is a foliation by great circles. If all such W are complex lines in $\mathbb{R}^4 = \mathbb{C}^2$ for some complex structure on the $\mathbb{R}^4$ in which S^3 is embedded, then all of these great circles are disjoint, M is the standard embedding of $\mathbb{CP}^1$ in $G_0(2,4)$, and the foliation is a Hopf fibration, and the corresponding curve M in $G_0(2,4) \subset \mathbb{CP}^3$ is defined by $z_0 = iz_1$ in addition to $z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0$. Other complex submanifolds of $G_0(2,4)$ do not correspond to even a singular foliation of S^3. For example, the curve $z_0 = 0$, which is also a hyperplane section of $G_0(2,4)$ and which is $G_0(2,3) \cong \mathbb{CP}^1$, lifts to $F_0(1,2,3) = T_1(S^2) \subset T_1(S^3)$, so does not correspond to a section over a dense subset of S^3.

The manifold $M = \{W \in G_0(2,4) | e_1 \in W\}$, which is dual to the previous submanifold, will also be calibrated by Φ, since at $W_0 \in M$, with basis chosen so that $W_0 = e_1 \wedge e_2 \in \Lambda_2(\mathbb{R}^4)$, the tangent plane satisfies $\xi_{2,2,3,4} = 1$. The current NS corresponding to a singular foliation will not be all of $\pi^{-1}(M)$, since that will be a double of the singular foliation by all great circles passing through $\pm e_1$. Instead, the current NS is formed from semicircular fibers of this bundle, from the fiber of $T_1(S^3)$ over $-e_1$ to that over $+e_1$. This current would minimize volume over all singular foliations of S^3 with two point singularities at $\pm e_1$, $-e_1$ having index -1 and e_1 having index 1. The minimum volume of such singular foliations is the same as that of the Hopf fibrations.

Case 2. $n > 1$

For $n > 1$ if M is the manifold

$$M := \{W \in G_0(2, 2n+2) | e_1 \in W\} := \{e_1 \wedge x | x \perp \{e_1\}, \|x\| = 1\},$$

then $M \cong S^{2n}$ is not the space of complex $2n$-planes in $\mathbb{R}^{2n+2}$ for any complex structure, and the corresponding "foliation" on S^{2n+1} will be singular. The tangent planes to M at each point clearly maximize the value of $e(\Omega^*)$. Note also that, in this case complex submanifolds of $G(2, 2n+2)$ are *not* calibrated by $e(\Omega^*)$.

In general, this singular distribution will indeed be calibrated by this form, so minimizes volume among all singular foliations with the same singular set; in this case, an antipodal pair of singular points, with indices ± 1 that are in each leaf of the singular foliation. Since the current in $T_1(S^{2n+1})$ actually defined by M consists of the unit tangent field to oriented semi-circles, longitudes, from $-e_1$ to $+e_1$ in S^{2n+1}, which has as a 2-fold cover the submanifold $S^{2n} \times S^1 = \pi^{-1}(M) \subset F_0(1, 2, \mathbb{R}^{2n+2}) = T_1(S^{2n+1})$, the mass-minimization property of the calibration compares the mass of this current, NS, to all other currents S with the same boundary (the two tangent fibers over $\pm e_1$, suitably oriented), which are homologous in that $NS - S$ is a boundary. This can be easily extended to all other currents with the same singular points and the same indices at those singular points, since any such current can be modified within the singular fibers to match the boundary of NS.

$\square$

3. 3-DIMENSIONAL FOLIATIONS OF S^7

3.1. **The calibration.** Note that the Grassmann bundle $G(3, S^7)$ of oriented 3-planes tangent to S^7 is isomorphic to the flag manifold of oriented lines within oriented 4-planes in $\mathbb{R}^8$, similarly to ([4]). This gives rise to the following diagram:

$$
\begin{array}{ccc}
G_o(3, S^7) & \longrightarrow & F_o(1, 4, \mathbb{R}^8) \\
h \uparrow \pi \downarrow & & \pi \downarrow \\
S^7 \searrow & & G(4, \mathbb{R}^8)
\end{array} \quad .
$$

$G_o(4, 8)$ has two universal 4-plane bundles, $U(4, 8)$ and $V(4, 8)$, defined by

$$
U(4, 8) \quad := \quad \cup_{x \in G_o(4,8)} x
$$

$$
V(4, 8) \quad := \quad \cup_{x \in G_o(4,8)} x^{\perp}.
$$

The respective Euler classes $E(U)$ and $E(V)$ satisfy $E(U) \cup E(V) = 0$ in $H^8(G_o(4, 8))$. Similarly, the respective first Pontryagin classes $P_1(U)$ and $P_1(V)$ satisfy the same relationship, $P_1(U) \cup P_1(V) = 0$. In particular, if ω is the universal connection on $U(4, 8)$ defined by Narasimhan and Ramanan (cf. [7]), and ω^* is the "dual" connection on $V(4, 8)$, then the associated Euler forms, $e(\Omega)$ and $e(\Omega^*)$, satisfy $e(\Omega) \wedge e(\Omega^*) = 0$ (respectively, the first Pontryagin forms). Then, consider the form

$$
\Phi := C \, Te(\omega) \wedge e(\Omega^*),
$$

which is well-defined on $F_o(1, 4, \mathbb{R}^8)$ as well as on the frame bundle $FU(4, 8)$ of oriented orthonormal frames on $U(4, 8)$, which is an $SO(3)$-principal bundle over $F_o(1, 4, \mathbb{R}^8)$. Here, $Te(\omega)$ is the transgressive Chern-Simons form corresponding to the Euler form $e(\Omega)$ of $U(4, 8)$ [3]. Because $d(Te(\omega)) = e(\Omega)$ (again, either as a form on the frame bundle, or on the associated bundle $F_o(1, 4, \mathbb{R}^8)$), we have that $d\Phi = 0$. The constant C is simply chosen so that the comass of Φ is one.

That Φ is well-defined on $F_o(1, 4, \mathbb{R}^8)$ is perhaps not obvious. However, the original version of the transgressive form $Te(\omega)$ was defined by Chern on the sphere bundle, not the frame bundle [2]. That same construction applies here. When restricted to vertical directions, those tangent to the 3-sphere fiber of $F_o(1, 4, \mathbb{R}^8) \to G_o(4, 8)$, $Te(\omega)$ is the volume form of the fibers.

3.2. **Calculations.** We will consider $G_o(4, 8)$ as $SO(8)/SO(4) \times SO(4)$, and the principal bundle $FU(4, 8)$ as $SO(8)/I \times SO(4)$. The universal connection ω on $FU(4, 8)$ can be defined as the truncation of the restriction of the Maurer-Cartan form on $o(8)$, denoted $\mu = [\mu_{ij}]$, to the tangents to $FU(4, 8)$. That is, the components of the connection ω_{ij} are defined for $i, j \in \{1, \ldots, 4\}$ and $\omega_{ij}(A) = A_{ij}$ for any

$$
A \in T_*(U(4, 8), (U_0, \{e_1, \ldots, e_4\})) = \left\{ A \in o(8) \,\middle|\, A = \begin{bmatrix} R & S \\ -S^t & 0 \end{bmatrix}, R \in o(4) \right\},
$$

if $U_0 = \mathbb{R}^4 \times 0 \subset \mathbb{R}^8$, with basis $\{e_1, \ldots, e_4\}$. By homogeneity, all calculations in $FU(4, 8)$ can be taken to be at this point.

The curvature Ω of this connection is given by $\Omega_{ij}(X, Y) = -\omega_{ij}([X, Y])$ for left-invariant vector fields that are horizontal at U_0, that is, of the form $\begin{bmatrix} 0 & S \\ -S^t & 0 \end{bmatrix}$. In terms of the Maurer-Cartan form, $\Omega_{ij} = + \sum_{k=5}^{8} \mu_{ik} \wedge \mu_{jk}$, for $i, j \in \{1, \ldots, 4\}$.

Similarly, the connection ω^* on the dual principal bundle $FV(4, 8) = SO(8)/SO(4) \times I$ at $U_0 = \mathbb{R}^4 \times 0 \subset \mathbb{R}^8$ is the restriction of the same Maurer-Cartan form μ to the other 4×4 block, and the curvature $\Omega_{kl}^* = \sum_{i=1}^{4} \mu_{ik} \wedge \mu_{il}$, for $k, l \in \{5, \ldots, 8\}$. Either of the tangent spaces to these principal bundles can be canonically embedded into the tangent space $o(8)$ of $SO(8)$ at the identity.

The Euler form $e(\Omega)$ of $FU(4,8)$ is the form

$$
\begin{aligned}
e(\Omega) \quad &:= \quad \frac{1}{2\pi^2} \left(\Omega_{12} \wedge \Omega_{34} - \Omega_{13} \wedge \Omega_{24} + \Omega_{14} \wedge \Omega_{23} \right) \\
&= \quad \frac{1}{2\pi^2} \left(\mu_{1k} \wedge \mu_{2k} \wedge \mu_{3l} \wedge \mu_{4l} - \mu_{1k} \wedge \mu_{3k} \wedge \mu_{2l} \wedge \mu_{4l} + \mu_{1k} \wedge \mu_{4k} \wedge \mu_{2l} \wedge \mu_{3l} \right),
\end{aligned}
$$

where the sum is taken over all $k, l \in \{5, \ldots, 8\}$. Dually, the Euler form $e(\Omega^*)$ of $FV(4,8)$ is the form

$$
\begin{aligned}
e(\Omega^*) \quad &:= \quad \frac{1}{2\pi^2} \left(\Omega_{56} \wedge \Omega_{78} - \Omega_{57} \wedge \Omega_{68} + \Omega_{58} \wedge \Omega_{67} \right) \\
&= \quad \frac{1}{2\pi^2} \left(\mu_{5i} \wedge \mu_{6i} \wedge \mu_{7j} \wedge \mu_{8j} - \mu_{5i} \wedge \mu_{7i} \wedge \mu_{6j} \wedge \mu_{8j} + \mu_{5i} \wedge \mu_{8i} \wedge \mu_{6j} \wedge \mu_{7j} \right),
\end{aligned}
$$

where the sum is taken over all $i, j \in \{1, \ldots, 4\}$.

Proposition 3.1. $e(\Omega) \wedge e(\Omega^*) \equiv 0$.

Proof. Each monomial in this product is of the form

$$
\mu_{1k} \wedge \mu_{2k} \wedge \mu_{3l} \wedge \mu_{4l} \wedge \mu_{5i} \wedge \mu_{6i} \wedge \mu_{7j} \wedge \mu_{8j}
$$

or a permutation thereof. k can be either 5, 6, 7 or 8. If k is, say, 5, then i cannot be 1 or 2, thus must be $i = 3$ or 4. Thus $l \neq 5, 6$, so $l = 7$ or 8, and finally, $j = 1$ or 2. No matter which choices are made, two of the indices between 1 and 4 will occur once, and the other two will occur three times, and similarly for the indices from 5 to 8. Thus, each monomial is determined by the multi-indices that occur with one index singly. For example, 2, 4, 6, and 8 occur singly, paired as $25, 47, 36,$ and 18 in exactly two terms,

$$
+\mu_{15} \wedge \mu_{25} \wedge \mu_{37} \wedge \mu_{47} \wedge \mu_{53} \wedge \mu_{63} \wedge \mu_{71} \wedge \mu_{81}, \text{ and}
$$

$$
+\mu_{17} \wedge \mu_{47} \wedge \mu_{25} \wedge \mu_{35} \wedge \mu_{51} \wedge \mu_{81} \wedge \mu_{63} \wedge \mu_{73}.
$$

However, using the fact that $\mu_{ik} = -\mu_{ki}$ and the exterior product, these terms cancel. Since all terms are permutations of these, all terms cancel in pairs. $\qquad\square$

Thus, the form $\Phi := C\, Te(\omega) \wedge e(\Omega^*)$ is indeed closed. That it is well-defined can be traced back to early versions of the Chern-Simons theory, such as [2]. Alternately, it can be directly verified from the local expression for $Te(\omega)$ in terms of the Maurer-Cartan form μ. That is, as a form on $F_o(1, 4, \mathbb{R}^8)$, at the point $x_0 := (e_1, W)$, $e_1 \in W = \mathbb{R}^4 \times \{0\} \subset \mathbb{R}^8$, since all the ω_{ij} tangent to $F_o(1, 4, \mathbb{R}^8)$ have one of $i = 1$ or $j = 1$,

$$
\begin{aligned}
Te(\omega) \quad &:= \quad \frac{1}{2\pi^2} \Bigl(\omega_{12}\Omega_{34} - \omega_{13}\Omega_{24} + \omega_{14}\Omega_{23} \\
&\qquad\qquad - \frac{1}{6} \left(\omega_{12} \left([\omega,\omega]\right)_{34} - \omega_{13} \left([\omega,\omega]\right)_{24} + \omega_{14} \left([\omega,\omega]\right)_{23} \right) \Bigr) \\
&= \quad \frac{1}{2\pi^2} \Bigl(\mu_{12} \wedge \mu_{3k} \wedge \mu_{4k} - \mu_{13} \wedge \mu_{2k} \wedge \mu_{4k} + \mu_{14} \wedge \mu_{2k} \wedge \mu_{3k} \\
&\qquad\qquad + \frac{1}{3} \left(\mu_{12} \wedge \mu_{13} \wedge \mu_{14} - \mu_{13} \wedge \mu_{12} \wedge \mu_{14} + \mu_{14} \wedge \mu_{12} \wedge \mu_{13} \right) \Bigr) \\
&= \quad \frac{1}{2\pi^2} \left(\mu_{12} \wedge \mu_{13} \wedge \mu_{14} + \mu_{12} \wedge \mu_{3k} \wedge \mu_{4k} - \mu_{13} \wedge \mu_{2k} \wedge \mu_{4k} + \mu_{14} \wedge \mu_{2k} \wedge \mu_{3k} \right),
\end{aligned}
$$

where the sum is over k from 5 to 8. As a left-invariant form on $SO(8)$, it is straightforward to see that it is invariant under the adjoint action of the isotropy subgroup $1 \times SO(3) \times I_4 \subset SO(8)$, so descends to a form on $F(1, 4, \mathbb{R}^8)$.

It remains to find the maximum of $\Phi(W)$ for 7-planes W in the total space of $\pi F(1, 4, \mathbb{R}^8) \to G(4, 8)$.

Not all vertical directions (those in the 3-sphere fiber of π) and combinations within $\Lambda_3(T_*(F(1,4,\mathbb{R}^8)))$ are detected by the form Φ above. That is, let $W \subset T_*(F(1,4,\mathbb{R}^8))$ be a 7-dimensional subspace. Then, for some basis of $T_*(F(1,4,\mathbb{R}^8))$, with vertical directions v_1, v_2, v_3 and horizontal basis $\{e_1, \ldots, e_{16}\}$, $W = (a_1 v_1 + h_1) \wedge (a_2 v_2 + h_2) \wedge (a_3 v_3 + h_3) \wedge (e_1 \wedge \cdots \wedge e_4)$ as a unit element of $\Lambda_7\left(T_*(F(1,4,\mathbb{R}^8))\right)$. $\|h_i\| = b_i = \sqrt{1 - a_i^2}$. This simply states that no more than 3 directions can have independent vertical components. At the point x_0, if we denote $a_i = \cos(\theta_i)$ and $b_i = \sin(\theta_i)$,

$$
\begin{aligned}
\Phi(W) &\leq C\left|a_1 a_2 a_3 + a_1 b_2 b_3 - b_1 a_2 b_3 + b_1 b_2 a_3\right| \left|e(\Omega^*)(e_1 \wedge \cdots \wedge e_4)\right| \\
&\leq C\left|\cos(\theta_1)\left(\cos(\theta_2)\cos(\theta_3) + \sin(\theta_2)\sin(\theta_3)\right) - \sin(\theta_1)\left(\cos(\theta_2)\sin(\theta_3) - \sin(\theta_2)\cos(\theta_3)\right)\right| \cdot \\
&\quad \cdot \left|e(\Omega^*)(e_1 \wedge \cdots \wedge e_4)\right| \\
&= C\left|\cos(\theta_1 + (\theta_2 - \theta_3))\right| \left|e(\Omega^*)(e_1 \wedge \cdots \wedge e_4)\right| \\
&\leq C\left|e(\Omega^*)(e_1 \wedge \cdots \wedge e_4)\right|,
\end{aligned}
$$

Thus the maximum is achieved when (among other values) all three θ_i are 0, as long as the remaining vectors form a 4-plane maximizing $e(\Omega^*)$. It is not clear whether other values of θ_i will achieve this maximum, since the mixed parts of $Te(\Omega)$ are only bounded by those values. However, the maximum is clearly achieved when all $\theta_i = 0$.

Let $\pi_U : F_o(1,4,\mathbb{R}^8) \to G_o(4,8)$ be the fibration associated with the unit sphere bundle of the universal bundle U, so that (x,W), where $x \in W$ is a unit vector in the 4-plane W, is mapped to $\pi_u(x,W) := W \in G_o(4,8)$. The other fibration $\pi_V : F_o(1,4,\mathbb{R}^8) \to G_o(4,8)$, associated with the dual bundle V, maps the same (x,W) onto $\pi_V(x,W) := W^\perp$. $e(\Omega^*)$, as a form on $F_o(1,4,\mathbb{R}^8)$, is the π_V-horizontal lift of the form $e(\Omega^*)$ on $G_o(4,8)$. That is clearly maximized on some collection of π_V-horizontal 4-planes tangent to $G_o(4,8)$ at W. $Te(\omega)$ is maximized on the 3-sphere fibers of W, that is $\{(x,W)\,|\, x \in W\}$, as described above. Since these two spaces are orthogonal, then

$$
\Phi := C\,Te(\omega) \wedge e(\Omega^*)
$$

will be maximized on any 7-plane which is the sum of a π_V-horizontal lift of a 4-plane maximizing $e(\Omega^*)$ (perpendicular to W) and the 3-plane tangent to the unit sphere in W at x. However, not all 4-planes orthogonal to W will maximize Φ.

Since

$$
\begin{aligned}
e(\Omega^*) &:= \frac{1}{2\pi^2}\left(\Omega_{56} \wedge \Omega_{78} - \Omega_{57} \wedge \Omega_{68} + \Omega_{58} \wedge \Omega_{67}\right) \\
&= \frac{1}{2\pi^2}\left(\mu_{5i} \wedge \mu_{6i} \wedge \mu_{7j} \wedge \mu_{8j} - \mu_{5i} \wedge \mu_{7i} \wedge \mu_{6j} \wedge \mu_{8j} + \mu_{5i} \wedge \mu_{8i} \wedge \mu_{6j} \wedge \mu_{7j}\right),
\end{aligned}
$$

if E_{ij} is the basis of tangent vectors dual to μ_{ij}, $e(\Omega^*)(E_{15}, E_{16}, E_{17}, E_{18}) = 3/2\pi^2$. It is straightforward to see that $e(\Omega^*)(E_{i5}, E_{i6}, E_{i7}, E_{i8}) = 3/2\pi^2$ for any $i = 1 \ldots 4$, and $e(\Omega^*)(E_{i5}, E_{i6}, E_{j7}, E_{j8}) = 1/2\pi^2$ for $i \neq j$, or, more generally, if $k_1, \ldots, k_4$ are a permutation of $5, \ldots, 9$, then when $i \neq j$, $e(\Omega^*)(E_{ik_1}, E_{ik_2}, E_{jk_3}, E_{jk_4}) = \pm 1/2\pi^2$, where the sign is the sign of the permutation. Finally, if $\{i_1, i_2, i_3, i_4\}$ consist of more than two distinct values (and not two pairs of values), or if $\{k_1, k_2, k_3, k_4\}$ does not consist of some permutation of $\{5,6,7,8\}$, then

$$
e(\Omega^*)(E_{i_1 k_1}, E_{i_2 k_2}, E_{i_3 k_3}, E_{i_4 k_4}) = 0.
$$

Theorem 3.2. *The singular foliation NS of S^7 consisting of all great 3-spheres containing a common great 2-sphere minimizes volume of all three-dimensional singular foliations on S^7 with that singular locus and limiting behavior, and provides a lower bound for the volume of all regular three-dimensional oriented foliations of S^7.*

Proof. For any decomposable, unit $\xi \in \Lambda_4(G(4,8), W_0)$ which is tangent to the variety $G(4,8)$ at W_0,

$$\xi = \sum_{i_1,\cdots,i_4, k_1 \leq \cdots \leq k_4} \xi_{i_1,\ldots,i_4,k_1,\ldots,k_4} E_{i_1 k_1} \wedge \cdots \wedge E_{i_4 k_4}.$$

Since ξ is decomposable, ξ satisfies the Plücker condition $\xi \wedge \xi = 0$, implying that, in particular (restricting to the case where $\{k_1,\ldots,k_4\} = \{5,6,7,8\}$ since otherwise $e(\Omega^*) = 0$), and denoting $\xi_{i,j,k,l,5,6,7,8}$ by $\xi_{i,j,k,l}$,

$$\xi_{1,1,1,1}\xi_{2,2,2,2} - \xi_{2,1,1,1}\xi_{1,2,2,2} - \xi_{1,2,1,1}\xi_{2,1,2,2} - \xi_{1,1,2,1}\xi_{2,2,1,2}$$
$$-\xi_{1,1,1,2}\xi_{2,2,2,1} + \xi_{1,1,2,2}\xi_{2,2,1,1} + \xi_{1,2,1,2}\xi_{2,1,2,1} + \xi_{1,2,2,1}\xi_{2,1,1,2} = 0.$$

and similarly for all other such combinations. Thus,

$$\left(\xi_{1,1,1,1} + \xi_{2,2,2,2} + \xi_{3,3,3,3} + \xi_{4,4,4,4}\right)^2$$
$$= \xi_{1,1,1,1}^2 + \xi_{2,2,2,2}^2 + 2\xi_{1,1,1,1}\xi_{2,2,2,2} + \cdots$$
$$= \xi_{1,1,1,1}^2 + \xi_{2,2,2,2}^2 + 2\xi_{2,1,1,1}\xi_{1,2,2,2} + 2\xi_{1,2,1,1}\xi_{2,1,2,2} + 2\xi_{1,1,2,1}\xi_{2,2,1,2}$$
$$+ 2\xi_{1,1,1,2}\xi_{2,2,2,1} - 2\xi_{1,1,2,2}\xi_{2,2,1,1} - 2\xi_{1,2,1,2}\xi_{2,1,2,1} - 2\xi_{1,2,2,1}\xi_{2,1,1,2} + \cdots$$
$$\leq \xi_{1,1,1,1}^2 + \xi_{2,2,2,2}^2 + \xi_{2,1,1,1}^2 + \xi_{1,2,2,2}^2 + \xi_{1,2,1,1}^2 + \xi_{2,1,2,2}^2 + \xi_{1,1,2,1}^2 + \xi_{2,2,1,2}^2$$
$$+ \xi_{1,1,1,2}^2 + \xi_{2,2,2,1}^2 + \xi_{1,1,2,2}^2 + \xi_{2,2,1,1}^2 + \xi_{1,2,1,2}^2 + \xi_{2,1,2,1}^2$$
$$+ \xi_{1,2,2,1}^2 + \xi_{2,1,1,2}^2 + \cdots$$
$$\leq 1,$$

since ξ is a unit. Thus, on any such ξ,

$$e(\Omega^*)(\xi) \leq 3/2\pi^2,$$

the maximum being achieved on those ξ so that $(\xi_{1,1,1,1,5,6,7,8} + \xi_{2,2,2,2,5,6,7,8} + \xi_{3,3,3,3,5,6,7,8} + \xi_{4,4,4,4,5,6,7,8}) = 1$. Those 4-planes, in contrast to the complex case studied by Gluck and Ziller, are *not* those which are tangent 4-planes in $T_*(G(4,8), W_0)$ to the quaternionic projective space $\mathbb{HP}^1$ under any quaternionic structure on $\mathbb{R}^8$ for which W_0 is quaternionic. Those 4-planes can be easily shown to evaluate to half the maximum possible value.

In fact, if M is the manifold

$$M := \{x \wedge e_2 \wedge e_3 \wedge e_4 \mid x \perp \{e_2, e_3, e_4\}, \|x\| = 1\},$$

then $M \cong S^4$, and the corresponding "foliation" on S^7 will be singular. The tangent planes to M at each point clearly maximize the value of $e(\Omega^*)$. The corresponding singular foliation on S^7 is the set of all great 3-spheres that are intersections of S^7 with a plane $W = span\{x, e_2, e_3, e_4\}$ for some unit $x \perp \{e_2, e_3, e_4\}$, which is singular on the S^2 common to all leaves. However, this singular distribution will indeed be calibrated by this form, so minimizes volume among, at least, all singular foliations with the same singular set; in this case, a totally-geodesic S^2 which is the intersection of any two leaves of the foliation.

As with the case for one-dimensional leaves, this singular foliation actually corresponds to half of the current $\pi^{-1}(M) \subset F_0(1,4,8) \cong G(3, S^7)$, since the leaf corresponding to the 4-plane $x \wedge e_2 \wedge e_3 \wedge e_4$ is the same set as that leaf corresponding to $(-x) \wedge e_2 \wedge e_3 \wedge e_4$ with the opposite orientation. The 3-plane common to all 4-planes separates each into two half-spaces. Choose the half-space consistent with a chosen orientation on the common 3-plane, which then restricts the fibers to hemispheres which still provides a singular foliation of S^7. Since this (non-cycle) current $NS \subset G_o(3, S^7)$ has boundary $S^2 \times S^4 \subset G_o(3, S^7)|_{S^2} \cong S^2 \times G_o(3,7)$, which is not itself a boundary, NS does not extend to a cycle. Thus, the fact that Φ calibrated NS only implies that NS represents a singular foliation on S^7 which is volume minimizing among foliations with the same singular locus.

However, similarly to [4], it follows that the full preimage $S := \pi^{-1}(M)$, which is also calibrated by Φ and is a cycle, minimizes mass among currents homologous to twice the homology class of a foliation (all foliations by 3-manifolds are homologous as maps into $G_o(3, S^7)$). If there were a (singular or regular) volume-minimizing foliation represented by a cycle C, then the mass of $2C$ could not be less than the mass of S, so that the mass of NS does represent a lower bound of volumes of foliations of dimension 3 on S^7.

$\square$

It remains an open question whether the Hopf fibration minimizes volume among 3-dimensional regular foliations of S^7. However, the Hopf fibration (a regular foliation) does have twice the volume of the singular foliation NS.

4. GENERALIZATIONS

It is a straightforward generalization of these computations to show that the corresponding sphere M maximizes the corresponding form $e(\Omega^*)$ in $G_o(4, 4n+4)$, showing that similar singular foliations by 3-manifolds minimize volume among all (singular) foliations of S^{4n+3} with the given singular set.

Theorem 4.1. *The singular foliation of S^{4n+3} consisting of all great 3-spheres containing a common great 2-sphere minimizes volume of all three-dimensional singular foliations on S^{4n+3} with that singular locus and limiting behavior, and provides a lower bound for the volume of all regular three-dimensional oriented foliations of S^{4n+3}.*

Similarly, the same methods will show that the Hopf fibration of S^{15} by great 7-spheres, the fibers of the Cayley projective plane, the fibers of the fibration

$$S^7 \quad \to \quad S^{15}$$
$$\downarrow$$
$$S^8 \quad ,$$

will not minimize volume among all singular foliations of that space as well, but rather the "longitudes", great 7-spheres foliating S^{15} except for a great 6-sphere common to all leaves, will be a volume-minimizing singular foliation.

Theorem 4.2. *The singular foliation NS of S^{15} consisting of all great 7-spheres containing a common great 6-sphere minimizes volume of all 7-dimensional singular foliations on S^{15} with that singular locus and limiting behavior, and provides a lower bound for the volume of all regular three-dimensional oriented foliations of S^{15}.*

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