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Preface

At its heart, this is a problem book about mathematical induction.

Why a book about induction? The answer is simple but compelling.

Mathematical induction, its equivalents complete induction and well-ordering, and its immediate consequence, the pigeonhole principle, are important proof techniques in mathematics. Indeed, they are not only important, but essential and ubiquitous. Every mathematician is familiar with mathematical induction, and every student of mathematics needs to be. Thus we have written this book to provide the reader with an introduction and a thorough exposure to these proof techniques.

To whom is it addressed? There are several audiences.

1. This book is well suited to be used for a course on mathematical induction. The author has used parts of this book for such a course. There is far more material in this book than can be covered in such a course, so instructors may pick their favorite topics from among the ones presented here.

2. Most theorem-proving courses include a segment on induction. Thus this book can be used as a supplement for such courses, providing additional explanation and additional problems to be solved.

3. Since this book contains a large collection of problems, it can be used in problem-solving courses. The author has often taught problem-solving courses ("coaching" for the Putnam competition), using some of these problems in these courses.

4. Students looking for interesting and challenging problems to cut their teeth on will find a variety of them here.

Let us next describe the plan of the book.

Despite its power, the basic idea of mathematical induction is quite simple. Thus we begin in Chapter 1 with an intuitive explanation of mathematical induction and its equivalents, and then proceed to formalize it. This basic idea appears in many variants, so we give a number of illustrative examples of its use.

The core of this book is Chapter 2, a large collection of problems consisting of results to be proved by induction or by the pigeonhole principle. (Henceforth when we say induction, we mean mathematical induction, complete induction, or well-ordering). These problems are deliberately presented *without* solutions to enable instructors to assign them to their students (and to keep students who read this book on their own honest).

A typical induction problem is a two-step problem. The first step is to find a pattern, and the second step is to prove that it holds. Induction is a proof technique, not a discovery technique, so it applies to the second step, not the first.

Many of our problems just involve the second step: We give a result, and the problem is to find a proof of it. Some involve both steps: the reader must first discover the pattern, and then prove that it holds. Some of our problems are relatively straightforward, while others require varying degrees of cleverness, ingenuity, and hard work. The author of this book thinks of problem-solving as fun. We hope the readers of this book will have fun attacking the problems here.

Our problems range from "old chestnuts" through original problems. Some of these are problems that are interesting but not important in themselves, being chosen to give the reader practice in proofs by induction. Others are important theorems that can be proved by induction. While we have mentioned that induction is ubiquitous in mathematics, we restrict ourselves here to elementary problems. By this we mean that most problems require little background, only material that any college (or advanced high school) student should know. There are some problems that involve calculus or linear algebra, but none more advanced than that. There are many problems that involve elementary number theory, as many of the basic theorems of number theory can be proved by induction. (For some of the number theory problems and results, the reader should be familiar with congruences.)

An inductive argument is often the acorn from which a mighty mathematical oak grows. Thus in some of those instances we have followed this growth, expanding on the mathematics in order to illustrate the consequences of the inductive argument (thereby showing the power of induction).

To say that a problem is elementary is not to say that it is easy. The problems here have varying levels of difficulty. There are some beautiful and important theorems that can be proved by induction, but whose proofs are just too difficult to expect students to be able to find on their own. Thus we include an exposition of some of these theorems and their proofs in Chapter 3. (Among these are some famous theorems by famous mathematicians.)

Finally, we have mentioned that mathematical induction, complete induction, and well-ordering are logically equivalent. The reader can take this on faith, and use whichever of these is most convenient in solving a particular problem. But taking things on faith is not a satisfactory way to proceed in mathematics, in the end, so we include an appendix proving this equivalence.

The reader of this book will learn some interesting and beautiful mathematics along the way, and it is one of our goals to present this. We even think that most professional mathematicians will find some items with which they were previously unfamiliar, so they should enjoy it, too.

We make some remarks about notation and language. Results in this book have three-level numbering, so that, for example, Theorem 3.1.9 is the 9th numbered item in Chapter 3, Section 1. The ends of proofs are marked by the symbol $\square$. In some cases we follow the statements of problems with additional remarks. In order to make clear where the problems themselves end, we mark the ends of all problems by the symbol $\diamond$.

We recall that, for any function $f(k)$, $\sum_{k=1}^{n} f(k)$ is the sum $f(1) + f(2) + \ldots + f(n)$ and $\prod_{k=1}^{n} f(k)$ is the product $f(1)f(2)\cdots f(n)$. We follow the standard convention that the empty sum is 0 and the empty product is 1, so that, for example, $\sum_{k=1}^{0} f(k) = 0$ and $\prod_{k=1}^{0} f(k) = 1$. We also recall that, as an analytic expression, x^y is undefined whenever $x = y = 0$. However, in this book whenever this situation comes up we will always be dealing with symbolic or combinatorial expressions, rather than analytic ones, and so we will understand that $0^0 = 1$ throughout.

Several of the problems in this book are modifications of problems that have appeared in Putnam competitions and the author thanks the Mathematical Association of America for permission to use them.

Finally, this book was begun while the author was physically at Lehigh, and was completed while the author was on sabbatical leave at the Mathematics Institute of the University of Göttingen. He thanks Lehigh for the time off to finish the book, and the Mathematics Institute for its hospitality during his visit.

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