1. $\frac{1}{2} + \frac{1}{4} - \frac{1}{3} =$

2. If the diagonal of a rectangle is 169 and one side s is 156, what is the length of a side perpendicular to s?

3. If the reciprocal of $x + 1$ is $x - 1$, then $x =$

4. A square whose area is 16 is partitioned into four congruent smaller squares. What is the area of the circle which passes through the centers of the four subsquares?

5. Let $a = 2^{1000}$, $b = 3^{600}$, and $c = 10^{300}$. Arrange a, b, and c in increasing order. For example, if $a < b < c$, write a, b, c.

6. Define a new binary operation, called $*$, on the real numbers by the formula $x * y = x + y + 5$. What real number e satisfies $x * e = x$ for all x?

7. What is the largest integer k such that $n^3 - n$ is divisible by k for all positive integers n?

8. When 270 is divided by the odd number n, the quotient is a positive prime number and the remainder is 0. What is n?

9. Let $x = 1/2$ (in radians). What is the value of $\log_3(\tan x) - \log_3(\sin x) + \log_3(\cos x)$?

10. Write an equation expressing a in terms of just b and c if $\frac{1}{x} + \frac{1}{y} = \frac{1}{a}$, $x + y = b$, and $x^2 + y^2 = c^2$.

11. A triangle has vertex A at $(0, 3)$, vertex B at $(4, 0)$, and vertex C at $(x, 5)$ for some x between 0 and 4. If the area of the triangle is 8, then what is the value of x?

12. Find the only real solution of $x^3 + 8x^2 + 12x - 385 = 0$.

13. There are two noncongruent rectangles with integer sides for each of which the area equals the perimeter. What is the sum of the areas of the two rectangles?

14. If $\displaystyle\sum_{k=1}^{n} g(k) = \frac{n}{3n-2}$, then write a simple expression for $g(k)$.

15. The numbers from 1 to 2002 are listed in the following order: First all numbers which are not divisible by 3 are listed in (increasing) order. Then all numbers which are divisible by 3 but not by 3^2 are listed in order. Then all numbers which are divisible by 3^2 but not by 3^3 are listed in order, etc. What is the last number in this list? (Give the entire number, not just its last digit.)

16. A lottery gives prizes which are powers of 11. Thus the possible prizes are 1, 11, 121, 1331, 14641, and 161051 dollars. They will give no more than 10 awards of any particular amount. They give out exactly $1,111,111. How many prizes do they award?

17. A circle passes through two adjacent vertices of a square and is tangent to the opposite side of the square. If the side length of the square is 2, what is the radius of the circle?

18. Two numbers are such that their difference, their sum, and their product are to each other as 1:7:24. Their product must equal what number?

19. A drawer contains n socks. When two are drawn randomly without replacement, the probability that both are red is $5/14$. What is the smallest possible value of n?

20. At how many minutes past 8:00 will the minute and hour hands form equal angles with the vertical for the first time?

21. How many positive integers are divisors of 2002?

22. Write a quadratic polynomial having as its two roots the numbers obtained by increasing each root of $x^2 - 2x - 5 = 0$ by the reciprocal of the other.

23. In racing over a given distance at uniform speed, Al can beat Bob by 20 meters, Bob can beat Chris by 10 meters, and Al can beat Chris by 28 meters. How many meters is the distance over which they are racing?

24. The number n has 2002 digits, all of which are 2. What is the greatest common divisor of n and 1111?

25. A regular polygon of n sides inscribed in a circle of radius r has area $3r^2$. What is the value of n?

26. What is the largest number of 0's that can occur at the end of $1^n + 2^n + 3^n + 4^n$ for any positive integer n?

27. Each side of triangle ABC has length 2. A circle with center at A and radius 1 cuts AB at M. A tangent to the circle from B and lying outside the triangle meets the circle at P. What is the area of the region bounded by BP, BM, and the minor arc MP?

28. What is the largest number of distinct positive integers such that no four of them have sum divisible by 4?

29. Chris and Dick take turns, with Chris going first. Chris tosses a fair coin at each of her turns, while Dick rolls a fair die at each of his turns. The game ends when either Chris tosses a Head, in which case Chris wins, or Dick rolls a 3, in which case Dick wins. What is the probability that Dick wins?

30. Suppose $a_1, \ldots, a_{2002}$ is a sequence of 2002 numbers, each of which equals 1 or -1. What is the smallest possible value of the sum $\sum_{i<j} a_i a_j$? (This is the sum of all products of pairs of distinct numbers in the sequence.)

31. If $a_0 = 1$, $a_1 = 3$, $a_2 = 7$, and $a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n$ for $n \geq 0$, then what is a general formula for a_n? (Your answer will be in terms of n.)

32. What is the largest integer n for which there exist points $P_1, \ldots, P_n$ in the plane and positive real numbers $r_1, \ldots, r_n$ such that the distance between P_i and P_j is $r_i + r_j$ for all $i \neq j$?

33. What is the number of solutions (x, y), with x and y nonnegative integers, of the equation $4^x - 9^y = 55$?

34. Suppose the parabola $y = x^2 + px + q$ is situated so that it has two arcs lying between the rays $y = x$ and $y = 2x$, $x \geq 0$. These two arcs are projected onto the x-axis, yielding segments s_L and s_R, with s_R to the right of s_L. What is the difference of the lengths $\ell(s_R) - \ell(s_L)$?

35. In triangle ABC, D is the midpoint of AB, while E lies on BC satisfying $BE = 2EC$. If $\angle ADC = \angle BAE$, then how many degrees are in $\angle BAC$?

36. If $2f(x) + f(1 - x) = x^2$ for all x, then $f(x) =$

37. What are the real roots of the equation $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$?

38. Let $f_1(x) = (x+7)^2(x^2+1)^3(x^5-4)^2$. For each $n > 1$, let $f_n(x) = f_{n-1}(x+1) - f_{n-1}(x)$. What is the smallest positive integer n such that $f_n(1) = f_n(2) = f_n(3) = \cdots = f_n(24)$?

39. A travel agency sells tickets to 100 destinations. If 1800 people buy tickets, what is the smallest number d such that there must be at least d destinations receiving the same number of these travelers?

40. Find the smallest possible value of $(x^2 + y^2)/(x - y)$ for all real numbers x and y satisfying $x > y > 0$ and $xy = 3$. (One method of doing this uses the Arithmetic-Geometric Mean Inequality, which states that if $a, b \geq 0$, then $\frac{a+b}{2} \geq \sqrt{ab}$ and equality holds if and only if $a = b$.)

SOLUTIONS TO 2001 EXAM

The number of people (out of 181) who got the problem right, and the number of the top 21 finishers who got the problem right, are listed in brackets after the correct answer.

1. 5/12. [175,21] It is $(6+3-4)/12$.

2. 65. [101.5,18] It is $13\sqrt{13^2-12^2} = 13\cdot 5$.

3. $\pm\sqrt{2}$. [99.5,14.5] $1/(x+1) = x-1$. Thus $x^2-1=1$. Thus $x^2=2$.

4. 2π. [116,17]Place the vertices of the big square at $(\pm 2, \pm 2)$. Then the circle is centered at (0,0) and passes through $(\pm 1, \pm 1)$. Thus the radius of the circle is $\sqrt{2}$.

5. b, c, a. [65,17] a, b, and c are the 100th powers of 1024, 729, and 1000, respectively.

6. -5. [162,21] e must satisfy $x+e+5=x$.

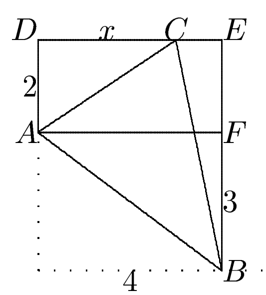
7. 6. [69,14] Since $2^3-2=6$, we must have $n\le 6$. Since $n^3-n=(n-1)n(n+1)$ is the product of three consecutive integers, of which at least one must be even and one a multiple of 3, n^3-n is always divisible by $2\cdot 3$.

8. 135. [130,19] Since $270 = 2\cdot 3^3\cdot 5$, the only way to write it as the product of a prime times an odd number is as $2\cdot 135$.

9. 0. [73,18] It is $\log_3(\tan x\cdot\cos x/\sin x) = \log_3(1) = 0$.

10. $a=(b^2-c^2)/(2b)$. [39,16] $b^2=(x+y)^2=c^2+2xy$. Thus $xy=(b^2-c^2)/2$. Now $a=xy/(x+y)=(b^2-c^2)/(2b)$.

11. 8/3. [17,8] In the diagram below the area of the triangle is $AFED + AFB - ADC - CEB = 8+6-x-\frac{5}{2}(4-x)$. If this equals 8, then $\frac{3}{2}x=4$.



12. 5. [110,18] Look first for integer solutions. They must be divisors of $385 = 5\cdot 7\cdot 11$. Find that 5 works. To see that this is the only solution, use division of polynomials to see that $x^3+8x^2+12x-385 = (x-5)(x^2+13x+77)$, and $x^2+13x+77$ is never 0.

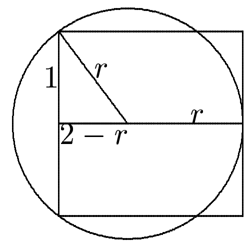
13. 34. [96,17] They are 3×6 and 4×4. They can be found by finding integer solutions of $L=2W/(W-2)$.

14. [1,1] $-2/((3k-2)(3k-5))$. Note that $g(n) = \frac{n}{3n-2} - \frac{n-1}{3(n-1)-2} = \frac{n(3n-5)-(n-1)(3n-2)}{(3n-2)(3n-5)} = -2/((3n-2)(3n-5))$. The desired answer is obtained by replacing n by k.

15. 1458. [49,12] The powers of 3 are 3, 9, 27, 81, 243, 729, 2187, etc. The last number in the list will be $2 \cdot 729$, since 279 is the highest 3-power less than or equal to 2002, and $2 \cdot 279$ is the highest multiple of 729 less than or equal to 2002.

16. 41. [37,8] $1111111 - 6 \cdot 161051 = 144805$. $144805 - 9 \cdot 14641 = 13036$. $13036 - 9 \cdot 1331 = 1057$. $1057 - 8 \cdot 121 = 89$. $89 - 8 \cdot 11 = 1$. Thus the number of awards is $6+9+9+8+8+1$.

17. 5/4. [21,13] The right triangle in the diagram below shows that $r^2 = 1 + (2-r)^2$, which implies $4r = 5$.



18. 48. [98,16] From $a - b = \frac{a+b}{7}$, we obtain $6a = 8b$. Substituting this in $24(a-b) = ab$ yields $(32-24)b = ab$. Thus $a = 8$ and then $b = 6$.

19. 8. [29,11] If r is the number of red socks, then the probability that both are red is $r(r-1)/(n(n-1))$. For this to equal 5/14, either n or $n-1$ must be divisible by 7. If $n = 7$, then $r(r-1) = 15$, which is not satisfied by an integer r. If $n = 8$, then $r(r-1) = 20$, so $r = 5$ works.

20. $18 + \frac{6}{13}$ or $10 + \frac{10}{11}$. [20,12] If it is x minutes before 8:20, then we must have $x = (20-x)/12$ and hence $x = 20/13$. The other solution allows the hands to be in a straight line. Then the number of minutes after 8:10 satisfies $x = (10+x)/12$.

21. 16. [39,13] $2002 = 2 \cdot 7 \cdot 11 \cdot 13$. Each factor can have exponent 0 or 1 in a divisor, so the answer is $2 \cdot 2 \cdot 2 \cdot 2$.

22. $x^2 - \frac{8}{5}x - \frac{16}{5}$. [8,3] If a and b were the roots of $x^2 - 2x - 5 = 0$, then we want the polynomial $(x - a - \frac{1}{b})(x - b - \frac{1}{a}) = x^2 - (a + b + \frac{1}{a} + \frac{1}{b})x + (ab + 2 + \frac{1}{ab})$. We have $a + b = 2$ and $ab = -5$. Since $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$, we obtain $x^2 - (2 - \frac{2}{5})x + (-5 + 2 - \frac{1}{5})$.

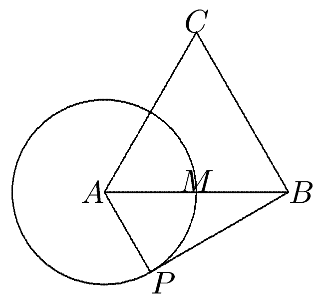
23. 100. [30,11] If the distance is d, and their speeds are s_A, s_B, and s_C, respectively, then $\frac{s_A}{s_B} = \frac{d}{d-20}$, etc. Since $\frac{s_A}{s_B}\frac{s_B}{s_C} = \frac{s_A}{s_C}$, we obtain $\frac{d}{d-20}\frac{d}{d-10} = \frac{d}{d-28}$. Cancel d and cross-multiply to obtain $d(d-28) = (d-10)(d-20)$, which simplifies to $2d = 200$.

24. 11. [46,14] $1111 = 11 \cdot 101$, and 101 is prime. The large number n equals $2 \cdot 11$ times a number with alternating 1's and 0's. But n is not divisible by 1111, since $n/1111$ begins 200 followed by 499 0200's with a remainder of 22.

25. 12. [12,7] We have $3r^2 = n \cdot \frac{1}{2}r^2 \sin(360/n)$, where the argument of sin is in degrees. So $\frac{6}{n} = \sin(360/n)$. Clearly $n = 12$ works.

26. 2. [43,14] $1^3 + 2^3 + 3^3 + 4^3 = 100$. For $n \geq 3$, the mod 8 value of $1^n + 2^n + 4^n$ is 1, while that of 3^n is 1 or 3. Thus $1^n + 2^n + 3^n + 4^n$ cannot be divisible by 8, and hence cannot be divisible by 1000.

27. $\sqrt{3}/2 - \pi/6$. [32,12] In the diagram below, we want the area of triangle ABP minus the area of the wedge AMP. Triangle ABP is a right triangle with hypotenuse 2 and one leg 1, so has area $\sqrt{3}/2$. This triangle shows that angle PAB is 60 degrees, and so the wedge has area $\pi/6$.



28. 6. [10,6] We just work with the mod 4 values, since distinct integers with any prescribed set of mod 4 values can always be found. $\{0,0,0,1,1,1\}$ works. Suppose a set of seven 0's, 1's, 2's, and 3's has no four summing to 0 mod 4. If at least four are even, then they must be 0,0,0,2 or 0,2,2,2, and there cannot be five even. The three odds cannot have both a 1 and a 3, for then 1,3,0,0 or 1,3,2,2 sum to 0 mod 4, so they would have to be 1,1,1 or 3,3,3, but then 1,1,2,0 or 3,3,2,0 sum to 0 mod 4. A similar argument, with all mod 4 values increased by 1, handles the case when four are odd.

29. 1/7. [13,10] Letting N denote the event that a number different from 3 is rolled, then Dick wins on the following sequences: T3, TNT3, TNTNT3, ... The probabilities of these are, respectively, $\frac{1}{12}$, $\frac{1}{12}\frac{5}{12}$, $\frac{1}{12}(\frac{5}{12})^2, \ldots$. The sum is $\frac{1}{12}/(1-\frac{5}{12}) = 1/(12-5)$.

30. -1001. [33,5] $(a_1 + \cdots + a_{2002})^2 = \sum a_i^2 + 2\sum_{i<j} a_i a_j$. It will always be true that $\sum a_i^2 = 2002$. Thus our desired sum equals $\frac{1}{2}(\sum a_i)^2 - 1001$. Its smallest possible value will occur when $\sum a_i = 0$.

31. $n^2 + n + 1$. [15.5,8.5] If you compute the next few terms, you can observe that the successive differences increase linearly. (You can then prove this, if you wish.) Thus the formula will be quadratic. If $a_n = An^2 + Bn + C$, then $C = 1$, $A + B + C = 3$, and $4A + 2B + C = 7$, from which one deduces $A = B = C = 1$.

32. 4. [4,1] Circles centered at P_i with radius r_i would be pairwise externally tangent to each other, and this can only happen to at most 4 circles.

33. 1. [61,13] $(2^x - 3^y)(2^x + 3^y) = 55$ implies either $2^x - 3^y = 5$ and $2^x + 3^y = 11$, or $2^x - 3^y = 1$ and $2^x + 3^y = 55$. The first case says $2^x = 8$ and $3^y = 3$, while the second case says $2^x = 28$ and $3^y = 27$. The first yields $(x, y) = (3, 1)$, while the second does not give an integer value for x.

34. 1. [14,2] Let $a < b$ be the roots of $x = x^2 + px + q$, and $c < d$ be the roots of $2x = x^2 + px + q$. We desire $(d - b) - (a - c) = (d + c) - (a + b) = (2 - p) - (1 - p) = 1$.

35. 90. [9,1] Let G denote the point of intersection of AE and CD, and let F be the midpoint of BE. Then $AG = GD$. Triangles GEC and DFC are similar, hence $CG = GD = AG$. Thus $\angle GCA = \angle GAC$, and so $\angle DAC = \angle DAG + \angle GAC =$

$\angle GDA + \angle GCA$, and the two sides of this equation sum to 180, since they are the angles of a triangle. Thus $\angle DAC = 90$.

36. $(x^2+2x-1)/3$. [5,4] Replace x by $1-x$ to get $2f(1-x)+f(x)=(1-x)^2$. Subtract this equation from 2 times the original equation to deduce $3f(x)=2x^2-(1-x)^2$.

37. $-9 \pm \sqrt{61}$. [1,1] Let $y=\sqrt{x^2+18x+45}$. Then $y^2-15=2y$, so $y=5$ or -3, but y cannot be negative. Thus $x^2+18x+45=25$, yielding the claimed result.

38. 19. [0,0] First note that if a polynomial $f(x)$ has leading term $a_k x^k$, then $f(x+1)-f(x)$ has leading term $ka_k x^{k-1}$. Since f_1 has degree 18, f_n will have degree $19-n$. Next note that the f_n in the conclusion must satisfy that the polynomial $f_n(x)-f_n(1)$ has at least 24 roots, and so must be identically 0, since it will not have degree ≥ 24. Thus we must have $n=19$ so that f_n is constant.

39. 3. [8,2] Suppose $a_1 \leq \cdots \leq a_{100}$ represents the number of travelers going to the various destinations. If no three a_i are equal, then the sequence $(a_1, \ldots, a_{100})$ is, in each position, $\geq$ the sequence $(0,0,1,1,\ldots,49,49)$, which has 2450 travelers. Thus it is impossible that no three destinations have the same number of travelers. However, the sequence $(0,0,0,1,1,1,\ldots,32,32,32,216)$, which sums to 1800, shows it is possible that no more than three destinations have the same number of travelers.

40. $2\sqrt{6}$. [0,0] Write $\frac{x^2+y^2}{x-y}=x-y+\frac{2xy}{x-y}$. Apply the AGMI with $a=x-y$ and $b=\frac{2xy}{x-y}$ to deduce $\frac{x^2+y^2}{x-y} \geq 2\sqrt{ab}=2\sqrt{6}$. To see that equality can be obtained, we must find x and y as above so that $a=b$. Thus we wish to solve $x-y=\frac{2xy}{x-y}$ and $xy=3$. This can be reduced to $x^2+\frac{9}{x^2}=12$, and hence $x=\sqrt{6\pm\sqrt{27}}$. Choose $x=\sqrt{6+\sqrt{27}}$ in order to have $x>y>0$.