1. $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} =$

2. The sum of three numbers is 17. The first is 2 times the second. The third is 5 more than the second. What is the value of the largest of the three numbers?

3. A chemist has 100 cc of 20% acid, the rest being water. She adds pure acid to make the solution $33\frac{1}{3}\%$ acid. How many ccs of water must she then add to return it to 20% acid?

4. An isosceles triangle has two sides of length 10 and one of length 12. What is its area?

5. What is the y-intercept of the straight line passing through (10,2) and (8,5)?

6. What is the length of the altitude drawn to the hypotenuse of a right triangle having legs equal to 5 and 12?

7. Which is the largest of the following: 1^{48}, 2^{42}, 3^{36}, 4^{30}, 5^{24}, 6^{18}, 7^{12}, 8^{6}, 9^{0}? (Write the exponential expression, not the large integer which it equals.)

8. Two bicyclists 2 miles apart start pedaling toward each other with speeds 9 and 10 miles per hour, respectively. A fly flies at 12 miles per hour from one bicycle to the other, turns around instantly, flies back, etc., until the bicyclists meet. How many miles did the fly fly?

9. A sequence is defined by $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 4$. Suppose $a_4 = 20$, $a_5 = 36$, and $a_7 = 121$. What is the value of a_1?

10. What is the number of sides of a regular polygon for which the number of diagonals is 44 or 45?

11. There are five cards, two red and three black. You draw two cards, without replacement. What is the probability that both have the same color?

12. The length of each leg of an isosceles triangle is $x+1$ and the length of the base is $3x-2$. Determine all possible values of x. (The triangle should be nondegenerate; i.e. not just a straight line. Note also that x need not be an integer; your answer should be an inequality.)

13. One laser blast will break asteroids larger than 20 kg into three pieces, each one third of the mass of the original. Asteroids smaller than 20 kg are shattered into dust by the laser. How many laser blasts would be required to reduce a 2000 kg asteroid to dust?

14. What is the radius of the smallest circle that contains both of the circles $x^2+y^2=1$ and $(x-1)^2+(y-2)^2=4$?

15. What is the number of different 7-digit numbers that can be made by rearranging the digits of 3053354? (Note that this includes the given number, and that the first digit of a number is never 0.)

16. In what positive base b does the equation $4 \cdot 12 = 103$ for multiplication of base-b numbers hold?

17. An octahedron is formed by connecting the centers of the faces of a cube. What is the ratio of the volume of the cube to that of the contained octahedron?

18. What is the largest number of pieces into which a circular pizza can be cut with 9 straight cuts?

19. If the odd numbers are grouped in the following way:

$$\{1\};\ \{3,5\};\ \{7,9,11\};\ \{13,15,17,19\};\ \ldots,$$

what is the sum of the numbers in the tenth group?

20. If $f(3x) = 3/(1+x)$ for all $x \neq -1$, then $3f(x) =$

21. Two students attempted to solve a quadratic equation $x^2 + bx + c = 0$. Although both students did the work correctly, one miscopied the middle term and obtained the solution set $\{2, 3\}$, while the other miscopied the constant term and obtained the solution set $\{2, 5\}$. What is the correct solution set?

22. Four circles of radius 1 are arranged so that each is tangent to two others, and their centers lie on vertices of a square of side 2. A small circle at the center of the square is tangent to the four circles. What is the radius of the small circle?

23. Find all values of x which satisfy $x^4 + 6ix^3 - 6x^2 - 6ix + 1 = (x + i)^4$, where $i^2 = -1$.

24. If the hypotenuse of a right triangle is 8 units, and the area is 8 square units, what is the tangent of the smallest angle of the triangle, written in the form $a + b\sqrt{c}$, with a, b, and c integers?

25. How many 0's occur at the end of the decimal expansion of $100^{100} - 100!$? (This number is 100 to the 100th power minus 100 factorial.)

26. What is the number of positive integers n for which there is a triangle with three positive acute angles and sidelengths 10, 24, and n?

27. Compute $S = \frac{1}{5} + \frac{1}{25} + \frac{2}{125} + \frac{3}{625} + \frac{5}{3125} + \cdots$, where each numerator is the sum of the two preceding numerators, and each denominator is 5 times the preceding one. Your answer should be a fraction.

28. Let $A = 1^{-4} + 2^{-4} + 3^{-4} + 4^{-4} + 5^{-4} + \cdots$ denote the sum of the reciprocals of the fourth powers of all positive integers, and $B = 1^{-4} + 3^{-4} + 5^{-4} + 7^{-4} + \cdots$ a similar sum for all odd positive integers. Express A/B as a fraction.

29. Let $ABCD$ be a rectangle with $BC = 2AB$, and let BCE be an equilateral triangle crossing through the rectangle. If M is the midpoint of EC, how many degrees are in angle CMD?

30. Let C denote the cylinder $x^2 + y^2 = 4$, $0 \leq z \leq 6$, of radius 2 and height 6. What is the length of the shortest path on the cylinder from the point $(2, 0, 6)$ to the point $(2, 0, 0)$ which passes through the point $(2, 0, 2)$ and passes twice around the cylinder?

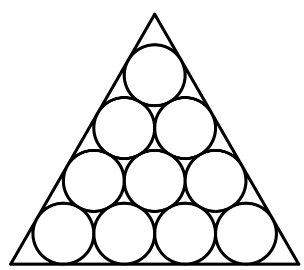
31. In triangle ABC, $AC = 6$ and $BC = 5$. Point D on AB divides it into segments of length $AD = 1$ and $DB = 3$. What is the length CD?

32. List all pairs (m, n) of positive integers for which $n! + 1 = (m! - 1)^2$.

33. An acute angle is formed by two lines of slope 1 and 7. What is the slope of the line which bisects this angle?

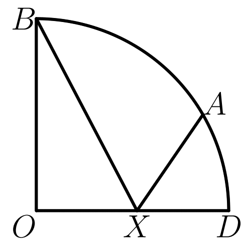
34. How many ordered triples (x, y, z) of positive integers satisfy $xyz = 4000$?

35. An equilateral triangle is filled to the max with n rows of congruent circles. (The case $n = 4$ is pictured below.) What is the limit as n approaches ∞ of the ratio (area in circles)/(area of triangle)?



36. List all 3-digit numbers abc for which the 6-digit number $579abc$ is divisible by 5, 7, and 9.

37. The figure below shows a quarter-circle of radius 1, with A chosen so that angle AOD is 30 degrees. What must be the distance OX so that the region bounded by AX, XB, and the arc AB occupies half the area of the quarter circle?



38. An equilateral triangle is inscribed in a circle. Let D and E be midpoints of two of its sides, and let F be the point where the line from D through E meets the circle. What is the ratio DE/EF?

39. How many pairs (n, m) satisfying $10 \leq m < n$ and $m+n \leq 99$ have the property that $m + n$ and $n - m$ have the same digits in reverse order? (This allows cases such as $(33, 27)$ where the sum and difference are 60 and 06.)

40. Parallel chords in a circle have length 12 and 16, and the distance between them is 7. Another chord is midway between them. What is its length?

SOLUTIONS TO 2005 CONTEST

The numbers in brackets are the number of people who answered the problem correctly, first out of the 31 people who scored at least 19, and then out of the other 222 people.

1. 31/30. [30,207] It is $(15 + 10 + 6)/30$.

2. 8. [29,173] Let x denote the second number. Then $2x+x+(x+5) = 17$, hence $4x = 12$ and $x = 3$. The other numbers are 6 and 8.

3. 80. [28,104] Let x be the amount of acid added at first. Then $(20 + x)/(100 + x) = 1/3$, which implies $x = 20$. Now we have 40 acid out of 120 altogether. To make it 20% acid, we must bring the total liquid up to 200, by adding 80 cc.

4. 48. [31,159] The altitude h satisfies $h^2 + 6^2 = 10^2$, and so $h = 8$. Thus the area is $(12 \cdot 8)/2$.

5. 17. [29,173] Decreasing x from 10 to 8 increased y by 3. Thus decreasing x 8 more will increase y four times as much; i.e. by 12, to 17. Or, $y - 2 = -\frac{3}{2}(x - 10)$ so when $x = 0$, then $y = 2 + 15$.

6. 60/13. [30,57] The hypotenuse equals 13. Calculating the area of the triangle in two ways yields $5 \cdot 12 = 13h$.

7. 4^{30}. [27,57] Raise each to the 1/6 power. Then we compare $3^6 = 729$, $4^5 = 1024$, and $5^4 = 625$. The others are smaller.

8. 24/19. [25,48] The bicylists are approaching each other at 19 mph, and so meet in 2/19 hours. Thus the fly travels $12 \cdot 2/19$ miles.

9. 4. [31,118] Find $a_6 = 121 - 36 - 20 = 65$, then $a_3 = 65 - 36 - 20 = 9$, then $a_2 = 36 - 20 - 9 = 7$, and finally $a_1 = 20 - 9 - 7 = 4$.

10. 11. [30,73] The number of diagonals of a regular n-gon is $\binom{n}{2} - n$. Then note that $\binom{11}{2} - 11 = 44$.

11. 2/5 or 0.4. [29,93] If the event is the two cards in order, then there are $5 \cdot 4$ ways of drawing the cards, of which $2 \cdot 1$ have both red, and $3 \cdot 2$ have both black. The answer is 8/20.

12. $\frac{2}{3} < x < 4$. [27,60] Must have $2(x+1) > 3x - 2$ hence $x < 4$, and the base must have positive length, which implies $x > 2/3$.

13. 364. [20,48] If $M = 2000$ denotes the initial mass, then 1 blast yields three of mass $M/3$, 3 more blasts yield 9 of mass $M/9$, 9 more blasts yield 27 of mass $M/27$, 27 more blasts yield 81 of mass $M/81$, 81 more blasts yield 243 of mass $M/243$, which are now less than $20 = M/100$ kg, but another 243 blasts are required to turn them to dust. Now add the total number of blasts.

14. $(3+\sqrt{5})/2$. [12,16] Extend the segment connecting the centers until it meets the two circles again. This will be a diameter of the containing circle. Its length equals the distance between the centers plus the sum of the radii, which is $\sqrt{5}+1+2$.

15. 360. [25,13] The 0 can be in any of 6 positions. Then the three 3's can be in any of $\binom{6}{3} = 20$ positions. Then the two 5's can be in any of $\binom{3}{2} = 3$ positions. Finally the position of the 4 is forced by the preceding choices. Hence the answer is $6 \cdot 20 \cdot 3$.

16. 5. [31,54] We must have $4(b+2) = b^2 + 3$, hence $b^2 - 4b - 5 = 0$, so, since b is positive, it equals 5.

17. 6 or 6 : 1. [16,14] Let the side length of the cube equal 1 unit. The octahedron is the union of two pyramids of height 1/2 on a base which is a square of sidelength $\sqrt{2}/2$ (since in a plane one side of the square could be considered to be connecting the points $(1/2, 0)$ and $(0, 1/2)$). The volume of each pyramid is $hA/3 = \frac{1}{2}\frac{1}{2}\frac{1}{3}$.

18. 46. [27,35] The maximum number of pieces that can be added by a cut is 1 greater than the number of lines that the new cut intersects. Hence the answer is $1 + 1 + 2 + \cdots + 9 = 1 + (9 \cdot 10)/2$.

19. 1000. [31,154] If you compute the first four sums 1, 8, 27, 64, you can probably guess that the sum of the ith group is i^3. One way to prove it would be to note that the ith group has i numbers and their average is i^2. To see this when i is odd, note that there will be $i(i-1)/2$ odd numbers preceding the group, and the middle entry will be the $((i+1)/2)$nd number in the group. It will thus equal $-1 + 2(i(i-1)/2 + (i+1)/2) = i^2$. A similar argument works when i is even.

20. $27/(3+x)$. [30,61] Since $f(3x) = 9/(3+3x)$, we have $f(x) = 9/(3+x)$. Now multiply by 3.

21. $\{1, 6\}$ or just 1,6. [30,88] Since the constant term is the product of the roots, this must be $2 \cdot 3 = 6$, while $-b$ equals the sum of the roots, and so this must be 7. Thus the polynomial is $x^2 - 7x + 6$.

22. $\sqrt{2} - 1$. [30,96] Let r denote the desired radius. A 45-45-90 triangle is formed with hypotenuse connecting the center of the little circle with the center of one of the four given circles, and other vertex at a point of tangency of that given circle with another. The hypotenuse is $1 + r$ and the legs are 1. Thus $(1 + r)^2 = 2$.

23. $0, 1, -1$. [19,28] $(x + i)^4 = x^4 + 4ix^3 - 6x^2 - 4ix + 1$. Equating this to the given expression yields $4x^3 - 4x = 6x^3 - 6x$, hence $2x(x^2 - 1) = 0$.

24. $2-\sqrt{3}$. [12,2] If x and y denote the legs, then $xy = 16$ and $x^2+y^2 = 64$. Adding and subtracting twice the first equation to the second yields $x+y = \sqrt{96} = 4\sqrt{6}$ and $x-y = \sqrt{32} = 4\sqrt{2}$. Thus $x = 2(\sqrt{6}+\sqrt{2})$ and $y = 2(\sqrt{6}-\sqrt{2})$. The desired tangent is $\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = (\sqrt{3}-1)^2/2 = (4 - 2\sqrt{3})/2$.

25. 24. [18,6] As 100^{100} has many more 0's at the end, we need here the number of 0's at the end of 100!. This will be the number of factors of 5 in its decomposition into primes, since there will be plenty of 2's to turn these into 10's. There are 20 multiples of 5 in 100! and four of them have a second factor of 5 (i.e. the multiples of 25).

26. 4. [18,21] The square of the largest side must be strictly less than the sum of the squares of the other two. Thus n must satisfy $476 < n^2 < 676$ and hence $n = 22$, 23, 24, or 25.

27. 5/19. [7,0] $5S = 1 + \frac{1}{5} + \frac{2}{25} + \frac{3}{125} + \frac{5}{625} + \cdots$. Subtract the series for S from this, obtaining $4S = 1 + \frac{1}{25} + \frac{1}{125} + \frac{2}{625} + \cdots = 1 + \frac{1}{5}S$. Thus $\frac{19}{5}S = 1$.

28. 16/15. [9,0] $B = A - (2^{-4} + 4^{-4} + 6^{-4} + \cdots) = A - A/16$.

29. 75. [24,41] Both CM and CD equal half a side of the triangle. Hence triangle CMD is isosceles. Since angle MCD equals $90 - 60$, the other angles of the triangle are $(180 - 30)/2$.

30. $4\sqrt{\pi^2+1} + 2\sqrt{4\pi^2+1}$. [14,2] Unroll the cylinder. The paths will be straight lines on the rectangle. The first time around, going from height 6 to height 2, will have length $\sqrt{(4\pi)^2 + 4^2} = 4\sqrt{\pi^2+1}$, while the second time around will have length $\sqrt{(4\pi)^2 + 2^2} = 2\sqrt{4\pi^2+1}$.

31. 11/2. [10,6] If x is the desired length and $\theta = \angle BDC$, then we have $x^2 + 1 + 2x\cos\theta = 36$ and $x^2 + 9 - 6x\cos\theta = 25$. Adding 3 times the first equation to the second yields $4x^2 + 12 = 133$.

32. (3,4). [14,28] We obtain $n! = m!(m! - 2)$, and so $n(n-1)\cdots(m+1) = m! - 2$. Clearly $m \geq 3$ and hence the right hand side is not divisible by 3 (since $m!$ is), and so, since the product of three consecutive odd integers is divisible by 3, we obtain $n = m+1$ or $m+2$. The equations become $m + 3 = m!$ or $m^2 + 3m + 4 = m!$. It is easy to check that the first equation has $m = 3$ as its only solution, while the second has no solutions.

33. 2. [13,12] Using the formula for the tangent of the difference of two angles, we obtain
$$\frac{x-1}{1+x} = \frac{7-x}{1+7x},$$
where x is the desired slope (or tangent). We obtain $7x^2 - 6x - 1 = -x^2 + 6x + 7$ and so $8x^2 - 12x - 8 = 0$ with solutions 2 and $-1/2$.

34. 210. [9,0] Since $4000 = 2^5 5^3$, we must have $x = 2^a 5^d$, $y = 2^b 5^e$, and $z = 2^c 5^f$, and our answer will be AB, where A is the number of ordered triples (a, b, c) of nonnegative integers such that $a + b + c = 5$, and B is the number of ordered triples (d, e, f) of nonnegative integers such that $d + e + f = 3$. Note that $a = 0$ has 6 possibilities for (b, c), namely $0 \le b \le 5$, $a = 1$ has 5 possibilities for (b, c), etc., down to $a = 5$ having one possible (b, c). Thus the number of possible (a, b, c) is $6 + 5 + 4 + 3 + 2 + 1 = 21$. Similarly the number of possible (d, e, f) is $4 + 3 + 2 + 1 = 10$.

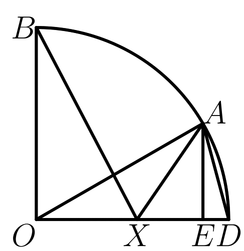
35. $\pi\sqrt{3}/6$. [11,0] Let the sidelength of the triangle equal 1 unit, and let the radius of the little circles equal r. Then $2(n-1)r + 2r\sqrt{3} = 1$. This is seen by consideration of the 30-60-90 triangle whose hypotenuse connects a vertex of the triangle with the nearest center of a circle. Thus the area of each circle is $\pi/(4(n+\sqrt{3}-1)^2)$, and there are $n(n+1)/2$ circles. The area covered by the circles is $\frac{\pi}{8}\frac{n(n+1)}{(n+\sqrt{3}-1)^2}$. This approaches $\pi/8$ as $n \to \infty$. Since the area of the triangle is $\sqrt{3}/4$, the answer follows.

36. 285, 600, 915 in any order. [13,6] Let $n = 579abc$. This n must be divisible by $5 \cdot 7 \cdot 9 = 315$. One can easily check that the 579000 is 30 more than a multiple of 315. Thus the valid abc will be 285, 600, and 915.

37. $\pi/6$. [4,0] Let $x = OX$ be the desired length. We will find x so that the area of the complement of the desired area is $\pi/8$. Triangle BOX has area $x/2$. Since $AE = 1/2$, triangle AXD has area $(1-x)/4$, and triangle OAD has area $1/4$. Thus the sliver between the line AD and the arc AD equals $\pi/12 - 1/4$. Our equation becomes

$$\tfrac{1}{4}(1-x) + \tfrac{\pi}{12} - \tfrac{1}{4} + \tfrac{x}{2} = \tfrac{\pi}{8},$$

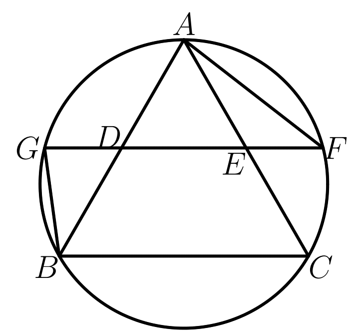
from which it follows that $x = \pi/6$.



38. $(1+\sqrt{5})/2$. [7,1] The triangles GDB and ADF are similar, since the angles at B and F subtend the same arc AG. This explains the second equality in the following string:

$$\frac{DE}{EF} = \frac{AD}{GD} = \frac{DF}{DB} = \frac{DF}{DE} = \frac{DE+EF}{DE}.$$

If x is the desired ratio, then this shows $x = 1 + \frac{1}{x}$ and hence x is as claimed.



39. 12. [3,1] We have $n+m = 10x+y$ and $n-m = 10y+x$, hence $2n = 11(x+y)$ and $2m = 9(x-y)$. Thus n is a multiple of 11, and m is a multiple of 9. The restrictions $10 \le m < n$ and $m+n < 100$ then imply that the following are the possible pairs: (22,18), (33,18), (33,27), (44,18), (44,27), (44,36), (55,18), (55,27), (55,36), (66,18), (66,27), and (77,18).

40. $\sqrt{249}$. [8,1] Let r denote the radius of the circle, and d_1 and d_2 the distances from the center of the circle to the chords of length 12 and 16, respectively. Each of the perpendiculars from the center to a chord makes a right triangle, with other leg equal to half the length of the chord. Thus $36 + d_1^2 = r^2 = 64 + d_2^2$. Assume the chords are on opposite sides of the center. (If you put them on the same side, you will get a negative value for one of the d's.) Since $d_1+d_2 = 7$, we obtain $7(d_1 - d_2) = 64 - 36 = 28$, hence $d_1 - d_2 = 4$. From this we obtain $d_1 = 11/2$ and $d_2 = 3/2$, and then $r^2 = 265/4$. The distance from our midway chord to the center is $(d_1 - d_2)/2 = 2$. If x denotes the length of the desired chord, then $(\frac{x}{2})^2 + 2^2 = r^2$, and so $(x/2)^2 = 249/4$.