Preface

Linear algebra is a beautiful and mature field of mathematics, and mathematicians have developed highly effective methods for solving its problems. It is a subject well worth studying for its own sake.

More than that, linear algebra occupies a central place in modern mathematics. Students in algebra studying Galois theory, students in analysis studying function spaces, students in topology studying homology and cohomology, or for that matter students in just about any area of mathematics, studying just about anything, need to have a sound knowledge of linear algebra.

We have written a book that we hope will be broadly useful. The core of linear algebra is essential to every mathematician, and we not only treat this core, but add material that is essential to mathematicians in specific fields, even if not all of it is essential to everybody.

This is a book for advanced students. We presume you are already familiar with elementary linear algebra, and that you know how to multiply matrices, solve linear systems, etc. We do not treat elementary material here, though in places we return to elementary material from a more advanced standpoint to show you what it really means. However, we do not presume you are already a mature mathematician, and in places we explain what (we feel) is the "right" way to understand the material. The author feels that one of the main duties of a teacher is to provide a viewpoint on the subject, and we take pains to do that here.

One thing that you should learn about linear algebra now, if you have not already done so, is the following: *Linear algebra is about vector spaces and linear transformations, not about matrices*. This is very much the approach of this book, as you will see upon reading it.

We treat both the finite and infinite dimensional cases in this book, and point out the differences between them, but the bulk of our attention is devoted to the finite dimensional case. There are two reasons: First, the strongest results are available here, and second, this is the case most widely used in mathematics. (Of course, matrices are available only in the finite dimensional case, but, even here, we almost always argue in terms of linear transformations rather than matrices.)

We regard linear algebra as part of algebra, and that guides our approach. But we have followed a middle ground. One of the principal goals of this book is to derive canonical forms for linear transformations on finite dimensional vector spaces, i.e., rational and Jordan canonical forms. The quickest and perhaps most enlightening approach is to derive them as corollaries of the basic structure theorems for modules over a principal ideal domain (PID). Doing so would require a good deal of background, which would limit the utility of this book. Thus our main line of approach does not use these, though we indicate this approach in an appendix. Instead we adopt a more direct argument.

We have written a book that we feel is a thorough, though intentionally not encyclopedic, treatment of linear algebra, one that contains material that is both important and deservedly "well known". In a few places we have succumbed to temptation and included material that is not quite so well known, but that in our opinion should be.

We hope that you will be enlightened not only by the specific material in the book but by its style of argument–we hope it will help you learn to "think like a mathematician". We also hope this book will serve as a valuable reference throughout your mathematical career.

Here is a rough outline of the text. We begin, in Chapter 1, by introducing the basic notions of linear algebra, vector spaces and linear transformations, and establish some of their most important properties. In Chapter 2 we introduce coordinates for vectors and matrices for linear transformations. In the first half of Chapter 3 we establish the basic properties of determinants, and in the last half of that chapter we give some of their applications. Chapters 4 and 5 are devoted to the analysis of the structure of a single linear transformation from a finite dimensional vector space to itself. In particular, in these chapters, we develop eigenvalues, eigenvectors, and generalized eigenvectors, and derive rational and Jordan canonical forms. In Chapter 6 we introduce additional structure on a vector space, that of a (bilinear, sesquilinear, or quadratic) form, and analyze these forms. In Chapter 7 we specialize the situation of Chapter 6 to that of a positive definite inner product on a real or complex vector space, and in particular derive the spectral theorem. In Chapter 8 we provide an introduction to Lie groups, which are central objects in mathematics and are a meeting place for algebra, analysis, and topology. (For this chapter we require the additional background knowledge of the inverse function theorem.) In Appendix A we review basic properties of polynomials and polynomial rings that we use, and in Appendix B we rederive some of our results on canonical forms of a linear transformation from the structure theorems for modules over a PID.

We have provided complete proofs of just about all the results in this book, except that we have often omitted proofs that are routine without comment.

As we have remarked above, we have tried to write a book that will be widely applicable. This book is written in an algebraic spirit, so the student of algebra will find items of interest and particular applications, too numerous to mention here, throughout the book. The student of analysis will appreciate the fact that we not only consider finite dimensional vector spaces, but also infinite dimensional ones, and will also appreciate our material on inner product spaces and our particular examples of function spaces. The student of algebraic topology will appreciate our dimension-counting arguments and our careful attention to duality, and the student of differential topology will appreciate our material on orientations of vector spaces and our introduction to Lie groups.

No book can treat everything. With the exception of a short section on Hilbert matrices, we do not treat computational issues at all. They do not fit in with our theoretical approach. Students in numerical analysis, for example, will need to look elsewhere for this material.

To close this preface, we establish some notational conventions. We will denote both sets (usually but not always sets of vectors) and linear transformations by script letters $\mathcal{A}, \mathcal{B}, \dots, \mathcal{Z}$. We will tend to use script letters near the front of the alphabet for sets and script letters near the end of the alphabet for linear transformations. $\mathcal{T}$ will always denote a linear transformation and $\mathcal{I}$ will always denote the identity linear transformation. Some particular linear transformations will have particular notations, often in boldface. Capital letters will denote either vector spaces or matrices. We will tend to denote vector spaces by capital letters near the end of the alphabet, and V will always denote a vector space. Also, I will almost always denote the identity matrix. $\mathbb{E}$ and $\mathbb{F}$ will denote arbitrary fields and $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ will denote the fields of rational, real, and complex numbers respectively. $\mathbb{Z}$ will denote the ring of integers. We will use $\mathcal{A} \subseteq \mathcal{B}$ to mean that $\mathcal{A}$ is a subset of $\mathcal{B}$ and $\mathcal{A} \subset \mathcal{B}$ to mean that $\mathcal{A}$ is a proper subset of $\mathcal{B}$. $A = (a_{ij})$ will mean that A is the matrix whose entry in the (i, j) position is a_{ij}. $A = [v_1 \mid v_2 \mid \cdots \mid v_n]$ will mean that A is the matrix whose ith column is v_i. We will denote the transpose of the matrix A by ${}^tA$ (not by A^t). Finally, we will write $\mathcal{B} = \{v_i\}$ as shorthand for $\mathcal{B} = \{v_i\}_{i \in I}$ where I is an indexing set, and $\sum c_i v_i$ will mean $\sum_{i \in I} c_i v_i$.

We follow a conventional numbering scheme with, for example, Remark 1.3.12 denoting the 12th numbered item in Section 1.3 of Chapter 1. We use $\square$ to denote the end of proofs. Theorems, etc., are set in italics, so the end of italics denotes the end of their statements. But definitions, etc., are set in ordinary type, so there is ordinarily nothing to denote the end of their statements. We use $\diamond$ for that.

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January 2010