1. 5 is 20% of what number?

2. What is the value of $1 - 2 + 3 - 4 - \cdots + 99 - 100$? (alternating signs)

3. A frog is at the bottom of a well 20 feet deep. It climbs up 3 feet every day, but slides back 2 feet each night. If it started climbing on January 1, on what date does it get to the top of the well?

4. How many integers between 2004 and 4002 are perfect squares?

5. Let the operation $*$ be defined by $a * b = a^2 + 3^b$. What is the value of $(2 * 0) * (0 * 1)$?

6. The radius of each small circle is $1/6$ of the radius of the large surrounding circle. The radius of the middle-sized circle is twice that of either small circle. What fraction of the area of the large circle lies outside the three interior circles?
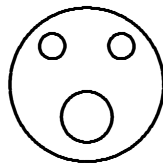

7. What is the number of square units enclosed by the polygon $ABCDE$?
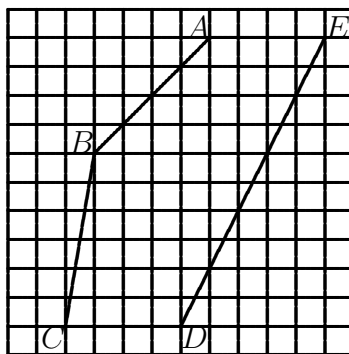
8. What is the value of $\dfrac{2^{2004} + 2^{2001}}{2^{2003} - 2^{2000}}$?

9. What is the area of the region of the plane satisfying $1 \le |x| + |y| \le 2$?

10. The square below can be filled in so that each row and each column contains each of the numbers 1, 2, 3, and 4 exactly once. What does x equal?

			1
	2		
		x	
1			4

11. Johnny was ill and had to take a test a day late. His 96 raised the class average from 71 to 72. How many students, including Johnny, took the test?

12. The pentagon $ABCDE$ has a right angle at A, $AB = AE$, and $ED = DC = CB = 1$. If $BE = 2$ and is parallel to CD, what is the area of the pentagon?
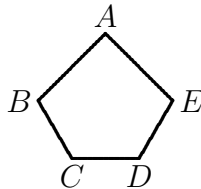

13. What value of x satisfies the equation $\sqrt{x + 10} + \sqrt[4]{x + 10} = 12$?

14. A quadratic polynomial f satisfies $f(x) \ge 0$ for all x, $f(1) = 0$, and $f(3) = 3$. What is $f(5)$?

15. In each hand, a girl holds a different end of a 26-inch long piece of light string that has a heavy bead on it. Initially her hands are together. How many inches apart must she pull her hands, keeping them at the same height, so that the bead moves upward by 8 inches?

16. Set $t_1 = 2$ and $t_{n+1} = (t_n - 1)/(t_n + 1)$. What is t_{203}?

17. If $a \neq b$ and $\left(\frac{a(1-b)}{b(1-a)}\right)^2 = 1$, what is the value of $(a+b)/(ab)$?

18. In the following figure, $AD = AE$, and F is the intersection of the extension of ED with the bisector of the angle C. If angle $B = 40$ degrees, how many degrees are there in angle CFE?
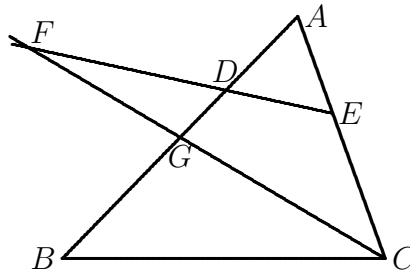
19. Let $N = 999{,}999{,}999{,}999{,}999{,}999$. How many 9's are in the decimal expansion of N^2.

20. If a, b, and c denote the solutions of the equation $x^3 - 2x^2 - 5x + 8 = 0$, what is the value of $\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}$?

21. How many integers n satisfy $n^4 + 6n < 6n^3 + n^2$?

22. Let P be the point $(8,0)$. Suppose PQ is tangent to the circle $x^2 + y^2 = 4$ at the point Q. What is the length PQ?

23. Find all ordered triples of positive integers (a, b, c) for which $a^3 - b^3 - c^3 = 3abc$ and $a^2 = 2(b + c)$.

24. What is the largest prime p such that there is a prime q for which $p+q$ and $p + 7q$ are both squares?

25. A square is partitioned into 30 non-overlapping triangles, so that each side of the square is a side of one of the triangles. We also require that the intersection of any two triangles is either empty, a common vertex, or a common edge. How many points in the interior of the square serve as vertices of one or more triangles?

26. How many 4-digit numbers (from 1000 to 9999) have at least one digit occurring more than once?

27. What is the sum of all positive integers x for which there exists a positive integer y with $x^2 - y^2 = 1001$?

28. A triangle has sides s_1, s_2, and s_3 of lengths 10, 9, and x, respectively. The angle between s_1 and s_3 is 60 degrees. What is the difference of the two possible values for x?

29. What is the sum of all 2-digit positive integers which exceed the product of their digits by 12?

30. In the diagram below a circular arc centered at one vertex of a square of side length 4 passes through two other vertices. A small circle is tangent to the large circle and to two sides of the square. What is the radius of the small circle?
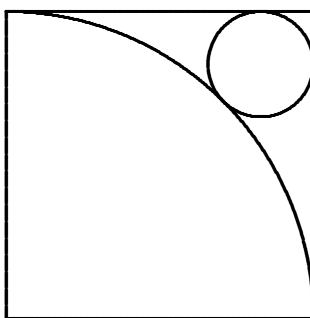
31. What is the sum of all positive integers n for which $2^8 + 2^{11} + 2^n$ is a perfect square?

32. What is the area of the shaded region in the diagram below?
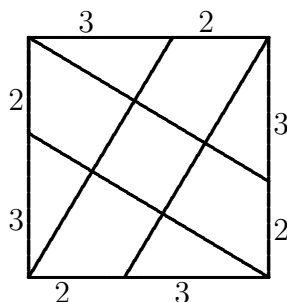
33. Suppose the function f satisfies $f(x) + f(x-1) = x^2$ and $f(19) = 94$. What is $f(94)$?

34. What is the number of degrees in the acute angle θ satisfying $\cos(\theta) = \frac{1}{2}\sqrt{2 + \sqrt{2}}$?

35. A number y is selected at random in the interval $[0, 10]$ (with uniform probability distribution). A right triangle with hypotenuse of length 10 is formed with one end of the hypotenuse at the point $(0, y)$ and the other on the nonnegative x-axis. The third vertex is at $(0,0)$. What is the probability that the area of the triangle is greater than 15?

36. What is the sum of the reciprocals of all positive integers none of whose prime divisors is greater than 3? That is, what is the value of

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{8} + \tfrac{1}{9} + \cdots .$$

37. How many ordered pairs (x, y) of integers satisfy $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$? (Note that both positive and negative integers are allowed.)

38. What is the smallest value of n for which the product

$$(2^2 - 1)(3^2 - 1)(4^2 - 1) \cdots (n^2 - 1)$$

is a perfect square?

39. Note that $649^2 - 13 \times 180^2 = 1$. Find an integer a satisfying $649 < a < 10^8$ such that $a^2 - 13b^2 = 1$ for some positive integer $b < a$.

40. Suppose triangle ABC is equilateral, and $AF = BD = CE = \frac{1}{3}AB$. What is the ratio of the area of triangle ABC to that of the shaded triangle?
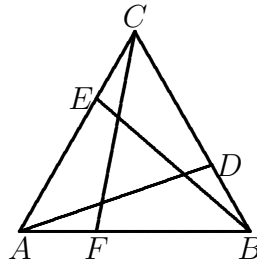
Solutions to 2004 exam

Numbers in brackets are (a) the number of people out of the top 30 who answered the question correctly, and (b) the number of people out of the other 139 who answered the question correctly.

1. 25. [30, 137] $5 = \frac{1}{5}x$ implies $x = 5 \cdot 5$.

2. -50. [29, 123] There are 50 pairs, each summing to -1.

3. Jan 18. [26, 72] It will get to height 3 on Jan 1, height 4 on Jan 2, ..., and height 20 on Jan 18.

4. 19. [26, 85] $44^2 = 1936$, $45^2 = 2025$, $63^2 = 3969$, and $64^2 = 4096$. Thus it is the integers from 45 to 63, inclusive.

5. 52. [26, 101] It is $(2^2 + 3^0) * (0^2 + 3^1) = 5 * 3 = 5^2 + 3^3$.

6. 5/6. [28, 85] Let the radius of the large circle be 1. Then the sum of the areas of the three circles is $\pi(\frac{1}{36} + \frac{1}{36} + \frac{1}{9}) = \pi/6$, while that of the large circle is π.

7. 50. [28, 97] The region can be divided into two trapezoids, divided by the horizontal line through B. The area of the top one is $4(4+6)/2 = 20$ and the area of the bottom one is $6(4+6)/2 = 30$.

8. 18/7. [26, 48] Divide top and bottom by 2^{2000}, getting $(16+2)/(8-1)$.

9. 6. [25, 16] It is the area inside a diamond with vertices at $(\pm 2, 0)$ and $(0, \pm 2)$ minus that in the diamond 1/4 the size (1/2 in each direction). Thus it is $\frac{3}{4} \cdot 4 \cdot \frac{1}{2} \cdot 2 \cdot 2$. Here we have considered the four quadrants.

10. 4. [30, 129] In the diagram below, the bottom row is forced, then the second column, then the top row, and then the third column.

2	4	3	①
	②	1	
	1	x	
①	3	2	④

11. 25. [26, 85] If x is the desired number of students, then $71(x-1)+96 = 72x$, and so $x = 96 - 71$.

12. $1 + \frac{3}{4}\sqrt{3}$. [21, 33] The right triangle ABE has area $\frac{1}{2} \cdot 2$. The vertical displacement from BE to CD is $\frac{1}{2}\sqrt{3}$. Thus the area of the bottom is $(2 - \frac{1}{2})\frac{1}{2}\sqrt{3}$.

13. 71. [26, 80] Let $y = \sqrt[4]{x + 10}$. Then $y^2 + y - 12 = 0$, so $y = 3$, since y cannot be negative. Thus $x + 10 = 3^4$.

14. 12. [18, 9] $f(x) = a(x - 1)^2$ and $4a = 3$. Thus $f(5) = \frac{3}{4}4^2 = 12$.

15. 24. [24, 64] The bead was initially 13 inches below the level of her hands. When it has moved up 8 inches, it will be 5 inches below the level of her hands, and will be 13 inches from each hand. Thus each hand will have moved $12(=\sqrt{13^2 - 5^2})$ inches out.

16. $-1/2$. [24, 28] $t_2 = 1/3$. $t_3 = -1/2$. $t_4 = -3$. $t_5 = 2 = t_1$. Thus the numbers repeat with period 4, and $t_{203} = t_3$.

17. 2. [20, 29] Either $a(1 - b) = b(1 - a)$, in which case $a = b$, which has been excluded, or $a(1 - b) = -b(1 - a)$. This gives $a + b = 2ab$, so $(a + b)/(ab) = 2$.

18. 20. [20, 52] Let the desired angle equal x, the base angles of the isosceles triangle ADE equal y, and the two halves of angle C equal z. Comparing triangles ADE and ABC yields $2y = 2z + 40$. Comparing triangles DFG and BCG yields $x + y = 40 + z$. Thus $x = 40 - 40/2$.

19. 17. [17, 16] $N^2 = (10^{18} - 1)^2 = 10^{36} - 2 \cdot 10^{18} + 1$. Its decimal expansion has 9's in positions 10^{19} through 10^{35}.

20. $-1/4$. [6, 0] The desired quantity equals $(a + b + c)/(abc)$. Since

$$(x - a)(x - b)(x - c) = x^3 - 2x^2 - 5x + 8,$$

we obtain $a + b + c = 2$ and $abc = -8$.

21. 4. [26, 35] We must have $n(n^2 - 1)(n - 6) < 0$. Thus the factors $n + 1$, n, $n - 1$, and $n - 6$ must be nonzero, and an odd number of them are negative. Hence $n = 2, 3, 4$, or 5.

22. $\sqrt{60}$ or $2\sqrt{15}$. [18, 19] If O is the origin, then POQ is a right triangle with hypotenuse 8 and one leg 2. The other leg is PQ which equals $\sqrt{60}$.

23. (2,1,1). [5, 11] The first equation says that a is the largest of the three, and the second equation that a is even. Thus $2a > b + c = a^2/2$, and hence $4 > a$. Thus $a = 2$, $b = 1$, and $c = 1$ is the only possibility.

24. 2. [12, 14] Let $p + q = a^2$ and $p + 7q = b^2$. Then $6q = (b - a)(b + a)$. Thus one of $b \pm a$ must be even, and hence both are even (since their difference is even). Since q is prime, we must have $q = 2$, then $b = 4$, $a = 2$, and $p = 2$.

25. 14. [8, 3] If there are v interior vertices, then the total number of degrees in all the angles (including the interior vertices and the four vertices of the square) is, on the one hand, $30 \cdot 180$ and, on the other hand, $360v + 4 \cdot 90$. Thus $30 = 2v + 2$.

26. 4464. [9, 4] There are 9000 4-digit numbers. To enumerate those with all digits distinct, we have 9 choices for the first digit (since 0 is excluded), then 9 for the second (i.e. all digits except the first), 8 for the third, and 7 for the fourth. Our answer is $9000 - 9 \cdot 9 \cdot 8 \cdot 7$.

27. 672. [8, 0] Since $x^2 - y^2 = (x+y)(x-y)$, we need the factorizations of $1001 = 7 \cdot 11 \cdot 13$. The pair $(x+y, x-y)$ can equal any of the following: (1001,1), (143,7), (91,11), or (77,13). Thus the sum of the x-values is $\frac{1}{2}(1002 + 150 + 102 + 90) = 672$.

28. $2\sqrt{6}$ or $\sqrt{24}$. [7, 5] The Law of Cosines yields $81 = 100 + x^2 - 2 \cdot 10x \cos(60)$. Thus $x^2 - 10x + 19 = 0$, so $x = \frac{1}{2}(10 \pm \sqrt{100 - 76})$.

29. 67. [24, 48] $(10a + b) - ab = 12$ implies $(10 - b)(a - 1) = 2$ and so the numbers are 39 and 28.

30. $12 - 8\sqrt{2}$. [10, 1] Let r denote the desired radius. The little 45-45-90 triangle in the top right corner with hypotenuse going from the center of the small circle to the top right vertex of the square and one leg along the top edge of the square implies $4\sqrt{2} - 4 - r = r\sqrt{2}$. (This is two ways of viewing the length of that hypotenuse.) Thus $r(\sqrt{2} + 1) = 4(\sqrt{2} - 1)$ and so $r = 4(2 + 1 - 2\sqrt{2})/(2 - 1)$.

31. 12. [10, 0] $2^8 + 2^{11} + 2^n = k^2$ implies $2^n = k^2 - (1+8)2^8 = k^2 - (3 \cdot 2^4)^2 = (k + 3 \cdot 2^4)(k - 3 \cdot 2^4)$. Thus there exist integers $a > b > 0$ such that $a + b = n$, $2^a = k + 3 \cdot 2^4$, and $2^b = k - 3 \cdot 2^4$. Hence $2^b(2^{a-b} - 1) = 3 \cdot 2^5$, from which we deduce $b = 5$ and $a = 7$. **Alternatively**, $1 + 2^u + 2^v$ with $0 < u < v$ is a perfect square if and only if $v = 2u - 2$. Then $2^8 + 2^{11} + 2^n$ must equal $2^{2i}(1 + 2^u + 2^{2u-2})$. Then $2i + u = 11$, $2i = 8$, so $2i + 2u - 2 = 12$.

32. 290/17 or 17 1/17. [6, 1] Let h denote the width of a strip, y the length of the leg of a white triangle emanating from a vertex of the square, and x the shorter leg of a white triangle. Then similar triangles yield $y/5 = x/3$, $y/3 = h/2$, and $(y - x)/3 = 2/(x + y + h)$. We obtain $y = \frac{5}{3}x$, then $h = \frac{10}{9}x$, then $2x/9 = 2/(\frac{34}{9}x)$ so $x = 9/\sqrt{34}$. Hence the shaded area is $10 + 10 - h^2 = 20 - 100/34 = 290/17$.

33. 4561. [4, 0] $f(94) = 94^2 - f(93) = 94^2 - 93^2 + f(92)$. Continuing this

way yields

$$
\begin{aligned}
f(94) &= 94^2 - 93^2 + \cdots + 22^2 - 21^2 + 20^2 - 94 \\
&= (94+93)(94-93) + \cdots + (22+21)(22-21) + 400 - 94 \\
&= 94 + 93 + \cdots + 22 + 21 + 306 \\
&= \tfrac{1}{2}(95 \cdot 94 - 21 \cdot 20) + 306 = 4561
\end{aligned}
$$

34. 22.5. [8, 2] We have $\cos(2\theta) = 2\cos^2(\theta) - 1 = 2(\tfrac{1}{4}(2+\sqrt{2})) - 1 = \tfrac{1}{2}\sqrt{2}$. We deduce 2θ equals 45 degrees.

35. $\tfrac{1}{5}\sqrt{10}$. [6, 0] We desire the probability that $y\sqrt{100 - y^2} > 30$. Equality is obtained if $y^2(100 - y^2) = 900$ and so $y^4 - 100y^2 + 900 = 0$. Hence $y^2 = 10$ or 90. The desired probability is $(\sqrt{90} - \sqrt{10})/10 = (3\sqrt{10} - \sqrt{10})/10 = \tfrac{1}{5}\sqrt{10}$.

36. 3. [12, 10] It is $\sum_{i,j} \frac{1}{2^i 3^j} = \sum_i \frac{1}{2^i} \sum_j \frac{1}{3^j} = \frac{1}{1-1/2} \frac{1}{1-1/3} = 3$.

37. 5. [7, 14] For $x, y \neq 0$, the equation reduces to $2(x + y) = xy$ or $(x-2)(y-2) = 4$. Thus $(x-2, y-2)$ equals $(2,2)$, $\pm(4,1)$, or $\pm(1,4)$. Note that it cannot equal $(-2,-2)$ since that corresponds to $(x, y) = (0,0)$.

38. 8. [12, 4] Writing $i^2 - 1 = (i-1)(i+1)$ and removing the square $(i+1)^2$ that occurs in $(i^2 - 1)((i+2)^2 - 1)$, we are asking for the smallest $n > 1$ for which $2n(n+1)$ is a perfect square. Here you can just try until you find that it works at $n = 8$. You can make the numbers a little smaller by letting $n = 2a$ or $2a + 1$ and removing the 2^2.

39. 842401. [0, 0] We have $(649 + 180\sqrt{13})(649 - 180\sqrt{13}) = 1$. Expand the squares of both factors, obtaining $(842401 + 233640\sqrt{13})(842401 - 233640\sqrt{13}) = 1$. Hence $a = 842401$ and $b = 233640$ works.

40. 7 or 7:1. [1, 0] Let α denote the area of the small triangles, β the area of the quadrilaterals, and γ the area of the desired small equilateral triangle, as indicated below. Since the area of triangle CFB is twice

the area of CAF, we obtain $2(2\alpha+\beta) = 2\beta+\alpha+\gamma$, and hence $\gamma = 3\alpha$. The Law of Cosines says $CF^2 = AF^2+(3AF)^2-2\cdot AF\cdot(3AF)\cos(60) = 7AF^2$. Since triangle ACF is similar to the little triangle in the lower left, their areas are proportional to $CF^2 : AF^2 = 7$. Thus $2\alpha+\beta = 7\alpha$ and so $\beta = 5\alpha$. The desired ratio is $(\gamma + 3\alpha + 3\beta)/\gamma = (3\alpha + 3\alpha + 15\alpha)/3\alpha = 7$.