Linear Algebra for the Young Mathematician

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10 9 8 7 6 5 4 3 2 1 24 23 22 21 20 19

Contents

Part II. Vector spaces with additional structure

Preface

Linear algebra is a subject that lies at the core of modern mathematics. Every mathematician, pure or applied, is intimately familiar with it. Every student of mathematics needs to become familiar with it, too.

Linear Algebra for the Young Mathematician is a careful, thorough, and rigorous introduction to linear algebra, one which emphasizes the basic ideas in the subject. Let us unpack this sentence. First of all, it is an introduction, presupposing no prior knowledge of the field. Second of all, it is careful and thorough—you should expect no less from any book. Thirdly, it is rigorous. We prove what we claim to be true. As a mathematician, you should demand this. But more than that, a good proof not only shows *that* something is true, it shows *why* it is true. This brings us to our last point. We emphasize the ideas of the subject, and provide both conceptual definitions and proofs, rather than calculational ones, so you will see and understand the logic of the field, and why its results are true. We freely admit that this conceptual approach means you will have to put more thought into learning the subject, but you will get a *lot* more out.

We presume that you, the reader, are a mathematician, but we interpret that term broadly. You could be a mathematician per se, or anyone in related fields (e.g., a statistician studying data modeling, a physicist studying quantum mechanics, or an engineer studying signal processing) with a serious interest in mathematical theory as well as in its applications to your field. We presume further that you, the reader, are a young mathematician, not necessarily in the sense of chronological years, but rather in the sense of mathematical years, i.e., that you are just starting your serious study of mathematics. The author is an old mathematician, in the sense that he has been thinking seriously about mathematics for over half a century. (Of course, this implies something about his chronological age as well.) Thus in addition to the rigorous statements in the book, we have taken considerable care, and space, to present what we think is a right viewpoint, developed over decades of thought and experience, from which to view the subject. We hope you will benefit from the wisdom we have accumulated. (Note we have said "a right viewpoint" rather than "the right viewpoint". We are not so presumptuous to claim that our viewpoint is the only right one, and you will undoubtedly go on to develop your own, which may differ. But at least we hope—indeed, expect—to provide you with a good starting point.)

What distinguishes this book from the many other introductory linear algebra books in existence? Of course, we think it is exceptionally well written. But beyond that, there are several aspects. First and foremost is our conceptual approach and our attempt to provide a viewpoint on the subject. Second is our coverage. We concentrate on the finite-dimensional case, where the results are strongest, but do not restrict ourselves to it. We consider vector spaces in general, and see to what extent results in the finite-dimensional case continue to hold in general. Of course, in the finite-dimensional case we have a very powerful computational tool, matrices, and we will learn how to do all the usual matrix computations, so that we can in practice solve the problems that arise. But we wish to emphasize, even in the finite-dimensional case: *Linear algebra is about vector spaces and linear transformations, not about matrices.* (In line with this emphasis, we do not deal with questions of how most accurately or efficiently to perform machine computations, leaving that for you to learn in a future numerical analysis course, if that is a direction you wish to pursue.) Even in the finite-dimensional case, we cover material often not covered in other introductory texts, and go further. Particularly noteworthy here is our thorough coverage of Jordan canonical form as well as the spectral theorem. Third, we wrote right at the start that linear algebra is a central subject in mathematics, and rather than having you simply have to take our word for it, we illustrate it by showing some of the connections between linear algebra and other parts of mathematics, in our case, with algebra and calculus.

On the one hand we will see that calculus "works" because the basic operations of calculus, evaluation of a function, differentiation, and definite integration, are linear transformations, and we will see how to most properly formulate some of the results of calculus in linear algebra terms. On the other hand, we will see how to use linear algebra to concretely solve some of the problems of algebra and calculus. (Some of these connections are given in separate sections of the text, while others are given as immediate applications of particular results in linear algebra.)

Here is, roughly speaking, the plan of the book.

We begin right away, in accord with our philosophy, by introducing vector spaces, in the situation of "usual" vectors (the kind you have seen in calculus), and linear transformations, in the situation of multiplication of vectors by matrices. (You will see that, throughout the book, we are much more likely to speak of "the linear transformation $\mathcal{T}_A$ that is multiplication by the matrix A" than we are to simply speak of "the matrix A".) This allows us to begin by posing the general questions we wish to answer in a relatively concrete context.

Usually, answering specific questions in linear algebra means solving systems of linear equations, and so we devote Chapter 2 entirely to that. (But, in line with our emphasis, we begin that chapter with a consideration of the geometry of linear systems, before considering the particulars of solution methods.)

The next three chapters, which deal with linear transformations, are the heart of the book. Here you will find key results in linear algebra. In Chapter 6 we turn our attention to determinants, an important theoretical and computational tool, before returning to the consideration of linear transformations in the next two chapters. Our analysis of linear transformations culminates in Jordan canonical form, which tells you all you want to know about a linear transformation (from a finite-dimensional complex vector space to itself, to be precise).

These first eight chapters (Part I of the book) deal with general vector spaces. The last two chapters (Part II) deal with vector spaces with additional structure—first that of bilinear or sesquilinear forms, and finally that of inner product spaces, culminating, in the finite-dimensional case, with the spectral theorem.

We have mentioned above that there are applications of linear algebra in various places in the text. Let us point some of them out:

Section 4.5: "Looking back at calculus".

In Section 5.2: A common generalization of finite Taylor polynomials and the Lagrange interpolation theorem with an application to cubic splines, and partial fraction decompositions.

In Section 5.5: Numerical integration and differentiation.

Sections 7.3 and 8.4: Solutions of higher-order linear differential equations and systems of first-order linear differential equations.

Section 9.5: Diagonalizing quadratic forms, and the "second derivative test" in multivariable calculus.

In Section 10.2: The method of Gaussian quadrature.

Some remarks on numbering and notation: We use three-level numbering, so that, for example, Theorem 8.1.6 is the 6th numbered item in Section 8.1. We denote the end of proofs by □, as is usual. Theorems, etc., are set in italics, so their end is denoted by the end of the italics. Definitions, etc., are set in roman type, so their end cannot be deduced from the typeface; we denote their end by ◊. Our mathematical notation is standard, though we want to point out that if A and B are sets, $A \subseteq B$ means A is a subset of B and $A \subset B$ means A is a proper subset of B (i.e., $A \subseteq B$ and $A \neq B$).

There are not enough typefaces to go around to distinguish different kinds of objects, and indeed one of the points of linear algebra is that it is logically impossible to do so. For example, we might want to regard a function $f(x)$ as a vector in a vector space—should we change notation to do so? But we will always use V to denote a vector space and we will always use $\mathcal{T}$ to denote a linear transformation.

Finally, we will remark that there is some (deliberate) repetition in the book, as in some cases we have introduced a concept in a specific situation and then generalized it later. Otherwise, the book should pretty much be read straight through, except that Section 9.4 will not be used again until Section 10.4, and that Section 9.5 will not be used in Chapter 10 at all. Also, while Chapter 8, Jordan canonical form, is the culmination of Part I, it is not necessary for the study of normal linear transformations in Part II.

There are several appendices. We use the notion of a field $\mathbb{F}$ in general, although all of our work in this book will be over the fields $\mathbb{Q}$ (the rational numbers), $\mathbb{R}$ (the real numbers), or $\mathbb{C}$ (the complex numbers). For those readers wishing to know more about fields, we have an optional Appendix A. Appendix B deals with properties of polynomials, and readers not already familiar with them will need to read it before Chapter 7, where these properties are used. We are doing algebra in this book, so we are dealing with finite sums throughout, but in analysis we need to deal with infinite sums and issues of convergence, and in the optional Appendix C we consider the basics of these. Finally, we have stated that linear algebra plays a central place in mathematics and its applications, and Appendix D is a guide to further reading for the reader who wishes to see further developments, aspects of linear algebra not treated here, or applications outside of pure mathematics.

February 2019 *Steven H. Weintraub*