2017 LEHIGH UNIVERSITY HIGH SCHOOL MATH CONTEST

1. What is the average of 3 and $\frac{1}{3}$?
2. Triangle ABC has a right angle at C, and $\angle A = 20^{o}$. If D lies on AC, and BD bisects angle B, how many degrees are in $\angle BDA$?
3. If $2a = 3b + 5$, what is $4^a/8^b$?
4. Twelve distinct points on a circle are connected in all possible ways by chords. How many chords are there?
5. Which of the following graphs pass through exactly two of the points $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$? (a) $x^2 + y^2 = 1$; (b) $y = x + 1$; (c) $y = 1 - x^2$; (d) $y = x$; (e) $y = 1/x$. Write the letters; for example, b, c.
6. Suppose $f(x) = x^4 - x^3 + ax + b$, $f(1) = 4$, and $f(2) = 6$. What is the ordered pair (a, b)?
7. Jo can mow a lawn in 40 minutes. Chris can mow the same lawn in 30 minutes. How many minutes will it take the two of them to mow it if both are mowing at the same time (and of course they mow without any duplication)?
8. Square $ABCD$ has sides of length 3. Side AB is extended through B to E with $BE = 1$. Segment DE intersects BC at point F. What is the area of triangle CDF?
9. What is the smallest positive integer which is divisible by all 1-digit primes but has no prime digits?
10. How many positive real solutions does the equation $x^3 + 9x^2 + 15x - 21 = 0$ have?
11. Define a function f on the set of integers by

$$f(n) = \begin{cases} n - 1 & \text{if } n \text{ is even} \\ n^2 - 1 & \text{if } n \text{ is odd.} \end{cases}$$

List all values of n for which $f(f(n)) = 8$.

12. Let E be the event that in two flips of a fair coin one is Head and one Tail. Let F be the event that in four flips of a fair coin two are Heads and two are Tails. Let G be the event that in five flips of a fair coin three are Heads and two are Tails. List these events (E, F, and G) in order of increasing probability. (Smallest to largest)

13. A rectangular solid, or cuboid, with sides of lengths 2, 10, and 22 is inscribed in a sphere. What is the side length of the cube that can be inscribed in that sphere?

14. For each way of arranging five points inside an equilateral triangle of side length 2, let k be the number of pairs of these points at distance ≤ 1 from one another. What is the smallest possible value of k?

15. For how many positive real values of x does $\log_4 x = 2\sin x$?

16. In triangle ABC point F lies on AC with the ratio $AF{:}FC = 2{:}3$. Point D is the midpoint of BF, and AD is extended to meet BC at point E. What is the ratio $BE{:}EC$?

17. The 4-digit base-6 number $abcd$ with $a > 0$ and d odd is a perfect square. List all possible values of c. (The letters are the digits of the base-6 number.)

18. What is the smallest number e for which there exist positive integers $a < b < c < d < e$ having the property that there is no quadrilateral with positive area having side lengths consisting of any four of the numbers a, b, c, d, and e?

19. If a and b are the solutions of $x^2 - 2x + 6 = 0$, what is the value of $a^4 + b^4$?

20. The product of two or more consecutive positive integers is $47AB74$; i.e., a 6-digit number beginning 47 and ending 74. List these consecutive integers having the desired product.

21. List all possible values for the area of a right triangle whose sides are in arithmetic progression and which has a side of length 60.

22. The distance between the centers of two circles is 15. One has radius 4, and the other has radius 5. What is the length of their common internal tangent?

23. Determine all positive real numbers x for which $\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$.

24. Find all ordered triples (a, b, c) which satisfy the equations

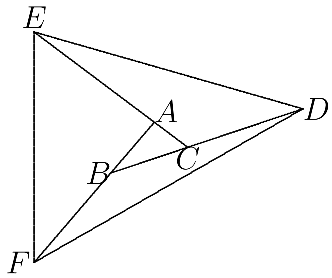
$$\begin{aligned} a^2 + 1 &= 3b + c \\ b^2 + 33 &= 7c - 3a \\ c^2 - 5 &= b - 3a. \end{aligned}$$

25. Arrange the numbers from 1 to 9 in the 3-by-3 grid below so that each number occurs once, the product of the entries in the first row is 12, product of entries in second row is 112, product of entries in the first column is 216, and product of entries in the second column is 12, as indicated. Write the ordered triple (A, B, C) of numbers corresponding to the indicated boxes.

A			12
	B		112
		C	
216	12		

26. If $\sin(x) + \cos(x) = \frac{1}{3}$, what is the value of $\sin^3 x + \cos^3 x$?

27. In the diagram below, which is not drawn to scale, $BD = 2BC$, $CE = 3CA$, and $AF = 4AB$. What is the ratio of the area of triangle DEF to that of triangle ABC?



28. What is the last (unit) digit of the smallest integer n for which $7n$ has 2017 digits?

29. How many 9-letter strings consisting entirely of the letters A and B do not contain the consecutive letters $ABBA$?

30. A circle of radius 1 and a circle of radius 3 in the upper half plane are tangent to each other and to the x-axis. The larger circle rolls around the smaller circle in the upper half plane until it is tangent on the other side to the little circle and to the x-axis. What is the area of the region passed through by all the points inside the circle as it rolls from one position to the other?

31. Circles of radius 5, 5, and 8 are tangent to one another. What is the radius of the circle which surrounds them, tangent to each?

32. Let $N = 1! \cdots 60!$ denote the product of the first 60 factorials. For which number k between 1 and 60, inclusive, is $N/k!$ a perfect square?

33. If two points are selected randomly and independently from the edges of a unit square, what is the probability that the distance between them is less than 1?

34. In how many ways can 530 be formed as the sum of numbers 1, 2, 4, 8, 16, 32, 64, and 128, allowing up to three occurrences of any of the eight numbers?

35. Find all pairs of integers (x, y) that satisfy $x^2 - y^4 = xy + 49$ and $y^2 = xy - 30$.

36. Define a sequence by $a_0 = 0$, $a_1 = 1$, and for $n \geq 2$, $6a_n = 11a_{n-1} - 5a_{n-2} + 11$. What integer is closest to a_{1000}?

37. For $n \geq 1$, let S_n be the set of the first n positive multiples of n. For example, $S_4 = \{4, 8, 12, 16\}$. What is the smallest integer that is contained in at least nine of the sets S_n?

38. What is the maximum value of $x^2 + 4xy - y^2$ for all (x, y) satisfying $x^2 + y^2 = 1$?

39. Let ABC be an isosceles right triangle with legs of length 2. Of all triangles having a vertex on each side of ABC with one of the vertices being the midpoint of a leg of ABC, there is one which has minimal perimeter. What is the length of its shortest side?

40. Find the minimum value of $x^2+6xy+9y^2+2z^2$ out of all positive numbers x, y, z which satisfy $xyz = \sqrt{3}$.

SOLUTIONS, annotated with the number of students (out of 55) who scored at least 22 and answered the question correctly.

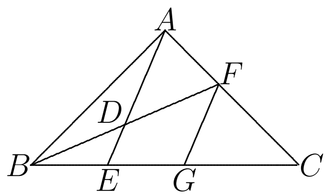
1. 5/3. [55] It is one half of 10/3.
2. 125. [54] Angle B is 70 degrees, so two angles in triangle BDA are 35 and 20. The answer is $180-35-20$.
3. 32. [54] $4^a/8^b = 2^{2a-3b} = 2^5$.
4. 66. [55] You can think of it as $\binom{12}{2} = 66$, or note that each of the 12 points has 11 chords coming from it, but each chord is counted twice in $12 \cdot 11$.
5. a,c,d. [55] (a) passes through (0,1) and (1,0). (b) passes through (0,1). (c) passes through (0,1) and (1,0). (d) passes through (0,0) and (1,1). (e) passes through (1,1).
6. $(-6, 10)$. [55] We have $4 = a+b$ and $6 = 8+2a+b$. Subtracting yields $a = -6$, and then $b = 10$.
7. 120/7. [55] In one minute they can do $\frac{1}{40} + \frac{1}{30}$ of the job. This is $\frac{3+4}{120}$. The number of minutes is the reciprocal of this.
8. 27/8. [53] By similar triangles, $BF/BE = 3/4$, so $BF = 3/4$ and $CF = 3 - \frac{3}{4} = \frac{9}{4}$. Thus the desired area is $\frac{1}{2} \cdot 3 \cdot \frac{9}{4} = \frac{27}{8}$.
9. 840. [54] It must be a multiple of $2 \cdot 3 \cdot 5 \cdot 7 = 210$. 210, 420, and 630 all have a prime digit.
10. 1. [48] The function is clearly increasing for positive values of x, so it has at most one positive solution. Since $f(0) = -21$ and $f(1) = 4$ it has a solution between 0 and 1.
11. 4 and -2. [48] For $f(m)$ to equal 8, we must have $m = \pm 3$. For $f(n)$ to equal ± 3, we must have $n = 4$ or -2.
12. G, F, E. [51] $\Pr(E) = \binom{2}{1}/2^2 = 1/2$. $\Pr(F) = \binom{4}{2}/2^4 = 6/16$. $\Pr(G) = \binom{5}{2}/2^5 = 10/32$.
13. 14. [47] The diagonal of the cuboid will be a diameter of the sphere, and has length $\sqrt{2^2 + 10^2 + 22^2} = \sqrt{588}$. This will also

have to be the diagonal of the cube, and so $3s^2 = 588$, hence $s = 14$.

14. 1. [46] By connecting the midpoints of its sides, the triangle can be partitioned into four equilateral triangles of side 1. Since there are five points and four regions, there must be at least one pair of points lying inside one of those regions, and hence $k \geq 1$. By choosing two points very close to the centroid of the triangle, and one point close to each of its vertices, we obtain a configuration with $k = 1$. Here we use the fact that the distance from the centroid to the vertices is $\frac{2}{3}\sqrt{3} > 1$.

15. 5. [43] Since $\log_4 x > 2$ when $x > 16$, the only possible solutions occur for $0 < x \leq 16$. The hump of $2\sin x$ for $0 \leq x \leq \pi$ will intersect the curve $y = \log_4 x$ once; the hump between 2π and 3π will intersect it twice; and the hump between 4π and 5π will intersect it twice.

16. 2:5. [45] Drop a line from F parallel to AE, meeting BC at G. One pair of similar triangles implies $BE = EG$, while another implies $EG : GC = 2 : 3$. Thus $BE : EC = 2 : 5$.



17. 0, 1, 2, 4. [37] We are basically just looking at the 6's digit in $(6B \pm 1)^2$ and in $(6B + 3)^2$. The first can give 0, 2, or 4 in the 6-place, while the second must give 1. One can check that 19^2, 21^2, 25^2, and 31^2 have four digits in base 6, with 6-digit 0, 1, 2, and 4, respectively.

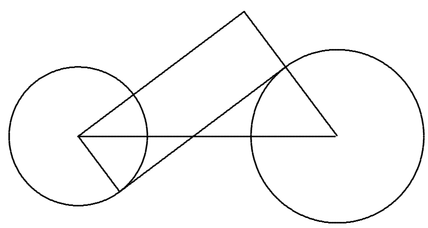
18. 11. [38] There exists a quadrilateral with positive area and side lengths $\alpha < \beta < \gamma < \delta$ if and only if $\delta < \alpha + \beta + \gamma$. To make the numbers as small as possible, we use 1, 2, 3, 6, and 11, since $6 = 1 + 2 + 3$ and $11 = 2 + 3 + 6$.

19. -8. [49] By Vieta, we have $a+b=2$ and $ab=6$. Then $a^2+b^2=(a+b)^2-2ab=4-12=-8$. Now $a^4+b^4=(a^2+b^2)^2-2(ab)^2=(-8)^2-72=-8$.

20. 77, 78, 79. [41] The product of two consecutive integers never ends in a 4, so there must be at least three integers in the product. Since the product is 2 mod 4, there cannot be two even integers in the product, so it must be a product of three numbers congruent to 1, 2, and 3 mod 4, respectively. $81\cdot 82\cdot 83$ is too large and ends in the wrong digit, while $73\cdot 74\cdot 75$ is too small and ends in the wrong digit. So it must be $77\cdot 78\cdot 79=474474$.

21. 864, 1350, and 2400. [49] If d is the common difference and x the middle side, then $(x-d)^2+x^2=(x+d)^2$, so $x=4d$ and $A=6d^2$. The sides are $3d$, $4d$, and $5d$, so $d=20$, 15, or 12, and A is as claimed.

22. 12. [44] Let P be the point of intersection of the line connecting their centers with the common internal tangent. Let x denote the distance from P to the center of the larger circle. Similar triangles yields $\frac{x}{5}=\frac{15-x}{4}$ and so $x=5\cdot 5/3$, while the distance from P to the smaller circle is $4\cdot 5/3$. The distances from P to the points of tangency are $\sqrt{(5\cdot 5/3)^2-5^2}$ and $\sqrt{(4\cdot 5/3)^2-4^2}$. The sum of these lengths is $(5+4)\sqrt{(5/3)^2-1}=3\sqrt{5^2-3^2}=12$. An alternate solution uses the right triangle below with hypotenuse 15 and one leg $5+4$.



23. 8 and $2^{-2/3}$. [33] We have $\log_4 x=\log_x 2+\frac{7}{6}$, so, letting $y=\log_2 x$, $\frac{y}{2}=\frac{1}{y}+\frac{7}{6}$. This quadratic equation has solutions $y=3$ and $y=-\frac{2}{3}$. Thus $x=8$ or $2^{-2/3}$.

24. $(-3, 2, 4)$. [21] Adding the equations and completing the squares reduces to $(a+3)^2 + (b-2)^2 + (c-4)^2 = 0$, from which the solution is immediate.

25. $(3, 2, 5)$. [53] The answer is given below. We can first fill in the 7 and the 5, and then the 4 since the last column needs $7 - 5$ powers of 2. Then we can fill in the 8 and 2 in the second row. Then the 6 and 1 in the second column, and finally the 3 and 9 in the first column.

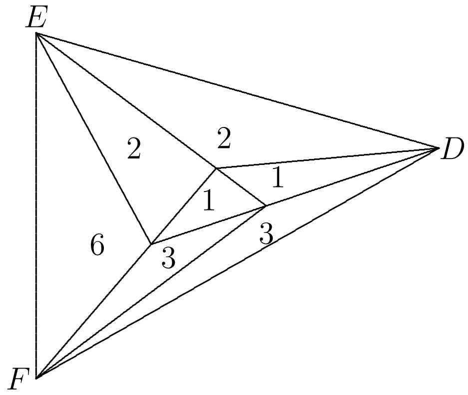
3	1	4	$2^2 \cdot 3$
8	2	7	$2^4 \cdot 7$
9	6	5	
$2^3 3^3$	$2^2 3$		

26. $13/27$. [47] First note that $\frac{1}{9} = (\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x$, and so $\sin x \cos x = \frac{1}{2}(\frac{1}{9} - 1) = -\frac{4}{9}$. Now

$$\begin{aligned} \tfrac{1}{27} &= (\sin x + \cos x)^3 \\ &= \sin^3 x + \cos^3 x + 3(\sin x + \cos x) \sin x \cos x, \end{aligned}$$

and so $\sin^3 x + \cos^3 x = \frac{1}{27} - 3 \cdot \frac{1}{3}(-\frac{4}{9}) = 13/27$.

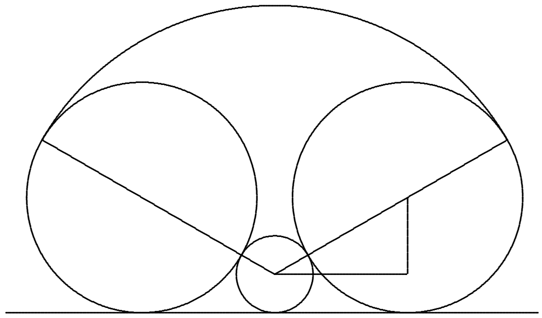
27. 18. [35] We use the fact that the area of a triangle is $\frac{1}{2}$ the product of two sides and the sine of the angle between them, and that $\sin(180 - \alpha) = \sin(\alpha)$. Thus the areas of triangles AEF, BDF, and CED are, respectively, 8, 6, and 3 times the area of triangle ABC. The ratio is $8 + 6 + 3 + 1$. Another method uses the fact that triangles with the same altitude have areas proportional to their bases. If we set the area of ABC to 1, then we can fill in the areas of other parts successively as in the diagram below.



28. 8. [33] We need $n \geq 10^{2016}/7$. Since $2016 = 6 \cdot 336$, this decimal number has 336 batches of 142857 before the decimal point, and continuing on past the decimal point. The smallest integer greater than that will end 142858.

29. 338. [19] The total number of strings is $2^9 = 512$. We count the number containing at least one ABBA. An ABBA may start in any of the first 6 positions, and the other positions can be filled in $2^5 = 32$ ways. These 192 sequences include some that are counted twice. Some have overlapping ABBA's. These must be of the form xyABBABBA, xABBABBAy, or ABBABBAxy. Since x and y can be A or B, there are 12 of these. There are also six sequences of the form xABBAABBA, ABBAxABBA, or ABBAABBAx which are counted twice. Altogether 18 sequences are counted twice. Thus $192 - 18 = 174$ sequences contain an ABBA, and so $512 - 174 = 338$ do not.

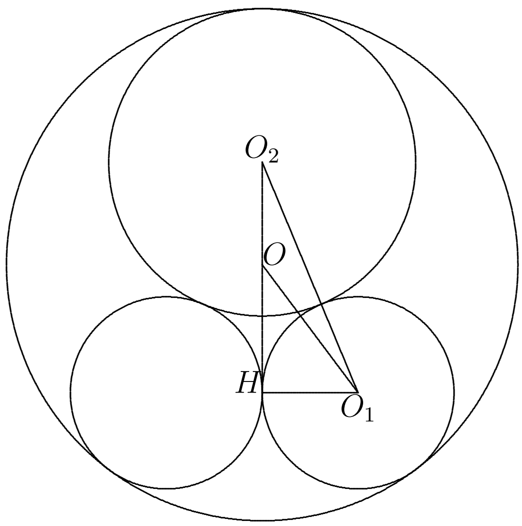
30. 25π. [15] In the diagram below, the desired region includes the two circles of radius 3, and the portion of the sector which lies outside the small circle. The portions of the two circles which lie outside the sector have total area 9π, while the area in the sector and outside the small circle has area $\frac{1}{3}\pi(7^2 - 1^2) = 16\pi$. Here we have used that the angle in the sector is 120 degrees, since the indicated triangle has height 2 and hypotenuse 4.



31. 40/3. [23] Let O_1 be the center of one of the radius-5 circles, and O_2 the center of the radius-8 circle. Let O be the center of the surrounding circle, with radius r. Let H be the point of tangency of the radius-5 circles. Then $O_1O_2 = 13$ and $O_1H = 5$, so $O_2H = 12$. Now $OO_2 = r - 8$, and $OO_1 = r - 5$. In right triangle OO_1H,

$$(r-5)^2 = 5^2 + (12 - (r-8))^2,$$

which simplifies to $30r = 400$.



32. 30. [18] Note that $N = (1!)^2 \cdot 2 \cdot (3!)^2 \cdot 4 \cdots (59!)^2 \cdot 60 = 30! \cdot (2^{15} 1! \cdot 3! \cdots (59!))^2$, so $N/30!$ is a perfect square. To see that

nothing else works, note that for $31 \le k \le 60$, $N/k!$ has an odd power of 31, and for $k \le 28$, $N/k!$ has an odd power of 29. Finally, $N/29! = N/30! \cdot 30$, which isn't a perfect square if $N/30!$ is.

33. $\frac{1}{4} + \frac{\pi}{8}$. [17] We may assume that the first point is along the bottom edge. If the second point is on the bottom edge, which happens 1/4 of the time, the distance between them is less than 1. (Note that having distance exactly equal to 1 occurs with probability 0.) If the second point is on the top row, the distance between them is greater than (or equal to) 1. If the second point is on either side, the the probability that the distance between them is less than 1 equals the fraction of the area in the unit square in the first quadrant which satisfies $x^2 + y^2 < 1$, which is $\pi/4$. As this happens with probability 1/2, we obtain as our answer $\frac{1}{4} + \frac{1}{2} \cdot \frac{\pi}{4}$.

34. 118. [1] The answer is the coefficient of x^{530} in the polynomial

$$
\begin{aligned}
& (1+x+x^2+x^3)(1+x^2+x^4+x^6)\cdots(1+x^{128}+x^{256}+x^{384}) \\
= & \frac{x^4-1}{x-1} \cdot \frac{x^8-1}{x^2-1} \cdot \frac{x^{16}-1}{x^4-1} \cdots \frac{x^{512}-1}{x^{128}-1} \\
= & \frac{x^{256}-1}{x-1} \cdot \frac{x^{512}-1}{x^2-1} \\
= & (1+x+x^2+\cdots+x^{255})(1+x^2+x^4+\cdots+x^{510}).
\end{aligned}
$$

We obtain x^{530} as $x^{2i}x^{530-2i}$ for $10 \le i \le 127$. There are 118 such values of i.

35. $(13, 3)$ and $(-13, -3)$. [9] Adding yields $(x-y)^2 - y^4 = 19$ so $(x-y-y^2)(x-y+y^2) = 19$. Thus the pair $(x-y+y^2, x-y-y^2)$ equals either $(19, 1)$ or $(-1, -19)$. The first case leads to $(x, y) = (13, 3)$ or $(7, -3)$, and the second to $(-7, 3)$ or $(-13, -3)$. One readily checks that only $\pm(13, 3)$ actually works.

36. 10940. [3] Let $b_n = a_n - a_{n-1}$, so $6b_n = 11 + 5b_{n-1}$ and $b_1 = 1$. Now let $c_n = b_n - 11$. Then $c_n = \frac{5}{6}c_{n-1}$ and $c_1 = -10$, so

$c_n = -10(5/6)^{n-1}$. Now a_{1000} equals

$$\sum_{n=1}^{1000}(a_n - a_{n-1}) = \sum_{n=1}^{1000}(c_n + 11)$$

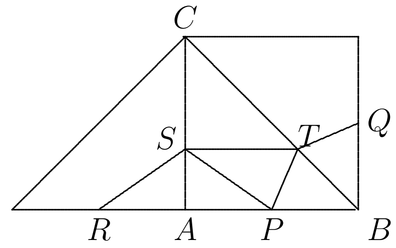
$$= 11000 - 10\sum_{n=1}^{1000}(5/6)^{n-1} \approx 11000 - 10/(1-\tfrac{5}{6}) = 11000 - 60.$$

37. 180. [18] Note that k is in S_n if and only if $m = \frac{k}{n}$ is a divisor of k with $m \leq \sqrt{k}$. Thus we seek the smallest integer k having at least 9 divisors $\leq \sqrt{k}$. The answer is 180, with divisors 1, 2, 3, 4, 5, 6, 9, 10, and 12. To discover this, you might guess that the number would be of the form $2^a 3^b 5^c$. For $k = 120$, there are only 10 integers $\leq \sqrt{k}$, and 7 and 9 fail to be divisors. For $k = 144$, out of the 12 possible, 5, 7, 10, and 11 fail. For 180, out of the 13 possible, 7, 8, 11, and 13 fail.

38. $\sqrt{5}$. [10] Let $x = \cos(\theta)$ and $y = \sin(\theta)$. Using trigonometric identities, we desire the maximum value of $\cos(2\theta) + 2\sin(2\theta)$. This is the maximum value of $X + 2Y$ on the unit circle. (Note that X and Y are not the same as x and y.) Of all lines $X + 2Y = c$, the highest one to intersect the circle does so on the line $Y = 2X$. Hence $5X^2 = 1$, so $X = 1/\sqrt{5}$ and $X + 2Y = 5/\sqrt{5}$. (It turns out that $(x, y) = (\sqrt{\frac{5+\sqrt{5}}{10}}, \sqrt{\frac{5-\sqrt{5}}{10}})$, but that was not part of the problem.)

39. $\frac{1}{4}\sqrt{10}$. [4] Let $A = (0, 0)$, $B = (2, 0)$, and $C = (0, 2)$. One vertex of the desired triangle is at $P = (1, 0)$. Reflect ABC about BC and AC. The reflections of point P are $Q = (2, 1)$ and $R = (-1, 0)$, respectively. If PST is one of the triangles being considered, its perimeter equals the length of the polygonal path $RSTQ$, which is shortest when it is the straight line from R to Q, of length $\sqrt{3^2 + 1^2}$. To find its shortest edge, suppose in the diagram below RQ is a straight line. Then BC has equation $y = 2 - x$, while RQ has equation $y = \frac{1}{3}(x + 1)$, so point T is at $(\frac{5}{4}, \frac{3}{4})$. The x-displacements of RS, ST, and TQ are 1, $\frac{5}{4}$, and

$\frac{3}{4}$, respectively. Thus TQ is shortest, with length $\frac{1}{4}$ of the total.



40. 18. [5] The expression equals $(x+3y)^2+2z^2$. By AM-GM, $(x+3y)^2 \geq 12xy$ with equality iff $x = 3y$. To minimize $12xy + 2z^2$ with $xyz = \sqrt{3}$, write it as $6xy + 6xy + 2z^2$, which, by AM-GM, is $\geq 3\sqrt[3]{6xy \cdot 6xy \cdot 2z^2}$ with equality iff $6xy = 2z^2$. This lower bound equals $3\sqrt[3]{72 \cdot (xyz)^2} = 18$. This minimum value is achieved when $z = x = \sqrt{3}$ and $y = \sqrt{3}/3$.