Mathematics 43 - Survey of Linear Algebra - 2018

Instructor: Professor Linghai Zhang.
Online Office Hours: Tuesday and Saturday, 8 PM - 9:30 PM, and by appointments.
Textbook: Linear Algebra and its Applications. Fifth Edition. By David C. Lay, Steven R. Lay and Judi J. McDonald.
Attendance is absolutely required!!!
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The Homework assignments: It is worth 100 points. There will be 13 homework assignments - one every week. We will count the best 10 assignments after dropping the lowest three. Every homework will be posted on the Coursesite 7 - 10 days before the deadline. The homework will be due on the following Sunday evening at 11:59 PM. No late homework will be accepted, even if you have a solid reason or have an emergency.
The First Midterm Exam: September 29, Thursday, 100 points.
The Second Midterm Exam: November 10, Thursday, 100 points.
The Final Exam: Middle of December, 200 points.
Total Score: 500 points.

$A-$ 451 - 470, A 471 - 500

$B-$ 401 - 416, B 417 - 433, $B+$ 434 - 450

$C-$ 351 - 366, C 367 - 383, $C+$ 384 - 400

D 301 - 350

F 0 - 300

Topics to be covered in **Mathematics 43**:

Chapter 1: Matrix Algebra, Systems of Equations

Definition of $m \times n$ matrices

Various matrices (identity, diagonal, symmetric, skew-symmetric, upper triangular, lower triangular)

Operations of matrices (addition, scalar multiplication, multiplication)

Elementary row operations

Reduced row echelon form

Rank

Gaussian elimination

Solutions of $m \times n$ systems of equations $A\mathbf{x} = \mathbf{b}$

Inverse matrix: Definition and properties of A^{-1}

Inverse matrix: Computations, Examples 2×2, 3×3, 4×4

Solutions of $n \times n$ systems of equations $A\mathbf{x} = \mathbf{b}$: $\mathbf{x} = A^{-1}\mathbf{b}$

Chapter 2: Determinants

Definition of determinants for 2×2 and 3×3 matrices

Cofactor C_{ij} and Minor M_{ij} of the element a_{ij}

Definition of determinants $\det(A)$ for $n \times n$ matrices

Properties of determinants

Computations of $\det(A)$ (2×2, 3×3, 4×4 matrices)

Computations of $\det(\lambda I - A)$ (2×2, 3×3, 4×4 matrices)

Adjoint matrix A^*

Properties of adjoint matrix

$$AA^* = A^*A = [\det(A)]I, \qquad \det(A)\det(A^*) = [\det(A)]^n$$

Determinants of adjoint matrix

$$\det(A^*) = [\det(A)]^{n-1}$$

The representation of inverse matrix

$$A^{-1} = \frac{1}{\det(A)}A^* \qquad \text{if} \qquad \det(A) \neq 0.$$

Determinants of $(2n+1) \times (2n+1)$ skew-symmetric matrices

Summary

Chapter 3: Vector Spaces

Vector spaces: Definition and elementary properties

Examples of vector spaces: $\mathbb{R}^n$, $M_n(\mathbb{R})$, $P_n(\mathbb{R})$, $C(\mathbb{R})$.

Subspaces of vector spaces $\mathbb{R}^n$ and $M_n(\mathbb{R})$.

Null space, Row space, Column space of a $m \times n$ matrix A

Spanning sets of the vector space $\mathbb{R}^n$.

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots + \alpha_m\mathbf{v}_m = \mathbf{b}$$

Applications of the system of equations $A\mathbf{x} = \mathbf{b}$

Linear independence and linear dependence

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots + \alpha_m\mathbf{v}_m = \mathbf{0}$$

Applications of the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$

Wronskian of functions defined on $\mathbb{R}$ or subset of $\mathbb{R}$

Basis and dimension of vector spaces. Examples: $\mathbb{R}^n$, $M_n(\mathbb{R})$, $P_n(\mathbb{R})$, $C(\mathbb{R})$.

Computations of dimensions of vector spaces: symmetric matrices in $M_n(\mathbb{R})$. Skew-symmetric matrices in $M_n(\mathbb{R})$.

Component vector $[\mathbf{v}]_{\mathcal{B}}$ of the vector $\mathbf{v} \in \mathbf{V}$ relative to a basis $\mathcal{B}$

Change-of-basis-matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$.

The rank-nullity theorem

Chapter 4: Linear Transformations

Definition and basic properties of linear transformation

The kernel of a linear transformation, one-to-one linear transformations

The range of a linear transformation, onto linear transformations

Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$

Linear transformation from $P_n(\mathbb{R})$ to $P_n(\mathbb{R})$

The existence of inverse linear transformation, isomorphisms

The rank-nullity theorem

Chapter 5: Eigenvalues, Eigenvectors and Diagonalization of Matrices

Definition of eigenvalues, eigenvectors, eigenspaces

Algebraic multiplicities, geometric multiplicities of eigenvalues

Diagonalization of matrices $T^{-1}AT = D$

Criterion of diagonalization

$$\det(A) = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n.$$

$$a_{11} + a_{22} + a_{33} + \cdots\cdots + a_{nn} = \lambda_1 + \lambda_2 + \lambda_3 + \cdots\cdots + \lambda_n$$

Eigenvalues and eigenvectors of the matrix $f(A)$, where

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots + a_m x^m,$$
$$f(A) = a_0 I + a_1 A + a_2 A^2 + a_3 A^3 + \cdots + a_m A^m$$

is a real polynomial function of x.

Eigenvalues of real matrices

Eigenvalues of real symmetric matrices

Chapter 6: Vector Spaces with Inner Products

Definition and properties of inner product

Orthogonality and Orthonormal sets

Orthogonal projections

The Gram-Schmidt orthogonal process

Least square solutions of system of equations

Chapter 7: Real Symmetric Matrices and Quadratic Forms

Eigenvalues and eigenvectors of symmetric matrices

Diagonalization of symmetric matrices

Projection matrices, Spectral decomposition

Quadratic forms of symmetric matrices

The Definition of a Real Vector Space: Let $\mathbf{V}$ be a nonempty set. There exist two operations: the addition $+$ and the scalar multiplication $\cdot$, in $\mathbf{V}$. For any two elements $\mathbf{u} \in \mathbf{V}$ and $\mathbf{v} \in \mathbf{V}$, for any real constant α, the addition $\mathbf{u} + \mathbf{v} \in \mathbf{V}$ and the scalar multiplication $\alpha\mathbf{u} \in \mathbf{V}$. That is

$$\mathbf{u} + \mathbf{v} \in \mathbf{V}, \qquad \alpha\mathbf{u} \in \mathbf{V}.$$

The set $\mathbf{V}$ is called a real vector space, if the following conditions are satisfied.
(1)

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u},$$

for all elements $\mathbf{u} \in \mathbf{V}$ and $\mathbf{v} \in \mathbf{V}$.
(2)

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}),$$

for all elements $\mathbf{u} \in \mathbf{V}$, $\mathbf{v} \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{V}$.
(3) There exists an element $\mathbf{0} \in \mathbf{V}$, such that

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u},$$

for all elements $\mathbf{u} \in \mathbf{V}$.
(4) For any element $\mathbf{u} \in \mathbf{V}$, there exists another element $\mathbf{u}' \in \mathbf{V}$, such that

$$\mathbf{u} + \mathbf{u}' = \mathbf{u}' + \mathbf{u} = \mathbf{0}.$$

(5)

$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v},$$

for all elements $\mathbf{u} \in \mathbf{V}$ and $\mathbf{v} \in \mathbf{V}$, for all real constants $\alpha \in \mathbb{R}$.
(6)

$$(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u},$$

(7)

$$(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u}) = \beta(\alpha\mathbf{u}),$$

for all elements $\mathbf{u} \in \mathbf{V}$ and for all real constants $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$.
(8)

$$1\mathbf{u} = \mathbf{u},$$

for all elements $\mathbf{u} \in \mathbf{V}$.

Example 1. Let $\mathbf{V} = \mathbb{R}^n$. Define the addition and scalar multiplication by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \cdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ \cdots \\ x_n + y_n \end{pmatrix},$$

$$\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \\ \cdots \\ \alpha x_n \end{pmatrix}.$$

The set $\mathbb{R}^n$ is a real vector space.

Example 2. Let $\mathbf{V} = M_n(\mathbb{R})$. Let

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \in M_n(\mathbb{R}).$$

Define the addition and scalar multiplication by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} & \cdots & a_{2n}+b_{2n} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} & \cdots & a_{3n}+b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & a_{n3}+b_{n3} & \cdots & a_{nn}+b_{nn} \end{pmatrix},$$

$$\alpha \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} & \cdots & \alpha a_{2n} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} & \cdots & \alpha a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha a_{n1} & \alpha a_{n2} & \alpha a_{n3} & \cdots & \alpha a_{nn} \end{pmatrix}.$$

Example 3. Let $\mathbf{V} = P_n(\mathbb{R})$. Let

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\cdots + a_nx^n \in P_n(\mathbb{R}), \\ g(x) &= b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots\cdots + b_nx^n \in P_n(\mathbb{R}), \end{aligned}$$

be any two vectors in $P_n(\mathbb{R})$. Define the addition and scalar multiplication by

$$\begin{aligned} & f(x) + g(x) \\ = \;& a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\cdots + a_nx^n \\ + \;& b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots\cdots + b_nx^n \\ = \;& (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + (a_3+b_3)x^3 + \cdots\cdots + (a_n+b_n)x \\ & \alpha f(x) \\ = \;& \alpha(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\cdots + a_nx^n) \\ = \;& (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + (\alpha a_3)x^3 + \cdots\cdots + (\alpha a_n)x^n, \end{aligned}$$

The set $P_n(\mathbb{R})$ is a real vector space.

Example 4. Let

$$C(\mathbb{R}) = \{f : f \text{ is a continuous function defined on } \mathbb{R}\}.$$

Define the addition and scalar multiplication by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x), \\ (\alpha f)(x) &= \alpha f(x), \end{aligned}$$

for all $x \in \mathbb{R}$.

Then the set C($\mathbb{R}$) is a real vector space.

The Definition of Real Vector Subspace: Let $\mathbf{V}$ be a real vector space. Let $\mathbf{W}$ be a subset of $\mathbf{V}$. $\mathbf{W}$ is called a real vector subspace of the vector space $\mathbf{V}$ if the following conditions are satisfied

$$\begin{aligned} &\mathbf{u} + \mathbf{v} \in \mathbf{W}, \\ &\alpha \mathbf{u} \in \mathbf{W}, \end{aligned}$$

for all vectors $\mathbf{u} \in \mathbf{W}$ and $\mathbf{v} \in \mathbf{W}$, for all real constants $\alpha \in \mathbb{R}$.

Examples Which of the following subsets is a vector subspcae of $M_n(\mathbb{R})$?

$$\begin{aligned} &\mathbf{W}_1 = \{A \in M_n(\mathbb{R}) : \det(A^T) = \det(A)\}. \\ &\mathbf{W}_2 = \{A \in M_n(\mathbb{R}) : A^T = A\}. \\ &\mathbf{W}_3 = \{A \in M_n(\mathbb{R}) : A^T = -A\}. \\ &\mathbf{W}_4 = \{A \in M_n(\mathbb{R}) : A^{-1} \text{ exists}\}. \\ &\mathbf{W}_5 = \{A \in M_n(\mathbb{R}) : \det(A) = 0\}. \\ &\mathbf{W}_6 = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\}. \end{aligned}$$

Basis of popular vector spaces:

For the vector space $\mathbb{R}^2$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

For the vector space $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

For the vector space $\mathbb{R}^4$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

For the vector space $\mathbb{R}^5$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The dimension of the vector space $\mathbb{R}^n$:

$$\dim(\mathbb{R}^n) = n.$$

For the vector space $P_1(\mathbb{R})$:

$$\mathcal{B} = \{1, x\}.$$

For the vector space $P_2(\mathbb{R})$:

$$\mathcal{B} = \{1, x, x^2\}.$$

For the vector space $P_3(\mathbb{R})$:

$$\mathcal{B} = \{1, x, x^2, x^3\}.$$

For the vector space $P_4(\mathbb{R})$:

$$\mathcal{B} = \{1, x, x^2, x^3, x^4\}.$$

For the vector space $P_n(\mathbb{R})$:

$$\mathcal{B} = \{1, x, x^2, x^3, \cdots, \cdots, x^n\}.$$

The dimension of the vector space $P_n(\mathbb{R})$:

$$\dim[P_n(\mathbb{R})] = n + 1.$$

For the vector space $M_2(\mathbb{R})$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

For the vector space $M_3(\mathbb{R})$:

$$\begin{aligned} \mathcal{B} = & \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right. \\ & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ & \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}. \end{aligned}$$

The dimension of the vector space $M_n(\mathbb{R})$:

$$\dim[M_n(\mathbb{R})] = n^2.$$

For the vector space (only symmetric matrices) $M_2(\mathbb{R})$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

For the vector space (only skew-symmetric matrices) $M_2(\mathbb{R})$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

0 .0 .

For the vector space (only symmetric matrices) $M_3(\mathbb{R})$:

$$\begin{aligned} \mathcal{B} \ &= \ \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}. \end{aligned}$$

For the vector space (only skew-symmetric matrices) $M_3(\mathbb{R})$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}.$$

For the vector space (only symmetric matrices) $M_4(\mathbb{R})$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}.$$

The dimension of the vector space $M_n(\mathbb{R})$:

$$\dim[M_n(\mathbb{R})] \text{ only symmetric matrices } = \frac{n(n+1)}{2}.$$

The dimension of the vector space $M_n(\mathbb{R})$:

$$\dim[M_n(\mathbb{R})] \text{ only skew-symmetric matrices } = \frac{n(n-1)}{2}$$

Summary of the Chapter on Vector Spaces

Examples of vector spaces:

$$\mathbb{R}^n, \qquad M_n(\mathbb{R}), \qquad P_n(\mathbb{R}), \qquad C(\mathbb{R}).$$

Subspaces of the Vector Space V

A subset $\mathbf{W}$ is a vector subspace of the vector space $\mathbf{V}$ if the following property is true. For any vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\cdots\cdots$, $\mathbf{v}_n$ in $\mathbf{W}$, for any real constants α_1, α_2, α_3, $\cdots\cdots$, α_n in $\mathbb{R}$, the linear combination

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots\cdots + \alpha_n\mathbf{v}_n \in \mathbf{W},$$

where $n \geq 1$ may be any positive integer.

Let A be a real $m \times n$ matrix. We may rewrite the matrix A in the following ways

$$A = \left(\; \mathbf{c}_1, \;\; \mathbf{c}_2, \;\; \mathbf{c}_3, \;\; \cdots\cdots \;, \;\; \mathbf{c}_n \; \right),$$

$$A = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \cdots\cdots \\ \mathbf{r}_m \end{pmatrix},$$

where $\mathbf{r}_i$ represents a row vector and $\mathbf{c}_j$ represents a column vector, the index $i = 1, 2, 3, \cdots, m$ and $j = 1, 2, 3, \cdots, n$.

The nullspace

$$NS(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

is a subspace of $\mathbb{R}^n$.

The row space

$$\begin{aligned} & RS(A) = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots\cdots, \mathbf{r}_m\} \\ = & \{\alpha_1\mathbf{r}_1 + \alpha_2\mathbf{r}_2 + \alpha_3\mathbf{r}_3 + \cdots + \alpha_m\mathbf{r}_m : \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}, \alpha_3 \in \mathbb{R}, \cdots, \alpha_m \in \end{aligned}$$

is a subspace of $\mathbb{R}^n$.

The column space

$$\begin{aligned} & CS(A) = \text{span}\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \cdots\cdots, \mathbf{c}_n\} \\ = & \{\alpha_1\mathbf{c}_1 + \alpha_2\mathbf{c}_2 + \alpha_3\mathbf{c}_3 + \cdots + \alpha_n\mathbf{c}_n : \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}, \alpha_3 \in \mathbb{R}, \cdots, \alpha_n \in \end{aligned}$$

is a subspace of $\mathbb{R}^m$.

For these concepts, basis and dimension are the most important points.

Example . Let A be a real $n \times n$ constant matrix, let the rank of A be equal to n. Then the nullspace, the row space and the column space of A are given, respectively, by

$$\begin{aligned} NS(A) &= \{\mathbf{0}\}, \dim[NS(A)] = 0 \\ RS(A) &= \mathbb{R}^n, \dim[RS(A)] = n \\ CS(A) &= \mathbb{R}^n, \dim[CS(A)] = n \end{aligned}$$

Example . Let A be a real $m \times n$ constant matrix, let $m > n$ and the rank of A be equal to n. Then the nullspace, the row space and the column space of A are given, respectively, by

$$\begin{aligned} & NS(A) = \{\mathbf{0}\}, \dim[NS(A)] = 0 \\ & RS(A) = \mathbb{R}^n, \dim[RS(A)] = n \\ & CS(A) \text{ is smaller than, it is a subspace of } \mathbb{R}^m, \dim[CS(A)] = n \end{aligned}$$

Example . Let A be a real $m \times n$ constant matrix, let $m < n$ and the rank of A be equal to m. Then the nullspace, the row space and the column space of A are given, respectively, by

$NS(A) = \mathbb{R}^n, \dim[NS(A)] = n - m$
$RS(A)$ is smaller than, it is a subspace of $\mathbb{R}^n, \dim[RS(A)] = m < n$
$CS(A) = \mathbb{R}^m, \dim[CS(A)] = m$

Let $\mathbf{V}$ be a real vector space, such as $\mathbb{R}^n\mathbb{R}^n$. Let $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots\cdots, \mathbf{v}_n\}$ be a set of vectors in $\mathbf{V}$. Then $\mathcal{S}$ is a basis if $\mathcal{S}$ is a spanning set and linearly independent.

For the vector space $\mathbb{R}^n$, if $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots\cdots, \mathbf{v}_n\}$, then define a matrix $A = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \ \cdots\cdots\ \ \mathbf{v}_n)$. If the determinant of the matrix

$$\det(A) \neq 0,$$

then
(1) $\mathcal{S}$ is a spanning set of $\mathbb{R}^n$.
(2) $\mathcal{S}$ is linearly independent.
(3) $\mathcal{S}$ is a basis of $\mathbb{R}^n$.

On the other hand, if

$$\det(A) = 0,$$

then
(1) $\mathcal{S}$ is not a spanning set of $\mathbb{R}^n$.
(2) $\mathcal{S}$ is linearly dependent.
(3) $\mathcal{S}$ is not a basis of $\mathbb{R}^n$.

For the vector space $P_n(\mathbb{R})$, if $\mathcal{S} = \{p_0(x), p_1(x), p_2(x), p_3(x), \cdots, p_n(x)\}$, and the Wronskian

$$\det\begin{pmatrix} p_0(x) & p_1(x) & p_2(x) & p_3(x) & \cdots\cdots & p_n(x) \\ p_0'(x) & p_1'(x) & p_2'(x) & p_3'(x) & \cdots\cdots & p_n'(x) \\ p_0''(x) & p_1''(x) & p_2''(x) & p_3''(x) & \cdots\cdots & p_n''(x) \\ p_0'''(x) & p_1'''(x) & p_2'''(x) & p_3'''(x) & \cdots\cdots & p_n'''(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ p_0^{(n-1)}(x) & p_1^{(n-1)}(x) & p_2^{(n-1)}(x) & p_3^{(n-1)}(x) & \cdots\cdots & p_n^{(n-1)}(x) \\ p_0^{(n)}(x) & p_1^{(n)}(x) & p_2^{(n)}(x) & p_3^{(n)}(x) & \cdots\cdots & p_n^{(n)}(x) \end{pmatrix} \neq 0,$$

then

(1) $\mathcal{S}$ is a spanning set of $P_n(\mathbb{R})$.

(2) $\mathcal{S}$ is linearly independent.

(3) $\mathcal{S}$ is a basis of $P_n(\mathbb{R})$.

For the vector space $M_n(\mathbb{R})$, if $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots\cdots, \mathbf{v}_{n^2}\}$, then define the real $n^2 \times n^2$ matrix $A = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots\cdots, \mathbf{w}_{n^2})$. If the determinant of the matrix

$$\det(A) \neq 0,$$

then

(1) $\mathcal{S}$ is a spanning set of $M_n(\mathbb{R})$.

(2) $\mathcal{S}$ is linearly independent.

(3) $\mathcal{S}$ is a basis of $M_n(\mathbb{R})$.

On the other hand, if

$$\det(A) = 0,$$

then

(1) $\mathcal{S}$ is not a spanning set of $M_n(\mathbb{R})$.

(2) $\mathcal{S}$ is linearly dependent.

(3) $\mathcal{S}$ is not a basis of $M_n(\mathbb{R})$.

The Rank-Nullity Theorem: Let A be a real $m \times n$ matrix. Then

$$\text{rank }(A) + \dim[NS(A)] = n.$$
$$\text{rank }(A) + \text{nullity} = n.$$

Here $\dim[NS(A)]$ is equal to the nullity.

Let

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots\cdots, \mathbf{v}_n\}$$

be a basis of the vector space $\mathbf{V} = \mathbb{R}^n$. Let $\mathbf{v} \in \mathbf{V}$. The component vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$ relative to the basis $\mathcal{B}$ is

$$[\mathbf{v}]_{\mathcal{B}} = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\cdots\ \mathbf{v}_n)^{-1}\mathbf{v}.$$

Let

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots\cdots, \mathbf{v}_n\},$$

and

$$\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots\cdots, \mathbf{w}_n\},$$

be two bases of the vector space $\mathbb{R}^n$. The change-of-basis-matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = C^{-1}B.$$

The change-of-basis-matrix from the basis $\mathcal{C}$ to the basis $\mathcal{B}$ is

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = B^{-1}C.$$

The change-of-basis-matrices satisfy the equation

$$P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}.$$

The Definition of A Linear Transformation: Let $\mathbf{V}$ and $\mathbf{W}$ be real vector spaces. The mapping $T : \mathbf{V} \to \mathbf{W}$ is called a linear transformation, if

$$\begin{aligned} & T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots\cdots + \alpha_n\mathbf{v}_n) \\ = \; & \alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \alpha_3 T(\mathbf{v}_3) + \cdots\cdots + \alpha_n T(\mathbf{v}_n), \end{aligned}$$

for all vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\cdots\cdots$, $\mathbf{v}_n$, and for all real constants α_1, α_2, α_3, $\cdots\cdots$, $\alpha_n \in \mathbb{R}$, where $n \geq 1$ is an integer.

Definition. Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Define the kernel, represented by $\mathrm{Ker}(T)$, of the linear transformation T by

$$\mathrm{Ker}(T) = \{\mathbf{v} \in \mathbf{V} : T(\mathbf{v}) = \mathbf{0}\}.$$

Note: The kernel $\mathrm{Ker}(T)$ of the linear transformation T is a subspace of $\mathbf{V}$.

Define the range, represented by $\mathrm{Rng}(T)$, of the linear transformation T by

$$\mathrm{Rng}(T) = \{T(\mathbf{v}) : \mathbf{v} \in \mathbf{V}\}.$$

Note: The range $\mathrm{Rng}(T)$ of the linear transformation T is a subspace of $\mathbf{W}$.

The linear transformation $T : \mathbf{V} \to \mathbf{W}$ is called a one-to-one transformation, if the kernel $\mathrm{Ker}(T) = \{\mathbf{0}\}$.

The linear transformation $T : \mathbf{V} \to \mathbf{W}$ is called an onto transformation, if the range $\mathrm{Rng}(T) = \mathbf{W}$.

The linear transformation $T : \mathbf{V} \to \mathbf{W}$ is called a linear isomorphism, if it is both one-to-one and onto. We write

$$\begin{aligned} T : \mathbf{V} \to \mathbf{W}, &\quad \mathbf{w} = T(\mathbf{v}), \\ T^{-1} : \mathbf{W} \to \mathbf{V}, &\quad \mathbf{v} = T^{-1}(\mathbf{w}). \end{aligned}$$

Let $T:\mathbf{V}\to\mathbf{W}$ be a linear transformation. The kernel Ker(T) is a subspace of the vector space $\mathbf{V}$, the range Rng(T) is a subspace of the vector space $\mathbf{W}$.

$$\dim[\text{ Ker }(T)]+\dim[\text{ Rng }(T)]=\dim(\mathbf{V}).$$

This is called the rank nullity theorem for linear transformations.

Important Examples of Linear Transformations

1. $T:\mathbb{R}^n\to\mathbb{R}^m$ is a linear transformation, it is given by

$$T(\mathbf{x})=A\mathbf{x},\qquad \mathbf{x}\in\mathbb{R}^n.$$

Here A is a real $m\times n$ matrix.

For this example, the kernel of the linear transformation T is the same as the null space of the matrix A.

The range of the linear transformation T is the same as the column space of the matrix A.

The linear transformation T is one-to-one, if the rank of A is n, where $m\geq n$.

The linear transformation T is onto, if the rank of A is m, where $m\leq n$.

The linear transformation T is one-to-one and onto, if the rank of A is n, where $m=n$.

The linear transformation T and the inverse linear transformation T^{-1}

$$\begin{aligned}T:\mathbb{R}^n\to\mathbb{R}^n,&\quad \mathbf{y}=T(\mathbf{x}),\\ T^{-1}:\mathbb{R}^n\to\mathbb{R}^n,&\quad \mathbf{x}=T^{-1}(\mathbf{y}).\end{aligned}$$

2. $T:P_n(\mathbb{R})\to P_n(\mathbb{R})$ is a linear transformation.

$$\begin{aligned}&T:P_n(\mathbb{R})\to P_n(\mathbb{R}),\\ &T(a_0+a_1x+a_2x^2+a_3x^3+\cdots\cdots+a_nx^n)\\ =\ &b_0+b_1x+b_2x^2+b_3x^3+\cdots\cdots+b_nx^n,\end{aligned}$$

where each b_i is a linear combination of all of the coefficients a_0, a_1, a_2, a_3, $\cdots\cdots$, a_n.

We may write it as

$$T\left[(1\ x\ x^2\ x^3\ \cdots\ x^n)\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \cdots \\ a_n \end{pmatrix}\right] = (1\ x\ x^2\ x^3\ \cdots\ x^n)A\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \cdots \\ a_n \end{pmatrix}.$$

where A is a real $n \times n$ matrix.

3. Let $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be a linear transformation, given by

$$T(A) = A - A^T, \qquad A \in M_n(\mathbb{R}).$$

(1) Find the kernel of the linear transformation T.

(2) Find the range of the linear transformation T.

First of all, let us find the kernel. Let $T(A) = \mathbf{0}$. Then

$$A - A^T = \mathbf{0}, \qquad A^T = A.$$

Therefore, the kernel

$$\text{Ker } (T) = \{A \in M_n(\mathbb{R}) : A^T = A\}$$

The kernel is the collection of all real $n \times n$ symmetric matrices.

Now let us find the range. Note that

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T).$$

Therefore, the range

$$\text{Rng } (T) = \{M \in M_n(\mathbb{R}) : M^T = -M\}.$$

This is the collection of all real $n \times n$ skew-symmetric matrices.

Let $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be a linear transformation, given by

$$T(A) = A + A^T, \qquad A \in M_n(\mathbb{R}).$$

(1) Find the kernel of the linear transformation T.

(2) Find the range of the linear transformation T.

First of all, let us find the kernel. Let $T(A) = \mathbf{0}$. Then

$$A + A^T = \mathbf{0}, \qquad A^T = -A.$$

Therefore, the kernel

$$\text{Ker } (T) = \{A \in M_n(\mathbb{R}) : A^T = -A\}$$

The kernel is the collection of all real $n \times n$ skew-symmetric matrices.

Now let us find the range. Note that

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T.$$

Therefore, the range

$$\text{Rng } (T) = \{M \in M_n(\mathbb{R}) : M^T = M\}.$$

This is the collection of all real $n \times n$ symmetric matrices.

Example. Let $T : P_4(\mathbb{R}) \to P_4(\mathbb{R})$ be a linear transformation, given by

$$\begin{aligned} & T(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) \\ = \; & a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 \\ + \; & (a_0 + a_1 + a_2 + a_3)x^3 + (a_0 + a_1 + a_2 + a_3 + a_4)x^4. \end{aligned}$$

Find the kernel and the range of the linear transformation.

Solution: To find the kernel, let us solve the equation

$$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) = 0.$$

That is

$$a_0 = 0, \qquad a_0 + a_1 = 0, \qquad a_0 + a_1 + a_2 = 0,$$
$$a_0 + a_1 + a_2 + a_3 = 0, \qquad a_0 + a_1 + a_2 + a_3 + a_4 = 0.$$

Solving this system of equations, we find that

$$a_0 = 0, \qquad a_1 = 0, \qquad a_2 = 0, \qquad a_3 = 0, \qquad a_4 = 0.$$

Therefore, the kernel is the trivial space:

$$\text{Ker }(T) = \{0\}.$$

Note that the right hand side of the linear transformation may be written as the linear combination of five linearly independent constant vectors:

$$\begin{aligned} & T(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) \\ = \; & a_0(1 + x + x^2 + x^3 + x^4) + a_1(x + x^2 + x^3 + x^4) \\ + \; & a_2(x^2 + x^3 + x^4) + a_3(x^3 + x^4) + a_4x^4. \end{aligned}$$

Note that a_0, a_1, a_2, a_3, a_4 are free variables. Moreover, the set

$$\begin{aligned} \mathcal{B} \; = \; & \{1 + x + x^2 + x^3 + x^4, \qquad x + x^2 + x^3 + x^4, \\ & x^2 + x^3 + x^4, \qquad x^3 + x^4, \qquad x^4\} \end{aligned}$$

is linearly independent. It is a spanning set and it is a basis of $P_4(\mathbb{R})$. Therefore, the range is given by

$$\text{Rng }(T) = P_4(\mathbb{R}).$$

Examples of Isomorphisms

1.

$$T : \mathbb{R}^4 \to M_2(\mathbb{R}),$$
$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

2.

$$\begin{aligned} &T : \mathbb{R}^4 \to P_3(\mathbb{R}), \\ &T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a + bx + cx^2 + dx^3. \end{aligned}$$

3.

$$\begin{aligned} &T : M_2(\mathbb{R}) \to P_3(\mathbb{R}), \\ &T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + bx + cx^2 + dx^3. \end{aligned}$$

4.

$$\begin{aligned} &T : M_3(\mathbb{R}) \to M_3(\mathbb{R}), \\ &T\begin{pmatrix} \alpha & a & b \\ a & \beta & c \\ b & c & \gamma \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}. \end{aligned}$$

5.

$$\begin{aligned} &T : M_3(\mathbb{R}) \to M_3(\mathbb{R}), \\ &T\begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}. \end{aligned}$$

6.

$$\begin{aligned} & T : C(\mathbb{R}) \to C(\mathbb{R}), \\ & T(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots\cdots + a_{2n-1}x^{2n-1} + a_{2n}x^{2n}) \\ = \;& a_0 + a_1\cos(x) + a_2\sin(x) + a_3\cos(2x) + a_4\sin(2x) \\ + \;& \cdots\cdots + a_{2n-1}\cos(nx) + a_{2n}\sin(nx), \end{aligned}$$

where $n \geq 0$ is any integer.

Let $T_i : \mathbf{V} \to \mathbf{W}$ be linear transformations, let α_i be real constants, where $i = 1, 2, 3, \cdots\cdots, n$. Then the linear combination of the linear transformations

$$\begin{aligned} & \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3 + \cdots + \alpha_n T_n : \mathbf{V} \to \mathbf{W}, \\ & (\alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3 + \cdots + \alpha_n T_n)(\mathbf{v}) \\ = \; & \alpha_1 T_1(\mathbf{v}) + \alpha_2 T_2(\mathbf{v}) + \alpha_3 T_3(\mathbf{v}) + \cdots + \alpha_n T_n(\mathbf{v}), \qquad \mathbf{v} \in \mathbf{V}, \end{aligned}$$

is also a linear transformation.

In particular, if $T_i : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations, given by

$$T_i(\mathbf{v}) = A_i \mathbf{v}, \qquad \mathbf{v} \in \mathbb{R}^n,$$

then

$$\alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3 + \cdots\cdots + \alpha_p T_p : \mathbb{R}^n \to \mathbb{R}^m,$$

is also a linear transformation, given by

$$\begin{aligned} & \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3 + \cdots\cdots + \alpha_p T_p : \mathbb{R}^n \to \mathbb{R}^m : \\ & (\alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3 + \cdots\cdots + \alpha_p T_p)(\mathbf{v}) \\ = \; & (\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \cdots\cdots + \alpha_p A_p)\mathbf{v}, \end{aligned}$$

for all $\mathbf{v} \in \mathbb{R}^n$.

Let $T : \mathbf{X} \to \mathbf{Y}$ be a linear transformation, let $S : \mathbf{Y} \to \mathbf{Z}$ be another linear transformation. Then

$$ST : \mathbf{X} \to \mathbf{Z}$$

is also a linear transformation, given by

$$(ST)(\mathbf{v}) = S(T(\mathbf{v})), \qquad \mathbf{v} \in \mathbf{X},$$

is also a linear transformation.

In particular, if

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

is a linear transformation, given by

$$T(\mathbf{v}) = A\mathbf{v}, \qquad \mathbf{v} \in \mathbb{R}^n,$$

and

$$S : \mathbb{R}^m \to \mathbb{R}^k,$$

is another linear transformation, given by

$$S(\mathbf{w}) = B\mathbf{w}, \qquad \mathbf{w} \in \mathbb{R}^m,$$

then

$$\begin{aligned} &ST : \mathbb{R}^n \to \mathbb{R}^k, \\ &(ST)(\mathbf{v}) = BA\mathbf{v}, \qquad \mathbf{v} \in \mathbb{R}^n, \end{aligned}$$

is also a linear transformation.

Let $\mathbf{V}$ and $\mathbf{W}$ be real vector spaces. Let

$$\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots\cdots, \mathbf{v}_n\},$$

be an order basis of the vector space $\mathbf{V}$, let

$$\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots\cdots, \mathbf{w}_m\},$$

be an ordered basis of the vector space $\mathbf{W}$. Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. We will define a real $m \times n$ matrix, denoted by $[T]^{\mathcal{C}}_{\mathcal{B}}$. The matrix is called the matrix of the linear transformation T relative to the order basis $\mathcal{B}$ of the vector space $\mathbf{V}$ and the ordered basis $\mathcal{C}$ of the vector space $\mathbf{W}$.

Let us find an equation involving the matrix $[T]_{\mathcal{B}}^{\mathcal{C}}$. Intuitively speaking, the matrix depends on the bases $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots\cdots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots\cdots, \mathbf{w}_m\}$ and the linear transformation T. For any vector $\mathbf{v} \in \mathbf{V}$, we have the linear combination

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots\cdots + \alpha_n\mathbf{v}_n,$$

where α_1, α_2, α_3, $\cdots$, α_n are real constants. Performing the linear transformation T to the linear combination, we have

$$\begin{aligned} & T(\mathbf{v}) = T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \cdots\cdots + \alpha_n\mathbf{v}_n) \\ = \ & \alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \alpha_3 T(\mathbf{v}_3) + \cdots\cdots + \alpha_n T(\mathbf{v}_n) \\ = \ & (T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), \cdots\cdots, T(\mathbf{v}_n)) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdots \\ \alpha_n \end{pmatrix}. \end{aligned}$$

Note that each vector $T(\mathbf{v}_j)$ is a linear combination of

$$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots\cdots, \mathbf{w}_m.$$

That is

$$\begin{aligned} & T(\mathbf{v}_j) = a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + a_{3j}\mathbf{w}_3 + \cdots\cdots + a_{mj}\mathbf{w}_m \\ = \ & (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots\cdots, \mathbf{w}_m) \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \cdots \\ a_{mj} \end{pmatrix}, \end{aligned}$$

where a_{1j}, a_{2j}, a_{3j}, $\cdots$, a_{mj} are real constants, $j = 1, 2, 3, \cdots, n$. In another word, we have

$$\begin{aligned} &(T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), \cdots\cdots, T(\mathbf{v}_n)) \\ = \ &(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots\cdots, \mathbf{w}_m) \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}. \end{aligned}$$

Therefore, we obtain the explicit representation

$$\begin{aligned} &T(\mathbf{v}) \\ = \ &(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots\cdots, \mathbf{w}_m) \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdots \\ \alpha_n \end{pmatrix}. \end{aligned}$$

Here, the matrix

$$[T]_{\mathcal{B}}^{\mathcal{C}} \stackrel{\text{def}}{=} \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}.$$

In particular, for the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, with the ordered basis

$$\begin{aligned} \mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\} \quad &\text{for} \quad \mathbb{R}^n, \\ \mathcal{C} = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots, \mathbf{w}_m\} \quad &\text{for} \quad \mathbb{R}^m, \end{aligned}$$

we have

$$[T]_{\mathcal{B}}^{\mathcal{C}} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots, \mathbf{w}_m)^{-1}(T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), \cdots, T(\mathbf{v}_n)).$$

Note that

$$(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots, \mathbf{w}_m),$$

is an $m \times m$ matrix, and its inverse matrix exists.

$$(T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), \cdots, T(\mathbf{v}_n))$$

is a real $m \times n$ matrix.

For the linear transformation

$$T : P_n(\mathbb{R}) \to P_m(\mathbb{R}),$$

we may the same ideas as above, with some slight modifications in some of the details, to find the matrix of linear transformation $[T]_{\mathcal{B}}^{\mathcal{C}}$.

For all other linear transformations, the equation

$$(T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3), \cdots\cdots, T(\mathbf{v}_n)) = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots\cdots, \mathbf{w}_n)[T]_{\mathcal{B}}^{\mathcal{C}},$$

is the main starting point to find the matrix of linear transformation $[T]_{\mathcal{B}}^{\mathcal{C}}$.

For linear transformations $T : \mathbf{V} \to \mathbf{W}$ between different kinds of vector spaces, such as

$$T : M_2(\mathbb{R}) \to P_3(\mathbb{R}),$$
$$T : M_2(\mathbb{R}) \to M_3(\mathbb{R}),$$

use

$$[T]_{\mathcal{B}}^{\mathcal{C}} = \left\{[T(\mathbf{v}_1)]_{\mathcal{C}}, [T(\mathbf{v}_2)]_{\mathcal{C}}, [T(\mathbf{v}_3)]_{\mathcal{C}}, \cdots\cdots, [T(\mathbf{v}_n)]_{\mathcal{C}}\right\},$$

where

$$\begin{aligned} T(\mathbf{v}_j) &= a_{1j}\mathbf{w}_1 + a_{2j}\mathbf{w}_2 + a_{3j}\mathbf{w}_3 + \cdots\cdots + a_{mj}\mathbf{w}_m \\ &= \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \cdots\cdots & \mathbf{w}_m \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \cdots \\ a_{mj} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 & \cdots\cdots & \mathbf{w}_m \end{pmatrix} [T(\mathbf{v}_j)]_{\mathcal{C}}, \end{aligned}$$

where

$$[T(\mathbf{v}_j)]_{\mathcal{C}} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \cdots \\ a_{mj} \end{pmatrix},$$

$j = 1, 2, 3, \cdots, n.$

$$\begin{aligned} \mathbf{v} \in \mathbf{V}\mathbf{v} \in \mathbf{V}\mathbf{v} \in \mathbf{V}\mathbf{v} \in \mathbf{V}[\mathbf{v}]_{\mathcal{B}} \\ \alpha\alpha\alpha\alpha \in \alpha \in \alpha \in \mathbb{R} \end{aligned}$$

$$\begin{gathered} () \\ () \\ () \\ () \\ () \\ () \end{gathered}$$

The First Midterm Exam - Mathematics 43 - 2018

There are twenty-five multiple choice problems in the Exam. Each problem is worth four points. The total is 100 points. The exam is 75 minutes. In each problem, there are four choices marked (A), (B), (C) and (D). Only one choice is correct. Choose the one you think is correct. If you do not know the answer to a problem, you may make a reasonable, best possible guess. No supporting work is necessary. No calculators, computers, cell phones, i-pads, i-touches or any other electronic devices are allowed in the exam. You are not allowed to ask for assistance from anybody else. You are not allowed to use any websites for assistance. Any student who cheats will receive an F as the Final Grade. This is Absolutely Firm! Therefore, be honest and solve all problems by yourself.

The First Midterm Exam 100 points
The Second Midterm Exam 100 points
The Final Exam 200 points
The Homework 100 points
Total 500 points

1. The reduced row echelon form of the augmented matrix of the system of equations $A\mathbf{x} = \mathbf{b}$ is a zero matrix, that is,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

How many free variables are there in the solutions?

(A) 1. (B) 3. (C) 5. (D) 7.

2. The reduced row echelon form of the augmented matrix of the system of equations $A\mathbf{x} = \mathbf{b}$ is given below

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

How many solutions are there to the system of equations?

(A) ∞. (B) 10. (C) 1. (D) 0.

3. Let A be a real $n \times n$ constant matrix, with the rank of A being equal to n. Which of the following statements is true?

(A) There exists infinitely many solutions to the system of equations $A\mathbf{x} = \mathbf{b}$, for every vector $\mathbf{b} \in \mathbb{R}^n$.

(B) There exists no solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for every vector $\mathbf{b} \in \mathbb{R}^n$.

(C) There exists a solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for some vectors, but not for all vectors $\mathbf{b} \in \mathbb{R}^n$.

(D) There exists a unique solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for every vector $\mathbf{b} \in \mathbb{R}^n$.

4. Which of the following statements is true?

(*A*) If $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$, then $A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$

(*B*) If $B = 4\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$, then $B^{-1} = 4\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

(*C*) If $C = 2\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$, then $C^{-1} = 2\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$

(*D*) If $D = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$, then $D^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

5. Let

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following is the rank of the matrix A?

(A) 1

(B) 2

(C) 3

(D) 4

6. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix},$$

where $\alpha > 0$ and $\beta > 0$ are positive constants, such that $\alpha > 3\beta$. What is the rank of A?
(A) 4. (B) 3. (C) 2. (D) 1.

7. Let A be a real $n \times n$ constant matrix with $\det(A) \neq 0$. Which of the following statements is true?

(A) There exists a unique solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for every vector $\mathbf{b} \in \mathbb{R}^n$.

(B) There exists no solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for any vector $\mathbf{b} \in \mathbb{R}^n$.

(C) The inverse matrix A^{-1} does not exist.

(D) The rank of A is less than n.

8. Let A be a real $n \times n$ constant matrix with $\det(A) = 0$. Which of the following statements is true?

(A) There exists infinitely many solutions to the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$.

(B) There exists a unique solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for every vector $\mathbf{b} \in \mathbb{R}^n$.

(C) The inverse matrix A^{-1} exist.

(D) The rank of A is equal to n.

9. Let α, β, γ and λ be real constants. Which of the following computations is correct?

$$(A) \quad \det \begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^3 + (\alpha^2 + \beta^2 + \gamma^2)\lambda.$$

$$(B) \quad \det \begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^3 + (\alpha^2 + \beta^2 + \gamma^2).$$

$$(C) \quad \det \begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^3 + (\alpha^2 + \beta^2 + \gamma^2)\lambda^2.$$

$$(D) \quad \det \begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^3 - (\alpha^2 + \beta^2 + \gamma^2)\lambda^2.$$

10. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following is equal to the determinant of A?

(A) 16
(B) -16
(C) 4
(D) -4

11. Let

$$A = \begin{pmatrix} \alpha\beta & -\alpha\beta & -\alpha\beta & -\alpha\beta \\ -\alpha\beta & \alpha\beta & -\alpha\beta & -\alpha\beta \\ -\alpha\beta & -\alpha\beta & \alpha\beta & -\alpha\beta \\ -\alpha\beta & -\alpha\beta & -\alpha\beta & \alpha\beta \end{pmatrix}.$$

Which of the following statements represents the determinant of the matrix A?

(A) $16(\alpha\beta)^4$

(B) $-16(\alpha\beta)^4$

(C) $4(\alpha\beta)^{24}$

(D) $-4(\alpha\beta)^{24}$

12. Let A and B be real $n \times n$ matrices. Let $\alpha \neq 0$ be a real nonzero constant. Which of the following statements is wrong?

(A) $\quad \det(A^T) = \det(A)$

(B) $\quad \det(\alpha A) = \alpha \det(A)$

(C) $\quad \det(AB) = \det(A)\det(B)$

(D) $\quad \det(A^{-1}) = \dfrac{1}{\det(A)}$, if $\det(A) \neq 0$

13. Let α, λ, a_i, b_i, c_i be real constants, where $i = 1, 2, 3$. Which of the following computations is NOT correct?

$$(A) \qquad \det \begin{pmatrix} \lambda-\alpha & \alpha & -\alpha \\ \alpha & \lambda-\alpha & -\alpha \\ -\alpha & -\alpha & \lambda-\alpha \end{pmatrix}$$

$$= \det \begin{pmatrix} \lambda-\alpha & \alpha & -\alpha \\ 2\alpha-\lambda & \lambda-2\alpha & 0 \\ \lambda-2\alpha & 0 & \lambda-2\alpha \end{pmatrix}$$

$$= (\lambda-2\alpha)^2 \det \begin{pmatrix} \lambda-\alpha & \alpha & -\alpha \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$(B) \qquad \det \begin{pmatrix} \lambda-\alpha & \beta & \beta \\ \beta & \lambda-\alpha & \beta \\ \beta & \beta & \lambda-\alpha \end{pmatrix}$$

$$= \det \begin{pmatrix} \lambda-\alpha & \beta & \beta \\ \alpha+\beta-\lambda & \lambda-\alpha-\beta & 0 \\ \alpha+\beta-\lambda & 0 & \lambda-\alpha-\beta \end{pmatrix}$$

$$= (\lambda-\alpha-\beta)^2 \det \begin{pmatrix} \lambda-\alpha & \beta & \beta \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$(C) \qquad \det\begin{pmatrix} a_1(\lambda+1) & b_1(\lambda+1) & c_1(\lambda+1) \\ a_2(\lambda+2) & b_2(\lambda+2) & c_2(\lambda+2) \\ a_3(\lambda+3) & b_3(\lambda+3) & c_3(\lambda+3) \end{pmatrix}$$

$$= (\lambda+1)(\lambda+2)(\lambda+3)\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$(D) \qquad \det\begin{pmatrix} a_1(\lambda-1) & a_2(\lambda-2) & a_3(\lambda-3) \\ b_1(\lambda-1) & b_2(\lambda-2) & b_3(\lambda-3) \\ c_1(\lambda-1) & c_2(\lambda-2) & c_3(\lambda-3) \end{pmatrix}$$

$$= (\lambda-1)(\lambda-2)(\lambda-3)\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

14. Let $A = (a_{ij})$ be a real $n \times n$ constant matrix. Let C_{ij} be the cofactor of the element a_{ij}, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$. Which of the following statement is wrong?

$$(A) \quad \sum_{k=1}^{n} a_{ik}C_{ik} = \det(A), \qquad \text{for all } i = 1, 2, 3, \cdots, n$$

$$(B) \quad \sum_{k=1}^{n} a_{kj}C_{kj} = 0, \qquad \text{for all } j = 1, 2, 3, \cdots, n$$

$$(C) \quad \sum_{k=1}^{n} a_{ik}C_{jk} = 0, \ \text{ for all } i = 1, 2, 3, \cdots, n; j = 1, 2, 3, \cdots, n; i \neq j$$

$$(D) \quad \sum_{k=1}^{n} a_{ki}C_{kj} = 0, \ \text{ for all } i = 1, 2, 3, \cdots, n; j = 1, 2, 3, \cdots, n; i \neq j$$

15. Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (C_{ij})$ be real $n \times n$ constant matrices, where $B = C^T$, and C_{ij} represents the cofactor of the element a_{ij}, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$. Which of the following statements is wrong?

$(A)\quad AB = BA = [\det(A)]I$

$(B)\quad AC = CA = [\det(A)]I$

$(C)\quad A^{10}B^5 = B^5A^{10} = [\det(A)]^5A^5$

$(D)\quad A^5B^{10} = B^{10}A^5 = [\det(A)]^5B^5$

16. Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (C_{ij})$ be real $n \times n$ constant matrices, where $\det(A) \neq 0$, $B = C^T$, and C_{ij} represents the cofactor of a_{ij}, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$. Which of the following statements is NOT true?

(A) $\quad A^{-1} = \dfrac{1}{\det(A)} B$

(B) $\quad B^{-1} = \dfrac{1}{\det(A)} A$

(C) $\quad \det(B) = \det(C) = [\det(A)]^n$

(D) $\quad \det(AB) = [\det(A)]^n$

17. Which of the following subsets is a subspace of $\mathbb{R}^4$?

$$(A) \quad \mathbf{W} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 > 0, x_2 > 0, x_3 > 0, x_4 > 0 \right\}.$$

$$(B) \quad \mathbf{W} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1^2 = x_2^2 = x_3^2 = x_4^2 \right\}.$$

$$(C) \quad \mathbf{W} = \left\{ \begin{pmatrix} x_1 \\ 2x_2 \\ 3x_3 \\ 4x_4 \end{pmatrix} : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}, x_4 \in \mathbb{R} \right\}.$$

$$(D) \quad \mathbf{W} = \left\{ \begin{pmatrix} x_1 + 10 \\ x_2 + 20 \\ x_3 + 30 \\ x_4 + 40 \end{pmatrix} : x_k \in \mathbb{R}, k = 1, 2, 3, 4 \right\}.$$

18. Which of the following subsets is NOT a subspace of the vector space $M_n(\mathbb{R})$?

$$(A) \quad \{A \in M_n(\mathbb{R}) : A^T = A\}.$$

$$(B) \quad \{A \in M_n(\mathbb{R}) : A^T = -A\}.$$

$$(C) \quad \{A \in M_n(\mathbb{R}) : \det(A) = 18\}.$$

$$(D) \quad \{A \in M_n(\mathbb{R}) : \det(A^T) = \det(A)\}.$$

19. Let α, β, λ be real constants, such that $\alpha \neq 0$, $\alpha + \beta \neq 0$, $\alpha - 2\beta \neq 0$, $\lambda \neq \alpha + \beta$, $\lambda \neq \alpha - 2\beta$. Which of the following sets of vectors is NOT a spanning set of $\mathbb{R}^3$?

$$(A) \quad \mathcal{A} = \left\{ \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} -\alpha \\ \alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} \right\}.$$

$$(B) \quad \mathcal{B} = \left\{ \begin{pmatrix} \alpha \\ -\beta \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ \alpha \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ -\beta \\ \alpha \end{pmatrix} \right\}.$$

$$(C) \quad \mathcal{C} = \left\{ \begin{pmatrix} 2\alpha \\ -\alpha \\ -\alpha \end{pmatrix}, \begin{pmatrix} -\alpha \\ 2\alpha \\ -\alpha \end{pmatrix}, \begin{pmatrix} -\alpha \\ -\alpha \\ 2\alpha \end{pmatrix} \right\}.$$

$$(D) \quad \mathcal{D} = \left\{ \begin{pmatrix} \lambda - \alpha \\ \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \lambda - \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \\ \lambda - \alpha \end{pmatrix} \right\}.$$

20. Given that the set

$$\mathcal{B} = \left\{ \begin{pmatrix} \lambda - \alpha \\ \beta \\ \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \lambda - \alpha \\ \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \\ \lambda - \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \\ \beta \\ \lambda - \alpha \end{pmatrix} \right\}$$

of vectors is a spanning set of the vector space $\mathbb{R}^4$. Which of the following conditions must be true?

$$\begin{array}{lll} (A) & \lambda = \alpha + \beta, & \lambda = \alpha - 3\beta \\ (B) & \lambda = \alpha + \beta, & \lambda \neq \alpha - 3\beta \\ (C) & \lambda \neq \alpha + \beta, & \lambda \neq \alpha - 3\beta \\ (D) & \lambda \neq \alpha + \beta, & \lambda = \alpha - 3\beta \end{array}$$

21. Given that the set of vectors

$$\mathcal{B} = \left\{ \begin{pmatrix} 3\alpha^2 \\ -\beta^2 \\ -\beta^2 \\ -\beta^2 \end{pmatrix}, \begin{pmatrix} -\beta^2 \\ 3\alpha^2 \\ -\beta^2 \\ -\beta^2 \end{pmatrix}, \begin{pmatrix} -\beta^2 \\ -\beta^2 \\ 3\alpha^2 \\ -\beta^2 \end{pmatrix}, \begin{pmatrix} -\beta^2 \\ -\beta^2 \\ -\beta^2 \\ 3\alpha^2 \end{pmatrix} \right\}$$

is linearly independent. Which of the following conditions must be true?

$$\begin{array}{lll} (A) & 3\alpha^2+\beta^2=0, & \alpha^2-\beta^2=0 \\ (B) & 3\alpha^2+\beta^2=0, & \alpha^2-\beta^2\neq 0 \\ (C) & 3\alpha^2+\beta^2>0, & \alpha^2-\beta^2=0 \\ (D) & 3\alpha^2+\beta^2>0, & \alpha^2-\beta^2\neq 0 \end{array}$$

22. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} \lambda^2 - \alpha^2 \\ \beta^2 \\ \beta^2 \\ \beta^2 \end{pmatrix}, \begin{pmatrix} \beta^2 \\ \lambda^2 - \alpha^2 \\ \beta^2 \\ \beta^2 \end{pmatrix}, \begin{pmatrix} \beta^2 \\ \beta^2 \\ \lambda^2 - \alpha^2 \\ \beta^2 \end{pmatrix}, \begin{pmatrix} \beta^2 \\ \beta^2 \\ \beta^2 \\ \lambda^2 - \alpha^2 \end{pmatrix} \right\}$$

be a basis of $\mathbb{R}^4$. Which of the following conditions must be true?

$$(A) \quad \lambda^2 = \alpha^2 + \beta^2, \qquad \lambda^2 = \alpha^2 - 3\beta^2.$$

$$(B) \quad \lambda^2 = \alpha^2 + \beta^2, \qquad \lambda^2 \neq \alpha^2 - 3\beta^2.$$

$$(C) \quad \lambda^2 \neq \alpha^2 + \beta^2, \qquad \lambda^2 = \alpha^2 - 3\beta^2.$$

$$(D) \quad \lambda^2 \neq \alpha^2 + \beta^2, \qquad \lambda^2 \neq \alpha^2 - 3\beta^2.$$

23. Let

$$\mathcal{S} = \left\{ \begin{pmatrix} \alpha^2 \\ -\beta^2 \\ \alpha^2 \\ -\beta^2 \end{pmatrix}, \begin{pmatrix} -\beta^2 \\ \alpha^2 \\ -\beta^2 \\ \alpha^2 \end{pmatrix}, \begin{pmatrix} \alpha^2 \\ -\beta^2 \\ \alpha^2 \\ -\beta^2 \end{pmatrix}, \begin{pmatrix} -\beta^2 \\ \alpha^2 \\ -\beta^2 \\ \alpha^2 \end{pmatrix} \right\},$$

where $\alpha > 0$ and $\beta > 0$ are positive constants. Which of the following statements about $\mathcal{S}$ is true?

(A) $\mathcal{S}$ is a basis of $\mathbb{R}^4$.

(B) $\mathcal{S}$ is linearly independent.

(C) $\mathcal{S}$ is a spanning set of $\mathbb{R}^4$.

(D) $\mathcal{S}$ is linearly dependent.

24. Let

$$\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots, \mathbf{v}_n\}$$

be a set of vectors in the vector space $\mathbb{R}^n$. Let $A = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_n)$ be the matrix made by using all vectors in $\mathcal{S}$. Let $\det(A) \neq 0$. Which of the following statements is wrong?

(A) $\mathcal{S}$ is a basis of $\mathbb{R}^n$.

(B) $\mathcal{S}$ is linearly independent.

(C) $\mathcal{S}$ is a spanning set of $\mathbb{R}^n$.

(D) $\mathcal{S}$ is linearly independent, but it is not a spanning set of the vector space $\mathbb{R}^n$.

25. Which of the following statements is NOT true?

$$(A) \quad \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis of $\mathbb{R}^4$ and the dimension of $\mathbb{R}^4$ is 4.

$$(B) \quad \mathcal{B} = \left\{ \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 20 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 30 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 40 \end{pmatrix} \right\}$$

is a basis of $M_2(\mathbb{R})$ and the dimension of $M_2(\mathbb{R})$ is 4.

$$(C) \quad \mathcal{C} = \{1, x, x^2, x^3, \cdots, x^n, \cdots\}$$

is a basis of $C(\mathbb{R})$ and the dimension of $C(\mathbb{R})$ is ∞.

$$(D) \quad \mathcal{D} = \{1, x, x^2, x^3, \cdots, x^{10}\}$$

is a basis of $P_{10}(\mathbb{R})$ and the dimension of $P_{10}(\mathbb{R})$ is 10.

Bonus 1 Let

$$A = \alpha \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix},$$

where $\alpha > 0$ is a positive constant. Which of the following statements is true?

$$\begin{array}{ll} (A) & A^{2000} = \alpha^{2000} A \\ (B) & A^{2010} = \alpha^{2010} A \\ (C) & A^{2020} = (2\alpha)^{2040} I \\ (D) & A^{2040} = (2\alpha)^{2040} I \end{array}$$

Bonus 2 Which of the following statements about the real vector space $C(\mathbb{R})$ is NOT true?

(A) $C(\mathbb{R})$ is a infinite-dimensional vector space

(B) $\mathcal{B} = \{1, x, x^2, x^3, \cdots\cdots\}$ is a basis of $C(\mathbb{R})$

(C) $\mathcal{C} = \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \cdots\cdots\}$ is a basis of $C(\mathbb{R})$

(D) $C(\mathbb{R})$ is an finite-dimensional vector space

Bonus 3 There are mistakes in the following computations. Let α, β, γ, δ be real constants, such that $\alpha < \beta < \gamma < \delta$. The Wronskian of the exponential functions $e^{\alpha x}$, $e^{\beta x}$, $e^{\gamma x}$, $e^{\delta x}$ is computed below.

$$\begin{aligned}
&\det\begin{pmatrix} e^{\alpha x} & e^{\beta x} & e^{\gamma x} & e^{\delta x} \\ \alpha e^{\alpha x} & \beta e^{\beta x} & \gamma e^{\gamma x} & \delta e^{\delta x} \\ \alpha^2 e^{\alpha x} & \beta^2 e^{\beta x} & \gamma^2 e^{\gamma x} & \delta^2 e^{\delta x} \\ \alpha^3 e^{\alpha x} & \beta^3 e^{\beta x} & \gamma^3 e^{\gamma x} & \delta^3 e^{\delta x} \end{pmatrix} \\
\overset{\text{step1}}{=}\ & e^{(\alpha+\beta+\gamma+\delta)x}\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{pmatrix} \\
\overset{\text{step2}}{=}\ & e^{(\alpha+\beta+\gamma+\delta)x}\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \beta-\alpha & \gamma-\alpha & \delta-\alpha \\ 0 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 & \delta^2-\alpha^2 \\ 0 & \beta^3-\alpha^3 & \gamma^3-\alpha^3 & \delta^3-\alpha^3 \end{pmatrix} \\
\overset{\text{step3}}{=}\ & (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)e^{(\alpha+\beta+\gamma+\delta)x} \\
\cdot\ & \det\begin{pmatrix} 1 & 1 & 1 \\ \alpha+\beta & \alpha+\gamma & \alpha+\delta \\ \alpha^2+\alpha\beta+\beta^2 & \alpha^2+\alpha\gamma+\gamma^2 & \alpha^2+\alpha\delta+\delta^2 \end{pmatrix} \\
\overset{\text{step4}}{=}\ & e^{(\alpha+\beta+\gamma+\delta)x}(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta).
\end{aligned}$$

In which step of the above computations, the first mistake was made?

(A) step 1

(B) step 2

(C) step 3

(D) step 4

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1. Given that

$$\mathcal{B} = \left\{ \begin{pmatrix} \alpha \\ -\beta \\ -\beta \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ -\beta \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

is a basis of $\mathbb{R}^4$. Which of the following is equal to the component vector $[\mathbf{v}]_{\mathcal{B}}$ of the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ relative to the basis $\mathcal{B}$?

$$(A) \quad + \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$(B) \quad - \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$(C) \quad + \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$(D) \quad - \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

2. Given two bases of the vector space $\mathbb{R}^4$:

$$\mathcal{B} = \left\{ \begin{pmatrix} \alpha \\ -\beta \\ -\beta \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ -\beta \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\},$$

where α and β are real constants, such that $\alpha+\beta > 0$ and $\alpha-3\beta > 0$.

Which of the following is equal to the change-of-basis-matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{C}$?

$$(A) \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}$$

$$(B) \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}^{-1}$$

$$(C) \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$(D) \quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^{-1}$$

3. Let α and β be positive constants, such that $\alpha^2 + \beta^2 > 0$ and $\alpha^2 - \beta^2 > 0$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

Which of the following statements is true?

(A) The dimension of the nullspace is 0, the dimension of the row space is 0 and the dimension of the column space is 0.

(B) The dimension of the nullspace is 2, the dimension of the row space is 2 and the dimension of the column space is 2.

(C) The dimension of the nullspace is 2, the dimension of the row space is 4 and the dimension of the column space is 6.

(D) The dimension of the nullspace is 1, the dimension of the row space is 3 and the dimension of the column space is 5.

4. Let α and β be positive constants, such that $\alpha^2 > \beta^2$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

Which of the following sets of vectors represent a basis of the column space of A?

$$(A) \quad \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$(B) \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 4 \end{pmatrix} \right\}$$

$$(C) \quad \left\{ \begin{pmatrix} \alpha^2 \\ -\beta^2 \\ \alpha^2 \\ -\beta^2 \end{pmatrix}, \begin{pmatrix} -\beta^2 \\ \alpha^2 \\ -\beta^2 \\ \alpha^2 \end{pmatrix} \right\}$$

$$(D) \quad \left\{ \begin{pmatrix} \alpha^2 \\ -\beta^2 \\ \alpha^2 \\ -\beta^2 \end{pmatrix}, \begin{pmatrix} \alpha^2 \\ -\beta^2 \\ \alpha^2 \\ -\beta^2 \end{pmatrix} \right\}$$

5. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, given by

$$T(\mathbf{x}) = A\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where A is an $n \times n$ constant matrix, such that $\det(A) \neq 0$. Which of the following statements in NOT true?

(A) The linear transformation T is one-to-one.

(B) The linear transformation T is onto.

(C) The inverse linear transformation T^{-1} exists and it is given by

$$T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n.$$

(D) None of the above.

6. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, given by

$$T(\mathbf{x}) = A\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where A is an $n \times n$ real matrix, such that $\det(A) = 0$. Which of the following statements is true?

(A) The linear transformation T is not one-to-one or onto.

(B) The linear transformation T is one-to-one.

(C) The linear transformation T is onto.

(D) The inverse linear transformation T^{-1} exists.

7. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, given by

$$T(\mathbf{x}) = A\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where A is an $m \times n$ real constant matrix. Let $m > n$ and let the rank of the matrix be equal to n. Which of the following statements is true?

(A) The linear transformation T is one-to-one.

(B) The linear transformation T is onto.

(C) The linear transformation T is one-to-one and onto.

(D) The inverse linear transformation T^{-1} exists.

8. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, given by

$$T(\mathbf{x}) = A\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where A is an $m \times n$ real constant matrix. Let $m < n$ and let the rank of A be equal to m. Which of the following statements is true?

(A) The linear transformation T is one-to-one.

(B) The linear transformation T is onto.

(C) The linear transformation T is one-to-one and onto.

(D) The inverse linear transformation T^{-1} exists.

9. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, given by

$$T(\mathbf{x}) = A\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where A is an $m \times n$ real constant matrix, $m > 1$ and $n > 1$ are positive integers. Let the rank of A be less than both m and n. Which of the following statements is NOT true?

(A) The linear transformation T is not one-to-one.

(B) The linear transformation T is not onto.

(C) The linear transformation T is neither one-to-one nor onto.

(D) The inverse linear transformation T^{-1} exists.

10. Let $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be a linear transformation, given by

$$T(A) = A^T - A, \text{ for all } A \in M_2(\mathbb{R}).$$

Which of the following sets represent a basis of the kernel $\mathrm{Ker}(T)$?

$$(A) \quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

$$(B) \quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$(C) \quad \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$(D) \quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

11. Let $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be a linear transformation, given by

$$T(A) = A^T - A, \qquad A \in M_n(\mathbb{R}).$$

Which of the following statements are true?

$$(A) \quad \dim[\mathrm{Ker}(T)] = \frac{n(n+1)}{2}, \qquad \dim[\mathrm{Rng}(T)] = \frac{n(n-1)}{2}$$

$$(B) \quad \dim[\mathrm{Ker}(T)] = \frac{n(n+1)}{2}, \qquad \dim[\mathrm{Rng}(T)] = \frac{n(n+1)}{2}$$

$$(C) \quad \dim[\mathrm{Ker}(T)] = \frac{n(n-1)}{2}, \qquad \dim[\mathrm{Rng}(T)] = \frac{n(n-1)}{2}$$

$$(D) \quad \dim[\mathrm{Ker}(T)] = \frac{n(n-1)}{2}, \qquad \dim[\mathrm{Rng}(T)] = \frac{n(n+1)}{2}$$

12. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation, given by

$$\begin{aligned} & T(a + bx + cx^2 + dx^3) \\ = \; & [(2a - b) + (3c - d)] + [(2a - b) - (3c - d)]x \\ + \; & [(2a - b) + (3c - d)]x^2 + [(2a - b) - (3c - d)]x^3 \end{aligned}$$

Which of the following sets of vectors represent a basis of the kernel and a basis of the range of the linear transformation T?

(A) $\mathcal{B} = \{1 + 2x, x^2 + 3x^3\}$, $\mathcal{C} = \{1 + x^2, x + x^3\}$
(B) $\mathcal{B} = \{1 + 4x, x^2 + 6x^3\}$, $\mathcal{C} = \{2 + x^2, 2x + x^3\}$
(C) $\mathcal{B} = \{1 + 6x, x^2 + 9x^3\}$, $\mathcal{C} = \{3 + x^2, 3x + x^3\}$
(D) $\mathcal{B} = \{1 + 8x, x^2 + 12x^3\}$, $\mathcal{C} = \{4 + x^2, 4x + x^3\}$

13. Which of the following linear transformations is NOT an isomorphism?

$$(A)\quad T:\mathbb{R}^3\to\mathbb{R}^3,\qquad T\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$(B)\quad T:\mathbb{R}^3\to\mathbb{R}^3,\qquad T\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}0&\alpha&\beta\\-\alpha&0&\gamma\\-\beta&-\gamma&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$(C)\quad T:\mathbb{R}^4\to M_2(\mathbb{R}),\qquad T\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

$$(D)\quad T:\mathbb{R}^4\to P_3(\mathbb{R}),\qquad T\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=a+bx+cx^2+dx^3$$

where α, β, γ are real constants.

14. Which of the following linear transformations is NOT an isomorphism?

$$(A)\quad T:\mathbb{R}^4\to\mathbb{R}^4,\qquad T\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}=\begin{pmatrix} a+b+c\\ b+c+d\\ c+d+a\\ d+a+b\end{pmatrix}$$

$$(B)\quad T:P_3(\mathbb{R})\to\mathbb{R}^4,\qquad T(a+bx+cx^2+dx^3)=\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}$$

$$(C)\quad T:\mathbb{R}^4\to\mathbb{R}^4,\qquad T\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}=\begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2\\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2\\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2\\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}$$

$$(D)\quad T:\mathbb{R}^4\to\mathbb{R}^4,\qquad T\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}=\begin{pmatrix} 1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}$$

15. Let α and β be positive constants, such that $\alpha^2 > 2\beta^2$. Define

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

Which of the following statements about the algebraic multiplicity and the geometric multiplicity of the eigenvalues is true?

(A) Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = \alpha^2 + \beta^2$ are equal to 2. Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_2 = \alpha^2 - 2\beta^2$ are equal to 2.

(B) Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = \alpha^2 + \beta^2$ are equal to 1. Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_2 = \alpha^2 - 2\beta^2$ are equal to 1.

(C) Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = \alpha^2 + \beta^2$ are equal to 1. Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_2 = \alpha^2 - 2\beta^2$ are equal to 2.

(D) Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = \alpha^2 + \beta^2$ are equal to 2. Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_2 = \alpha^2 - 2\beta^2$ are equal to 1.

16. Let α and β be positive constants, such that $\alpha^2 > 3\beta^2$. Define

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

Which of the following statements about the algebraic multiplicity of the eigenvalues is true?

(A) The first eigenvalue is $\lambda_1 = \alpha^2 + \beta^2$ and the algebraic multiplicity is 3. The second eigenvalue is $\lambda_2 = \alpha^2 - 3\beta^2$ and the algebraic multiplicity is 1.

(B) The first eigenvalue is $\lambda_1 = \alpha^2 + \beta^2$ and the algebraic multiplicity is 1. The second eigenvalue is $\lambda_2 = \alpha^2 - 3\beta^2$ and the algebraic multiplicity is 3.

(C) The first eigenvalue is $\lambda_1 = \alpha^2 + \beta^2$ and the algebraic multiplicity is 3. The second eigenvalue is $\lambda_2 = \alpha^2 - 3\beta^2$ and the algebraic multiplicity is 3.

(D) The first eigenvalue is $\lambda_1 = \alpha^2 + \beta^2$ and the algebraic multiplicity is 1. The second eigenvalue is $\lambda_2 = \alpha^2 - 3\beta^2$ and the algebraic multiplicity is 1.

17. Let A and B be $n \times n$ real constant matrices and $\det(B) > 0$. Which of the following statements is NOT true?

(A) $\det(\lambda I - A) = \det(\lambda I - A^2)$

(B) $\det(\lambda I - A) = \det(\lambda I - A^T)$

(C) $\det(\lambda I - AB) = \det(\lambda I - BA)$

(D) $\det(\lambda I - B^{-1}AB) = \det(\lambda I - A)$

18. Let A be an $n \times n$ real constant matrix. Let

$$\lambda_1, \qquad \lambda_2, \qquad \lambda_3, \qquad \cdots, \qquad \lambda_n$$

be the eigenvalues of A. Which of the following statements in NOT true?

(A) $\det(A) = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n$

(B) $\det(\lambda I - A) = -\det(\lambda I - A)$

(C) $a_{11} + a_{22} + a_{33} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n$

(D) $\det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \cdots (\lambda - \lambda_n)$

19. Let A be an $n \times n$ real constant matrix. Let

$$\lambda_1, \qquad \lambda_2, \qquad \lambda_3, \qquad \cdots, \qquad \lambda_n$$

be the eigenvalues of A. Which of the following is true?

$$\begin{array}{ll} (A) & \det(A) = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n \\ (B) & \det(A) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_n^2 \\ (C) & \det(A) = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n \\ (D) & \det(A) = \lambda_1^2 \lambda_2^2 \lambda_3^2 \cdots \lambda_n^2 \end{array}$$

20. Let A and B be $n \times n$ real constant matrices, such that $A = T^{-1}AT$, where T is also an $n \times n$ constant matrix and T^{-1} exists. Which of the following statements about the eigenvalues and eigenvectors of A and B is true?

(A) The eigenvalues and eigenvectors of A and B are the same.

(B) The eigenvalues and eigenvectors of A and B are different.

(C) The eigenvalues of A and B are different.

(D) The eigenvalues of A and B are the same.

21. Let A be an $n \times n$ real constant matrix. Which of the following statements must be true is the matrix A is diagonalizable?

(A) The algebraic multiplicity is equal to the geometric multiplicity, for each eigenvalue λ of A.

(B) The algebraic multiplicity is equal to the geometric multiplicity, for one eigenvalue λ of A.

(C) The algebraic multiplicity is larger than the geometric multiplicity, for each eigenvalue λ of A.

(D) The algebraic multiplicity is smaller than the geometric multiplicity, for each eigenvalue λ of A.

22. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following matrices is equal to $T^{-1}AT$?

$$(A) \quad \begin{pmatrix} \alpha-3\beta & 0 & 0 & 0 \\ 0 & \alpha+\beta & 0 & 0 \\ 0 & 0 & \alpha+\beta & 0 \\ 0 & 0 & 0 & \alpha+\beta \end{pmatrix}$$

$$(B) \quad \begin{pmatrix} \alpha+\beta & 0 & 0 & 0 \\ 0 & \alpha+\beta & 0 & 0 \\ 0 & 0 & \alpha+\beta & 0 \\ 0 & 0 & 0 & \alpha-3\beta \end{pmatrix}$$

$$(C) \quad \begin{pmatrix} \alpha-3\beta & 0 & 0 & 0 \\ 0 & \alpha-3\beta & 0 & 0 \\ 0 & 0 & \alpha-3\beta & 0 \\ 0 & 0 & 0 & \alpha+\beta \end{pmatrix}$$

$$(D) \quad \begin{pmatrix} \alpha+\beta & 0 & 0 & 0 \\ 0 & \alpha+\beta & 0 & 0 \\ 0 & 0 & \alpha-3\beta & 0 \\ 0 & 0 & 0 & \alpha-3\beta \end{pmatrix}$$

23. Let A be a 4×4 real constant matrix. Two complex eigenvalues of A are given:

$$\lambda_1 = \alpha - \beta \mathrm{i}, \qquad \lambda_2 = \alpha^2 - \beta^2 \mathrm{i},$$

where α and β are real nonzero constants. What are the other complex eigenvalues?

$$\begin{array}{lll} (A) & \lambda_3 = \alpha + \beta \mathrm{i}, & \lambda_4 = \alpha^2 - \beta^2 \mathrm{i} \\ (B) & \lambda_3 = \alpha + \beta \mathrm{i}, & \lambda_4 = \alpha^2 + \beta^2 \mathrm{i} \\ (C) & \lambda_3 = \alpha - \beta \mathrm{i}, & \lambda_4 = \alpha^2 - \beta^2 \mathrm{i} \\ (D) & \lambda_3 = \alpha - \beta \mathrm{i}, & \lambda_4 = \alpha^2 + \beta^2 \mathrm{i} \end{array}$$

24. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix} \mathbf{u}$$

Which of the following represents the solution of the system of differential equations?

$$(A) \qquad \mathbf{u}(t) = C_1 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + C_3 e^{(\alpha-3\beta)t} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(B) \qquad \mathbf{u}(t) = C_1 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + C_3 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 e^{(\alpha-3\beta)t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(C) \qquad \mathbf{u}(t) = C_1 e^{(\alpha-3\beta)t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \; C_3 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(D) \qquad \mathbf{u}(t) = C_1 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{(\alpha-3\beta)t} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \; C_3 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 e^{(\alpha+\beta)t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

25. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix} \mathbf{u}$$

Which of the following represents the solution of the system of differential equations?

$$(A) \qquad \mathbf{u}(t) = C_1 e^{(2\alpha^2+2\beta^2)t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{(2\alpha^2+2\beta^2)t} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \; C_3 e^{(2\alpha^2-2\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 e^{(2\alpha^2-2\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(B) \qquad \mathbf{u}(t) = C_1 e^{(2\alpha^2-2\beta^2)t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{(2\alpha^2-2\beta^2)t} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \; C_3 e^{(2\alpha^2+2\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 e^{(2\alpha^2+2\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(C) \qquad \mathbf{u}(t) = C_1 e^{(2\alpha^2+2\beta^2)t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 e^{(2\alpha^2-2\beta^2)t} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \; C_3 e^{(2\alpha^2+2\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 e^{(2\alpha^2-2^2\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(D) \qquad \mathbf{u}(t) = C_1 e^{(2\alpha^2-2\beta^2)t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \; C_3 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 e^{(2\alpha^2+2\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Mathematics 43 - The Final Examination - 2018

This is the Final Exam of Linear Algebra.

There are 50 multiple choice problems in the Final Examination. Each problem is worth 4 points. The Final Exam is 200 points and it is 3 hours. In each problem, there are four choices, marked with (A), (B), (C) and (D). Only one choice is correct in every problem. On the answer sheet, write down the answer you think is correct. If you do not know the answer to a problem, you may make the possible, best guess. No supporting work is necessary for any problem.

There are three bonus problems at the end of the Final Exam. For some of the Bonus Problems, you will need to show important supporting work to receive credits. Partial credits will be given, it depends on how much valuable work you will do.

In this exam, A, B, C represent real matrices; A^T represents the transposed matrix of A; α, β, γ, λ, μ represent real nonzero constants; $m \geq 1$ and $n \geq 1$ represent positive integers; $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, ξ and η represent real vectors.

Hint 1: *Let*

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

There are two real eigenvalues to the matrix: $\lambda_1 = \alpha^2 + \beta^2$ *and* $\lambda_2 = \alpha^2 - 2\beta^2$. *The eigenvectors corresponding to the eigenvalues are given by*

$$(1) \quad \lambda_1 = \alpha^2 + \beta^2, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$(2) \quad \lambda_2 = \alpha^2 - 2\beta^2, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Hint 2: *Let*

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

There are two real eigenvalues to the matrix: $\lambda_1 = \alpha^2 + \beta^2$ *and* $\lambda_2 = \alpha^2 - 3\beta^2$. *The eigenvectors corresponding to the eigenvalues are given by*

$$(1) \quad \lambda_1 = \alpha^2 + \beta^2, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix},$$

$$(2) \quad \lambda_2 = \alpha^2 - 3\beta^2, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Hint 3: *Let*

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

There are three eigenvalues to the matrix: $\lambda_1 = 2\alpha^2 + 2\beta^2$, $\lambda_2 = 2\alpha^2 - 2\beta^2$, $\lambda_3 = 0$. *The eigenvectors corresponding to these eigenvalues are given by*

$$(1) \quad \lambda_1 = 2\alpha^2 + 2\beta^2, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix},$$

$$(2) \quad \lambda_2 = 2\alpha^2 - 2\beta^2, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$(3) \quad \lambda_3 = 0, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Hint 4: Let

$$U = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Then

$$U^T = U, \qquad U^2 = I.$$

Chapter 1

1. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Which of the following integers is the rank of A?

(A) 1
(B) 2
(C) 3
(D) 4

2. Let A be a real $n \times n$ constant matrix. Which of the following statements is NOT equivalent to the others?

(A) There exists a unique solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$, for each fixed vector $\mathbf{b} \in \mathbb{R}^n$.

(B) The rank of A is less than n.

(C) The inverse matrix A^{-1} exists.

(D) The determinant of A is $\det(A) \neq 0$.

3. Let $A = (a_{ij})$ be a real $m \times n$ constant matrix, let $\mathbf{b} \in \mathbb{R}^m$ be a real constant vector, let $A^{\#}$ represent the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$. Which of the following statements is NOT true?

(A) There exists a unique solution to the system $A\mathbf{x} = \mathbf{b}$, if both the rank of $A^{\#}$ and the rank of A are equal to n.

(B) There exists infinitely many solutions to the system $A\mathbf{x} = \mathbf{b}$, if both the rank of $A^{\#}$ and the rank of A are equal to k and $k < n$.

(C) There exists no solution to the system $A\mathbf{x} = \mathbf{0}$, if the rank of A is less than n.

(D) There exists no solution to the system $A\mathbf{x} = \mathbf{b}$, if the rank of $A^{\#}$ is larger than the rank of A.

Old 3. Let A be a real $n \times n$ matrix. Which of the following statements is NOT true?
(A) $A^T - A$ must be a skew-symmetric matrix.
(B) $A + A^T$ must be a symmetric matrix.
(C) $AA^T - A^TA$ must be a skew-symmetric matrix.
(D) $AA^T + A^TA$ must be a symmetric matrix.

4. Consider the system of linear equations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \begin{pmatrix} 10 \\ 9 \\ 7 \\ 4 \end{pmatrix}.$$

Which of the following is the solution of the system?

$$(A) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(B) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(C) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$(D) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 9 \\ 7 \\ 4 \end{pmatrix}$$

5. Let

$$A = 4\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following matrices is equal to the inverse matrix A^{-1}?

$$(A) \quad A^{-1} = \frac{1}{32}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$(B) \quad A^{-1} = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$(C) \quad A^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$(D) \quad A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Chapter 2

6. Let

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Which of the following statements is true?

$$\begin{array}{ll} (A) & \det(A) = -1 \\ (B) & \det(A) = -2 \\ (C) & \det(A) = 1 \\ (D) & \det(A) = 2 \end{array}$$

7. Let

$$A = \begin{pmatrix} \alpha & -\alpha & \alpha \\ -\alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \end{pmatrix}.$$

Which of the following computations is true?

$$\begin{array}{ll} (A) & \det(\lambda I - A) = (\lambda + \alpha)(\lambda - 2\alpha)^2 \\ (B) & \det(\lambda I - A) = (\lambda + \alpha)^2(\lambda - 2\alpha) \\ (C) & \det(\lambda I - A) = (\lambda - \alpha)^2(\lambda + 2\alpha) \\ (D) & \det(\lambda I - A) = (\lambda - \alpha)(\lambda + 2\alpha)^2 \end{array}$$

8. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

Which of the following computations is true?

$$\begin{array}{rl} (A) & \det(A) = (\alpha^2 + \beta^2)(\alpha^2 - 2\beta^2)^2 \\ (B) & \det(A) = (\alpha^2 + \beta^2)^2(\alpha^2 - 2\beta^2) \\ (C) & \det(A) = (\alpha^2 - \beta^2)^2(\alpha^2 + 2\beta^2) \\ (D) & \det(A) = (\alpha^2 - \beta^2)(\alpha^2 + 2\beta^2)^2 \end{array}$$

9. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}.$$

Which of the following computations is true?

(A) $\det(\lambda I - A) = (\lambda - \alpha + \beta)(\lambda - \alpha - 3\beta)^3$

(B) $\det(\lambda I - A) = (\lambda - \alpha + \beta)^3(\lambda - \alpha - 3\beta)$

(C) $\det(\lambda I - A) = (\lambda - \alpha - \beta)^3(\lambda - \alpha + 3\beta)$

(D) $\det(\lambda I - A) = (\lambda - \alpha - \beta)(\lambda - \alpha + 3\beta)^3$

10. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

Which of the following computations is true?

$$\begin{array}{ll} (A) & \det(\lambda I - A) = (\lambda - \alpha^2 - \beta^2)^3(\lambda - \alpha^2 + 3\beta^2) \\ (B) & \det(\lambda I - A) = (\lambda - \alpha^2 - \beta^2)(\lambda - \alpha^2 + 3\beta^2)^3 \\ (C) & \det(\lambda I - A) = (\lambda - \alpha^2 - \beta^2)(\lambda - \alpha^2 - 3\beta^2)^3 \\ (D) & \det(\lambda I - A) = (\lambda - \alpha^2 - \beta^2)^3(\lambda - \alpha^2 - 3\beta^2) \end{array}$$

Chapter 3

11. Given that

$$\left\{ \begin{pmatrix} \alpha \\ -\beta \\ -\beta \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ -\beta \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

is a spanning set of $\mathbb{R}^4$. Which of the following statements is true?

$$\begin{array}{lll} (A) & \alpha+\beta \neq 0, & \alpha-3\beta \neq 0 \\ (B) & \alpha+\beta \neq 0, & \alpha-3\beta = 0 \\ (C) & \alpha+\beta = 0, & \alpha-3\beta = 0 \\ (D) & \alpha+\beta = 0, & \alpha-3\beta \neq 0 \end{array}$$

12. Let

$$\mathcal{S} = \{1, x, x^2, x^3, \cdots\cdots, x^8, x^9, x^{10}\}.$$

Which of the following statements about $\mathcal{S}$ is NOT true?

(A) $\mathcal{S}$ is a basis of the vector space $C(\mathbb{R})$

(B) $\mathcal{S}$ is a spanning set of the vector space $P_{10}(\mathbb{R})$

(C) $\mathcal{S}$ is a basis of the vector space $P_{10}(\mathbb{R})$

(D) $\mathcal{S}$ is linearly independent

13. Which of the following is NOT a spanning set of $M_2(\mathbb{R})$?

$$(A)\quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$(B)\quad \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \right\}$$

$$(C)\quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

$$(D)\quad \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right\}$$

14. Which of the following is NOT a basis of the vector space $\mathbb{R}^4$?

$$(A) \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$(B) \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$(C) \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$(D) \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

15. Which of the following statements is NOT true?

$$\begin{array}{ll} (A) & \dim(\mathbb{R}^n) = n \\ (B) & \dim[M_n(\mathbb{R})] = n^2 \\ (C) & \dim[P_n(\mathbb{R})] = n + 1 \\ (D) & \dim[C(\mathbb{R})] = n^2 + n \end{array}$$

Chapter 4

16. Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Which of the following statements is NOT true?

(A) The inverse linear transformation T^{-1} exists if T is one-to-one or onto.
(B) The sum of the dimension of the kernel and the dimension of the range is equal to the dimension of the vector space $\mathbf{V}$. That is,

$$\dim[\ \mathrm{Ker(T)}\ ] + \dim[\ \mathrm{Rng(T)}\ ] = \dim \mathbf{V}.$$

(C) The kernel $\mathrm{Ker}(T)$ is a subspace of the vector space $\mathbf{V}$.
(D) The range $\mathrm{Rng}(T)$ is a subspace of the vector space $\mathbf{W}$.

17. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, given by

$$T(\mathbf{x}) = A\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where A is a real symmetric matrix. All eigenvalues are positive. Which of the following statements is NOT true?

(A) The linear transformation T is one-to-one.
(B) The determinant $\det(A) = 0$.
(C) The linear transformation T is onto.
(D) The inverse linear transformation T^{-1} exists.

18. Let $T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$ be a linear transformation, given by

$$T(A) = A + A^T, \qquad A \in M_3(\mathbb{R}).$$

Which of the following is a basis of the kernel of the linear transformation T?

$$(A) \quad \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

$$(B) \quad \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

$$(C) \quad \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

$$(D) \quad \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

19. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation, given by

$$P(a + bx + cx^2 + dx^3) = a + bx + cx^2 + dx^3.$$

Which of the following statements is true?

(A) The range of $T =$ span $\{x^3\}$
(B) The range of $T =$ span $\{x^2, x^3\}$
(C) The range of $T =$ span $\{x, x^2, x^3\}$
(D) The range of $T =$ span $\{1, x, x^2, x^3\}$

20. Which of the following linear transformations is NOT an isomorphism?

$$(A) \quad T : \mathbb{R}^4 \to P_3(\mathbb{R}),$$

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a + bx + cx^2 + dx^3.$$

$$(B) \quad T : \mathbb{R}^4 \to M_2(\mathbb{R}),$$

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$(C) \quad T : M_2(\mathbb{R}) \to P_3(\mathbb{R}),$$

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + bx + cx^2 + dx^3.$$

$$(D) \quad T : \mathbb{R}^4 \to \mathbb{R}^4,$$

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Chapter 5

21. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

There are two real eigenvalues to the matrix A. Which of the following choices are the eigenvalues of A?

$$\begin{array}{lll} (A) & \lambda_1 = \alpha^2 + \beta^2, & \lambda_2 = \alpha^2 - 2\beta^2 \\ (B) & \lambda_1 = \alpha^2 + \beta^2, & \lambda_2 = \alpha^2 + 2\beta^2 \\ (C) & \lambda_1 = \alpha^2 - \beta^2, & \lambda_2 = \alpha^2 - 2\beta^2 \\ (D) & \lambda_1 = \alpha^2 - \beta^2, & \lambda_2 = \alpha^2 + 2\beta^2 \end{array}$$

22. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

There are two real eigenvalues to the matrix A. Which of the following choices are the eigenvalues of A?

$$\begin{array}{lll} (A) & \lambda_1 = \alpha^2 + \beta^2, & \lambda_2 = \alpha^2 + 3\beta^2 \\ (B) & \lambda_1 = \alpha^2 + \beta^2, & \lambda_2 = \alpha^2 - 3\beta^2 \\ (C) & \lambda_1 = \alpha^2 - \beta^2, & \lambda_2 = \alpha^2 - 3\beta^2 \\ (D) & \lambda_1 = \alpha^2 - \beta^2, & \lambda_2 = \alpha^2 + 3\beta^2 \end{array}$$

23. Consider the linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix} \mathbf{u}.$$

Which of the following is the solution of the system of differential equations?

$$(A) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$+ \ C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 \exp[(\alpha^2-2\beta^2)t] \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$(B) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$+ \ C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 \exp[(\alpha^2-2\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$(C) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$+ \ C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_3 \exp[(\alpha^2-2\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(D) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$+ \ C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 \exp[(\alpha^2-2\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

24. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix} \mathbf{u}.$$

Which of the following is the solution of the system of differential equations?

$$(A) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \; C_3 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 \exp[(\alpha^2-3\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(B) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$+ \; C_3 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 \exp[(\alpha^2-3\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(C) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$+ \; C_3 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_4 \exp[(\alpha^2-3\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(D) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$+ \; C_3 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + C_4 \exp[(\alpha^2-3\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

25. Consider the linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix} \mathbf{u}.$$

Which of the following is the solution of the system of differential equations?

$$(A) \qquad \mathbf{u}(t) = C_1 \exp[(2\alpha^2 - 2\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \; C_3 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 \exp[(2\alpha^2 + 2\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(B) \qquad \mathbf{u}(t) = C_1 \exp[(2\alpha^2 - 2\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \; C_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + C_4 \exp[(2\alpha^2 + 2\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$(C) \qquad \mathbf{u}(t) = C_1 \exp[(2\alpha^2 - 2\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \ C_3 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 \exp[(2\alpha^2 + 2\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(D) \qquad \mathbf{u}(t) = C_1 \exp[(2\alpha^2 - 2\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$+ \ C_3 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + C_4 \exp[(2\alpha^2 + 2\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Chapter 6

26. Let

$$\mathbf{a} = \begin{pmatrix} 7 \\ 24 \\ 25 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 7 \\ 24 \\ 25 \end{pmatrix}.$$

What is the scalar product of **a** and **b** equal to?

(A) 1150.
(B) 1250.
(C) 1350.
(D) 1450.

27. Let $\alpha < 0$ and $\beta < 0$ be negative constants. What is the angle θ between the following two vectors

$$\mathbf{a} = \begin{pmatrix} \alpha^{10} \\ \alpha^{10} \\ -\alpha^{10} \\ -\alpha^{10} \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -\beta^{20} \\ -\beta^{20} \\ \beta^{20} \\ \beta^{20} \end{pmatrix}?$$

(A) $\theta = \pi/3$
(B) $\theta = \pi/2$
(C) $\theta = \pi$
(D) $\theta = 2\pi/3$

28. Which of the following is NOT an orthogonal basis of $\mathbb{R}^4$?

$$(A)\quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$(B)\quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$(C)\quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

$$(D)\quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

29. Let $\alpha > 0$ and $\beta > 0$ be positive constants, such that $\alpha = 10 + 10\beta$. What is the projection of the vector $\begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \alpha \end{pmatrix}$ onto the vector $\begin{pmatrix} \beta \\ \beta \\ \beta \\ \beta \end{pmatrix}$?

(A) $\begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \alpha \end{pmatrix}$

(B) $\begin{pmatrix} 2\alpha \\ 2\alpha \\ 2\alpha \\ 2\alpha \end{pmatrix}$

(C) $\begin{pmatrix} \beta \\ \beta \\ \beta \\ \beta \end{pmatrix}$

(D) $\begin{pmatrix} 2\beta \\ 2\beta \\ 2\beta \\ 2\beta \end{pmatrix}$

30. Let

$$\mathcal{S} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\} \subset \mathbb{R}^{10},$$

be an nontrivial orthonormal set. Define the 10×10 matrix

$$A = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5)(\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5)^T.$$

Which of the following statements is true?

(A) The rank of the matrix A is 10.

(B) The matrix A is a projection matrix.

(C) The inverse matrix A^{-1} of A exists.

(D) All eigenvalues of the matrix A are negative.

Old What is the projection of the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ onto the vector

$$\begin{pmatrix} 4 \\ 3 \\ -2 \\ -1 \end{pmatrix}?$$

$$(A) \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$(B) \quad \begin{pmatrix} 4 \\ 3 \\ -2 \\ -1 \end{pmatrix}$$

$$(C) \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(D) \quad \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

31. Let $\mathcal{B} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_m\}$ be an orthogonal basis of the subspace

$$\mathcal{E} \overset{\text{def}}{=} \text{ span } \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_m\} \subset \mathbb{R}^n.$$

Let the vector $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{a} \notin$ span $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_m\}$. The projection of $\mathbf{a}$ onto $\mathcal{E} =$ span$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_m\}$ is given by

$$\mathbf{a}_0 = \frac{\mathbf{a} \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{a} \cdot \mathbf{a}_2}{\mathbf{a}_2 \cdot \mathbf{a}_2}\mathbf{a}_2 + \frac{\mathbf{a} \cdot \mathbf{a}_3}{\mathbf{a}_3 \cdot \mathbf{a}_3}\mathbf{a}_3 + \cdots + \frac{\mathbf{a} \cdot \mathbf{a}_m}{\mathbf{a}_m \cdot \mathbf{a}_m}\mathbf{a}_m.$$

Which of the following statements is NOT true?

(A) $\mathbf{a}_0 \in \mathcal{E} =$ span $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_m\}$

(B) $\|\mathbf{a}\|^2 = \|\mathbf{a}_0\|^2 + \|\mathbf{a} - \mathbf{a}_0\|^2$

(C) $\mathbf{a} - \mathbf{a}_0 \in \mathcal{E}$

(D) $\|\mathbf{a}_0\|^2 \leq \|\mathbf{a}\|^2$

32. The projection of the vector $\mathbf{a} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ onto the subspace

$$\text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

is given by which of the following vectors?

$$(A) \quad \begin{pmatrix} -a \\ -b \\ -c \\ -d \end{pmatrix}$$

$$(B) \quad \begin{pmatrix} -2a \\ -2b \\ -2c \\ -2d \end{pmatrix}$$

$$(C) \quad \begin{pmatrix} 2a \\ 2b \\ 2c \\ 2d \end{pmatrix}$$

$$(D) \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

33. Let α, β, γ be real nonzero constants, such that $\alpha^2+\beta^2=\gamma^2$. Let $\mathbf{a}_0$ be the projection of the vector $\begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix}$ onto the subspace

$$\operatorname{span}\left\{ \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix} \right\}.$$

The projection $\mathbf{a}_0$ is equal to which of the following vectors?

(A) $\begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix}$

(B) $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

(C) $\begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix}$

(D) $\begin{pmatrix} -\alpha \\ -\beta \\ -\gamma \end{pmatrix}$

34. Which of the following matrices is NOT a projection matrix?

$$(A)\quad \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$(B)\quad \frac{1}{4}\begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$(C)\quad \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

$$(D)\quad \frac{1}{4}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

35. Which of the following matrices is a projection matrix?

$$(A) \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$$(B) \quad B = \begin{pmatrix} -4 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$(C) \quad C = \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$(D) \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

36. Consider the system of linear equations

$$A\mathbf{x} = \mathbf{b},$$

where A is an $m \times n$ real matrix, $\mathbf{b} \in \mathbb{R}^m$ is real constant vector. Let $\mathbf{x}_0$ represent a least square solution of the system. Which of the following statements is NOT true?

$$\begin{array}{ll} (A) & A^T A\mathbf{x}_0 = A^T\mathbf{b} \\ (B) & \|A\mathbf{x}_0 - \mathbf{b}\| = \min\limits_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\| \\ (C) & A\mathbf{x}_0 = \mathbf{b}_0 \\ (D) & A\mathbf{x}_0 = \mathbf{b} \end{array}$$

where $\mathbf{b}_0$ represents the projection of $\mathbf{b}$ onto the column space of A.

37. Consider the system of linear equations

$$\begin{pmatrix} 10 & 10 \\ 20 & 20 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 300 \\ 50 \end{pmatrix}.$$

Which of the following sets represents the least square solutions of the system?

$$(A) \quad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \qquad x + y = 8 \right\}$$

$$(B) \quad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \qquad x + y = 16 \right\}$$

$$(C) \quad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \qquad x + y = 32 \right\}$$

$$(D) \quad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \qquad x + y = 64 \right\}$$

38. Let α, β, γ be real nonzero constants, such that $\alpha^2+\beta^2=\gamma^2$. Consider the system of linear equations

$$\begin{pmatrix} \alpha & \alpha \\ \beta & \beta \\ \gamma & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (\alpha^2+\beta^2+\gamma^2) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Which of the following is the least square solution of the system of equations?

$$(A) \quad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x+y=1 \right\}$$

$$(B) \quad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x+y=\alpha^2+\beta^2+\gamma^2 \right\}$$

$$(C) \quad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x+y=(\alpha^2+\beta^2+\gamma^2)^2 \right\}$$

$$(D) \quad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x+y=-\alpha^2-\beta^2-\gamma^2 \right\}$$

Chapter 7

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39. Let

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following is the best, complete description of this matrix?

(A) It is a real square matrix.
(B) It is a real symmetric matrix.
(C) It is a real symmetric, orthonormal matrix and $A^2 = I$.
(D) It is a real symmetric, orthonormal matrix and $A^2 = A$.

40. Let A be a real $n \times n$ constant matrix. Which of the following statements is true about the matrix $AA^T - A^TA$?

(A) All eigenvalues of $AA^T - A^TA$ are positive.
(B) All eigenvalues of $AA^T - A^TA$ are negative.
(C) The matrix $AA^T - A^TA$ is skew-symmetric.
(D) The matrix $AA^T - A^TA$ is symmetric.

41. Let A be a real $m \times n$ matrix. Let λ and μ be real distinct eigenvalues of the matrix $A^T A$, let ξ and η be eigenvectors of $A^T A$ corresponding to the eigenvalues λ and μ, respectively, that is,

$$A^T A\xi = \lambda\xi, \qquad A^T A\eta = \mu\eta, \qquad \lambda \neq \mu.$$

Which of the following statements is true?

$$\begin{array}{ll} (A) & \xi \cdot \eta > 0 \\ (B) & \xi \cdot \eta = 0 \\ (C) & \xi \cdot \eta < 0 \\ (D) & \xi = r\eta \end{array}$$

for some real nonzero constant $r \neq 0$.

42. Which of the following is an orthonormal matrix?

$$(A) \quad U_1 = \frac{1}{6}\begin{pmatrix} 3\sqrt{2} & \sqrt{6} & 2\sqrt{3} \\ 3\sqrt{2} & \sqrt{6} & 2\sqrt{3} \\ 0 & -2\sqrt{6} & 2\sqrt{3} \end{pmatrix}$$

$$(B) \quad U_2 = \frac{1}{6}\begin{pmatrix} 3\sqrt{2} & \sqrt{6} & 2\sqrt{3} \\ -3\sqrt{2} & \sqrt{6} & 2\sqrt{3} \\ 0 & 2\sqrt{6} & 2\sqrt{3} \end{pmatrix}$$

$$(C) \quad U_3 = \frac{1}{6}\begin{pmatrix} 3\sqrt{2} & \sqrt{6} & 2\sqrt{3} \\ -3\sqrt{2} & \sqrt{6} & 2\sqrt{3} \\ 0 & -2\sqrt{6} & 2\sqrt{3} \end{pmatrix}$$

$$(D) \quad U_4 = \frac{1}{6}\begin{pmatrix} 3\sqrt{2} & \sqrt{6} & 2\sqrt{3} \\ -3\sqrt{2} & \sqrt{6} & 2\sqrt{3} \\ 0 & -2\sqrt{6} & -2\sqrt{3} \end{pmatrix}$$

43. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix},$$

$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

Which of the following is correct?

$$(A) \quad U^TAU = \begin{pmatrix} \alpha^2 - 3\beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2+\beta^2 & 0 & 0 \\ 0 & 0 & \alpha^2+\beta^2 & 0 \\ 0 & 0 & 0 & \alpha^2+\beta^2 \end{pmatrix}$$

$$(B) \quad U^TAU = \begin{pmatrix} \alpha^2+\beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2 - 3\beta^2 & 0 & 0 \\ 0 & 0 & \alpha^2+\beta^2 & 0 \\ 0 & 0 & 0 & \alpha^2+\beta^2 \end{pmatrix}$$

$$(C) \quad U^TAU = \begin{pmatrix} \alpha^2+\beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2+\beta^2 & 0 & 0 \\ 0 & 0 & \alpha^2 - 3\beta^2 & 0 \\ 0 & 0 & 0 & \alpha^2+\beta^2 \end{pmatrix}$$

$$(D) \quad U^TAU = \begin{pmatrix} \alpha^2+\beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2+\beta^2 & 0 & 0 \\ 0 & 0 & \alpha^2+\beta^2 & 0 \\ 0 & 0 & 0 & \alpha^2 - 3\beta^2 \end{pmatrix}$$

44. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix},$$

$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following diagonalizations is true?

$$(A) \quad U^T A U = \begin{pmatrix} 2\alpha^2 + 2\beta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha^2 - 2\beta^2 \end{pmatrix}$$

$$(B) \quad U^T A U = \begin{pmatrix} 2\alpha^2 - 2\beta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha^2 + 2\beta^2 \end{pmatrix}$$

$$(C) \quad U^T A U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\alpha^2 + 2\beta^2 & 0 & 0 \\ 0 & 0 & 2\alpha^2 - 2\beta^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(D) \quad U^T A U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\alpha^2 - 2\beta^2 & 0 & 0 \\ 0 & 0 & 2\alpha^2 + 2\beta^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

45. Given that the real symmetric matrix

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

is positively definite. Which of the following conditions must be true?

$$\begin{array}{lll} (A) & \alpha^2+\beta^2>0, & -\alpha^2-2\beta^2>0 \\ (B) & \alpha^2+\beta^2>0, & -2\alpha^2-\beta^2>0 \\ (C) & \alpha^2+\beta^2>0, & \alpha^2-2\beta^2>0 \\ (D) & \alpha^2+\beta^2>0, & \alpha^2-2\beta^2<0 \end{array}$$

46. Given that the real symmetric matrix

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

is positively definite. Which of the following conditions must be true?

$(A) \quad \alpha^2 + \beta^2 > 0, \qquad -\alpha^2 - 3\beta^2 > 0$

$(B) \quad \alpha^2 + \beta^2 > 0, \qquad -\alpha^2 - 3\beta^2 > 0$

$(C) \quad \alpha^2 + \beta^2 > 0, \qquad \alpha^2 - 3\beta^2 < 0$

$(D) \quad \alpha^2 + \beta^2 > 0, \qquad \alpha^2 - 3\beta^2 > 0$

47. Let $\alpha > 0$ and $\beta > 0$ be positive constants. Given that the real symmetric matrix

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}$$

is indefinite. Which of the following conditions must be true?

$$\begin{array}{lll} (A) & \alpha + \beta > 0, & \alpha - 3\beta > 0 \\ (B) & \alpha + \beta > 0, & \alpha - 3\beta < 0 \\ (C) & \alpha + \beta < 0, & \alpha - 3\beta > 0 \\ (D) & \alpha + \beta < 0, & \alpha - 2\beta < 0 \end{array}$$

48. Let $\alpha > 0$ and $\beta > 0$ be positive constants. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

Let

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}.$$

Which of the following statements is true?

(A) $\max\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + 3\beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + \beta^2$

(B) $\max\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + 3\beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 - \beta^2$

(C) $\max\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + \beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 - 3\beta^2$

(D) $\max\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + \beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + 3\beta^2$

49. Let α and β be real nonzero constants. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

Define

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}.$$

Which of the following statements is true?

(A) $\quad \max\limits_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + 2\beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + \beta^2$

(B) $\quad \max\limits_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + 2\beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 - \beta^2$

(C) $\quad \max\limits_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + \beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + 2\beta^2$

(D) $\quad \max\limits_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + \beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 - 2\beta^2$

50. Let α and β be positive constants, such that $2\alpha^2 + 2\beta^2 > 0$ and $2\alpha^2 - 2\beta^2 < 0$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

Define

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}.$$

Which of the following statements is true?

(A) $\max_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 + 2\beta^2, \quad \min_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 - 2\beta^2$

(B) $\max_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 + 2\beta^2, \quad \min_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 + 2\beta^2$

(C) $\max_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 - 2\beta^2, \quad \min_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 + 2\beta^2$

(D) $\max_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 - 2\beta^2, \quad \min_{\mathbf{x}\in\mathbb{R}^3, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 - 2\beta^2$

Bonus Problem 1: Let U be a real $n \times n$ constant, orthonormal matrix, that is

$$U^T U = I.$$

Compute

$$\|U\mathbf{x}\|^2 - \|\mathbf{x}\|^2$$

for all $\mathbf{x} \in \mathbb{R}^n$. Show important supporting work to receive credits.

Bonus Problem 2: Let $\alpha > 0$ and $\beta > 0$ be positive constants. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix},$$

$$B = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

Define

$$U = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following computations is correct?

$$(A) \quad U^TAU + U^TBU = \begin{pmatrix} \alpha^2+\beta^2 & 0 & 0 & 0 \\ 0 & 3\alpha^2-5\beta^2 & 0 & 0 \\ 0 & 0 & 3\alpha^2+3\alpha^2 & 0 \\ 0 & 0 & 0 & \alpha^2+\beta^2 \end{pmatrix}$$

$$(B) \quad U^TAU + U^TBU = \begin{pmatrix} 3\alpha^2-5\beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2+\beta^2 & 0 & 0 \\ 0 & 0 & \alpha^2+\beta^2 & 0 \\ 0 & 0 & 0 & 3\alpha^2+3\beta^2 \end{pmatrix}$$

$$(C) \quad U^TAU + U^TBU = \begin{pmatrix} \alpha^2-3\beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2+\beta^2 & 0 & 0 \\ 0 & 0 & \alpha^2+\beta^2 & 0 \\ 0 & 0 & 0 & \alpha^2+\beta^2 \end{pmatrix}$$

$$(D) \quad U^TAU + U^TBU = \begin{pmatrix} 2\alpha^2-2\beta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha^2+2\beta^2 \end{pmatrix}$$

Bonus Problem 3: Let $\alpha > 0$ and $\beta > 0$ be positive constants. Let

$$\begin{aligned} A &= \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}, \\ B &= \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}. \end{aligned}$$

Let

$$U = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

Evaluate/Compute

$$U^T (AB)^m U,$$

where $m \gg 1$ is a sufficiently large positive integer. Show important supporting work to receive credits.

This is the end of the Final Exam of Linear Algebra.

Mathematics 43 - The First Midterm Examination - 2023

There are twenty-five multiple choice problems in the First Midterm Exam. There are five bonus problems (they are not mandatory to solve). Each problem is worth four points. The total is 100 points (it is 120 points if the bonus problems are included). The exam is 75 minutes. In each problem, there are four choices marked (A), (B), (C) and (D). Only one choice is correct. Choose the one you think is correct. If you do not know the answer to a problem, you may make a reasonable, best possible guess. No supporting work is necessary. In the exam, we use $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$ and $\lambda \neq 0$ to represent various real nonzero constants or positive constants.

No calculators, computers, cell phones, i-pads, i-touches or any other electronic devices are allowed in the exam. Students are not allowed to ask for assistance from anybody else. Students are not allowed to use any websites for assistance. Any student who cheats will receive an F as the Final Grade. This is Absolutely Firm! Therefore, be honest and solve all problems by yourself.

The First Midterm Exam: 100 points
The Second Midterm Exam: 100 points
The Final Exam: 200 points
The Homework Assignments: 100 points
Total Score: 500 points

ANSWER SHEET

PRINT NAME

1 (   ) 2 (   ) 3 (   ) 4 (   ) 5 (   )
6 (   ) 7 (   ) 8 (   ) 9 (   ) 10 (   )
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56 (   ) 57 (   ) 58 (   )

Chapter 1: Linear Systems of Equations and Inverse Matrix. There are 13 problems in Chapter 1.

1. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Which of the following matrices is the reduced row echelon form of A?

$$(A) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(B) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(C) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(D) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. Let the matrix

$$A = \begin{pmatrix} 2 & 0 & 2 & 3 \\ 2 & 0 & 2 & 3 \\ 2 & 0 & 2 & 3 \\ 2 & 0 & 2 & 3 \end{pmatrix}.$$

Which of the following integers is equal to the rank of the matrix A?

(A) 0.

(B) 1.

(C) 2.

(D) 3.

3. The reduced row echelon form of the augmented matrix of the linear system of equations $A\mathbf{x} = \mathbf{b}$ is given below

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

How many free variables are there in the solutions?

(A) 1.

(B) 2.

(C) 3.

(D) 4.

4. The reduced row echelon form of the augmented matrix of the linear system of equations $A\mathbf{x} = \mathbf{b}$ is given below

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix}.$$

How many free variables are there in the solutions?

(A) 0.

(B) 1.

(C) 2.

(D) 5.

5. The augmented matrix of the linear system of equations $A\mathbf{x} = \mathbf{b}$ is given below

$$\begin{pmatrix} 10 & 20 & 30 & 40 & 50 & 60 \\ 10 & 20 & 30 & 40 & 50 & 120 \\ 10 & 20 & 30 & 40 & 50 & 600 \end{pmatrix}.$$

How many solutions are there to the system of equations?

(A) ∞.

(B) 0.

(C) 1.

(D) 2.

6. The reduced row echelon form of the augmented matrix of the linear system of equations $A\mathbf{x} = \mathbf{b}$ is given below

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

How many solutions are there to the system of equations?

(A) 0.

(B) 1.

(C) ∞.

(D) 2.

7. Consider the linear system of equations

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 14 \\ 30 \end{pmatrix}.$$

Which of the following is the solution of the system?

$$(A) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

$$(B) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}$$

$$(C) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(D) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

8. Consider the linear system of equations

$$\begin{pmatrix} \alpha & -\alpha & \alpha \\ -\alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Which of the following is a solution of the system?

$$(A) \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$(B) \quad \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}$$

$$(C) \quad \begin{pmatrix} 3 \\ 6 \\ 5 \end{pmatrix}$$

$$(D) \quad \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

9. Let

$$A = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Which of the following matrices is equal to the inverse matrix A^{-1}?

$$(A) \quad A^{-1} = 2\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$(B) \quad A^{-1} = 4\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$(C) \quad A^{-1} = 8\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$(D) \quad A^{-1} = 16\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

10. Which of the following statements is NOT true?

$$(A)\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

$$(B)\quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$(C)\quad \begin{pmatrix} \alpha & -\alpha & \alpha \\ -\alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \end{pmatrix}^{-1} = \frac{1}{2\alpha}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$(D)\quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

In this problem, a, b, c and d are real constants, such that $ad \neq bc$. Additionally, $\alpha \neq 0$ is a real nonzero constant.

11. Let A be a real $n \times n$ constant matrix. Which of the following statements is NOT equivalent to the others?

(A) There exists a unique solution to the linear system of equations $A\mathbf{x} = \mathbf{b}$, for each fixed constant vector $\mathbf{b} \in \mathbb{R}^n$.

(B) The inverse matrix A^{-1} does not exist.

(C) The determinant $\det(A) \neq 0$.

(D) The rank of A is precisely equal to n.

12. Let A be a real $n \times n$ constant matrix. Which of the following statements is NOT equivalent to the others?

(A) The rank of A is less than n.

(B) The inverse matrix A^{-1} does not exist.

(C) There exists a unique solution to the linear system of equations $A\mathbf{x} = \mathbf{b}$, for each fixed constant vector $\mathbf{b} \in \mathbb{R}^n$.

(D) The determinant $\det(A) = 0$.

13. Let A be a real $m \times n$ constant matrix, let $\mathbf{b} \in \mathbb{R}^n$ be a real constant vector, let $A^\#$ represent the augmented matrix of the linear system of equations $A\mathbf{x} = \mathbf{b}$. Which of the following statements is NOT true?

(A) There exists a unique solution to the linear system $A\mathbf{x} = \mathbf{b}$, if both the rank of $A^\#$ and the rank of A are equal to n.

(B) There exist infinitely many solutions to the system $A\mathbf{x} = \mathbf{b}$, if both the rank of $A^\#$ and the rank of A are equal to k and $k < n$.

(C) There exists no solution to the linear system of equations $A\mathbf{x} = \mathbf{b}$, if the rank of A is less than the rank of $A^\#$.

(D) There exists no solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$, if the rank of A is less than n.

Chapter 2: Determinants. There are 12 problems in Chapter 2.

14. Which of the following statements is true?

$$(A) \quad \det\begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^3 + (\alpha^2+\beta^2+\gamma^2)\lambda$$

$$(B) \quad \det\begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^2 + (\alpha^2+\beta^2+\gamma^2)\lambda^2$$

$$(C) \quad \det\begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^2 - (\alpha^2+\beta^2+\gamma^2)\lambda^2$$

$$(D) \quad \det\begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^3 - (\alpha^2+\beta^2+\gamma^2)\lambda$$

15. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Which of the following statements is true?

$$\begin{array}{ll} (A) & \det(A) = 32 \\ (B) & \det(A) = 16 \\ (C) & \det(A) = 4 \\ (D) & \det(A) = 1 \end{array}$$

16. Let

$$A = \begin{pmatrix} \lambda - \alpha & \alpha & -\alpha \\ \alpha & \lambda - \alpha & -\alpha \\ -\alpha & -\alpha & \lambda - \alpha \end{pmatrix}.$$

Which of the following statements is true?

$$\begin{array}{ll} (A) & \det(A) = (\lambda + 2\alpha)(\lambda - \alpha)^2 \\ (B) & \det(A) = (\lambda + 2\alpha)^2(\lambda - \alpha) \\ (C) & \det(A) = (\lambda + \alpha)(\lambda - 2\alpha)^2 \\ (D) & \det(A) = (\lambda + \alpha)^2(\lambda - 2\alpha) \end{array}$$

17. Let

$$A = \begin{pmatrix} \lambda - \alpha & \beta & \beta & \beta \\ \beta & \lambda - \alpha & \beta & \beta \\ \beta & \beta & \lambda - \alpha & \beta \\ \beta & \beta & \beta & \lambda - \alpha \end{pmatrix}.$$

Which of the following statements is true?

$(A)\quad \det(A) = (\lambda - \alpha - \beta)^3(\lambda - \alpha + 2\beta)$

$(B)\quad \det(A) = (\lambda - \alpha - \beta)(\lambda - \alpha + 2\beta)^3$

$(C)\quad \det(A) = (\lambda - \alpha - \beta)(\lambda - \alpha + 3\beta)^3$

$(D)\quad \det(A) = (\lambda - \alpha - \beta)^3(\lambda - \alpha + 3\beta)$

18. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

Which of the following statements is true?

$(A)\quad \det(\lambda I - A) = (\lambda - \alpha^2 - \beta^2)(\lambda - \alpha^2 + 3\beta^2)^3$

$(B)\quad \det(\lambda I - A) = (\lambda - \alpha^2 - \beta^2)^3(\lambda - \alpha^2 + 3\beta^2)$

$(C)\quad \det(\lambda I - A) = (\lambda - \alpha^2 - \beta^2)^3(\lambda - \alpha^2 - 3\beta^2)$

$(D)\quad \det(\lambda I - A) = (\lambda - \alpha^2 - \beta^2)(\lambda - \alpha^2 - 3\beta^2)^3$

19. Let $A = (a_{ij})$ be a real $n \times n$ constant matrix. Let C_{ij} represent the cofactor of the element a_{ij}, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$. Which of the following statements is NOT true?

(A) $\displaystyle\sum_{k=1}^{n} a_{ik}C_{ik} = \det(A), \qquad \text{for all } i = 1, 2, 3, \cdots, n$

(B) $\displaystyle\sum_{k=1}^{n} a_{kj}C_{kj} = \det(A), \qquad \text{for all } j = 1, 2, 3, \cdots, n$

(C) $\displaystyle\sum_{k=1}^{n} a_{ik}C_{jk} = 1, \text{ for all } i \neq j, i = 1, 2, 3, \cdots, n; j = 1, 2, 3, \cdots, n$

(D) $\displaystyle\sum_{k=1}^{n} a_{ki}C_{kj} = 0, \text{ for all } i \neq j, i = 1, 2, 3, \cdots, n; j = 1, 2, 3, \cdots, n$

20. Which of the following statements is NOT true?

$$(A) \quad \det \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \lambda & \mu & \nu & \xi \\ a & b & c & d \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$$

$$(B) \quad \det \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \alpha & \beta & \gamma & \delta \\ a & b & c & d \\ p & q & r & s \end{pmatrix} = 0$$

$$(C) \quad \det \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ a & b & c & d \\ 12 & 24 & 36 & 48 \\ -6 & -12 & -18 & -24 \end{pmatrix} = 0$$

$$(D) \quad \det \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \lambda & \mu & \nu & \xi \\ a & b & c & d \\ p & q & r & s \end{pmatrix} = 0$$

In this problem, α, β, γ, δ, λ, μ, ν, ξ, a, b, c, d, p, q, r, s represent various real constants.

21. Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (C_{ij})$ be three real $n \times n$ constant matrices, where $B = C^T$ represents the adjoint matrix of A, C_{ij} represents the cofactor of the element a_{ij}, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$. Which of the following statements is NOT true?

$$
\begin{array}{ll}
(A) & C^{10}A^{10} = A^{10}C^{10} = [\det(A)]^{10}I \\
(B) & A^{10}B^{15} = B^{15}A^{10} = [\det(A)]^{10}B^{5} \\
(C) & A^{25}B^{20} = B^{20}A^{25} = [\det(A)]^{20}A^{5} \\
(D) & AB = BA = [\det(A)]I
\end{array}
$$

22. Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (C_{ij})$ be three real $n \times n$ constant matrices, where $B = C^T$ represents the adjoint matrix of A, C_{ij} represents the cofactor of the element a_{ij}, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$. Which of the following statements is NOT true?

- (A) $\det(AB) = \det(BA) = [\det(A)]^n$
- (B) $\det(A) = \det(B) = \det(C)$
- (C) $\det(A) \neq 0,$ if and only if the inverse matrix A^{-1} exists
- (D) $\det(B) = \det(C) = [\det(A)]^{n-1}$

23. Which of the following statements is NOT true?

$$(A)\quad \det\begin{pmatrix} \lambda_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & \lambda_2 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & \lambda_3 & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \lambda_1\lambda_2\lambda_3\cdots\lambda_n$$

$$(B)\quad \det\begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ a_{21} & \lambda_2 & 0 & \cdots & 0 \\ a_{31} & a_{32} & \lambda_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & \lambda_n \end{pmatrix} = \lambda_1\lambda_2\lambda_3\cdots\lambda_n$$

$$(C)\quad \det\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_n \end{pmatrix} = \lambda_1\lambda_2\lambda_3\cdots\lambda_n$$

$$(D)\quad \det\begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \lambda_1\lambda_2\lambda_3\cdots\lambda_n$$

In this problem, λ_1, λ_2, λ_3, $\cdots$, λ_n, a_{11}, a_{12}, a_{13}, $\cdots$, a_{1n}, a_{21}, a_{22}, a_{23}, $\cdots$, a_{2n}, $\cdots$, a_{n1}, $\cdots$, a_{nn} represent various real constants.

24. Which of the following statements is NOT true?

$$(A)\qquad \det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \det\begin{pmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

$$(B)\qquad \det\left\{\alpha\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}\right\} = \alpha^4 \det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$(C)\qquad \det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$= \det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41}+\lambda a_{11} & a_{42}+\lambda a_{12} & a_{43}+\lambda a_{13} & a_{44}+\lambda a_{14} \end{pmatrix}$$

$$(D)\qquad \det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = -\det\begin{pmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \end{pmatrix}$$

In this problem, λ represents a real constant, a_{ij} represent various real constants, where $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

25. Which of the following statements is NOT true?

$$
\begin{array}{ll}
(A) & \det(AB^{-1}) = [\det(A)][\det(B)]^{-n} \\
(B) & \det(A^T) = \det(A) \\
(C) & \det(A^{-1}) = \dfrac{1}{\det(A)}, \text{ if } \det(A) \neq 0 \\
(D) & \det(AB) = [\det(A)][\det(B)]
\end{array}
$$

Problems 26 - 30 are Bonus Problems, with 4 points each.

26. Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants. Which of the following computations is NOT true?

$$(A)\quad \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = (\alpha^2 + 2\beta^2) \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$(B)\quad \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = (\alpha^2 + \beta^2) \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$(C)\quad \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = (\alpha^2 + \beta^2) \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(D)\quad \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (\alpha^2 - 3\beta^2) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

27. Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants. Which of the following computations is NOT true?

$$(A)\quad \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = (\alpha^2+\beta^2) \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$(B)\quad \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$(C)\quad \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = (2\alpha^2+2\beta^2) \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(D)\quad \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (2\alpha^2-2\beta^2) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

28. Let $\alpha \neq 0$ be a real nonzero constant. Let

$$A = \begin{pmatrix} \alpha^2 & \alpha^2 & \alpha^2 & \alpha^2 \\ \alpha^2 & \alpha^2 & -\alpha^2 & -\alpha^2 \\ \alpha^2 & -\alpha^2 & -\alpha^2 & \alpha^2 \\ \alpha^2 & -\alpha^2 & \alpha^2 & -\alpha^2 \end{pmatrix}.$$

What is the rank of the matrix A^{2023}?

(A) 4.

(B) 3.

(C) 2.

(D) 1.

29. Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants, such that $\alpha^2 > \beta^2 > 0$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

What is the rank of the matrix A^{2468}?

(A) 2.

(B) 4.

(C) 6.

(D) 8.

30. Let

$$A = \begin{pmatrix} \lambda - \alpha & \beta & \beta \\ \beta & \lambda - \alpha & \beta \\ \beta & \beta & \lambda - \alpha \end{pmatrix}.$$

Suppose that the real nonzero constants α, β and λ satisfy these conditions: $\lambda \neq \alpha + \beta$ and $\lambda \neq \alpha - 2\beta$. Which of the following statements is true?

$$(A) \quad A^{-1} = \frac{1}{(\lambda - \alpha - \beta)(\lambda - \alpha + 2\beta)} \cdot \begin{pmatrix} \lambda - \alpha + \beta & -\beta & -\beta \\ -\beta & \lambda - \alpha + \beta & -\beta \\ -\beta & -\beta & \lambda - \alpha + \beta \end{pmatrix}$$

$$(B) \quad A^{-1} = \frac{1}{(\lambda - \alpha - \beta)(\lambda - \alpha + 2\beta)} \cdot \begin{pmatrix} \lambda - \alpha + \beta & \beta & \beta \\ \beta & \lambda - \alpha + \beta & \beta \\ \beta & \beta & \lambda - \alpha + \beta \end{pmatrix}$$

$$(C) \quad A^{-1} = \frac{1}{(\lambda - \alpha - \beta)(\lambda - \alpha + 2\beta)} \cdot \begin{pmatrix} \lambda + \alpha + \beta & \beta & \beta \\ \beta & \lambda + \alpha + \beta & \beta \\ \beta & \beta & \lambda + \alpha + \beta \end{pmatrix}$$

$$(D) \quad A^{-1} = \frac{1}{(\lambda - \alpha - \beta)(\lambda - \alpha + 2\beta)} \cdot \begin{pmatrix} \lambda + \alpha + \beta & -\beta & -\beta \\ -\beta & \lambda + \alpha + \beta & -\beta \\ -\beta & -\beta & \lambda + \alpha + \beta \end{pmatrix}$$

Mathematics 43 - The Second Midterm Examination - 2023

There are twenty-five multiple choice problems in the Second Midterm Exam. There are five bonus problems (they are not mandatory to solve). Each problem is worth four points. The total is 100 points (it is 120 points if the bonus problems are included). The exam is 75 minutes. In each problem, there are four choices marked (A), (B), (C) and (D). Only one choice is correct. Choose the one you think is correct. If you do not know the answer to a problem, you may make a reasonable, best possible guess. No supporting work is necessary. In the exam, we use $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$ and $\lambda \neq 0$ to represent various real nonzero constants or positive constants.

No calculators, computers, cell phones, i-pads, i-touches or any other electronic devices are allowed in the exam. Students are not allowed to ask for assistance from anybody else. Students are not allowed to use any websites for assistance. Any student who cheats will receive an F as the Final Grade. This is Absolutely Firm! Therefore, be honest and solve all problems by yourself.

The First Midterm Exam: 100 points
The Second Midterm Exam: 100 points
The Final Exam: 200 points
The Homework Assignments: 100 points
Total Score: 500 points

Chapter 3: Vector Spaces. There are 10 problems in Chapter 3.

1. Which of the following subsets is a subspace of $M_n(\mathbb{R})$?

$$\begin{array}{ll} (A) & \mathcal{A} = \left\{ A \in M_n(\mathbb{R}) : A^T = A \right\} \\ (B) & \mathcal{B} = \left\{ A \in M_n(\mathbb{R}) : A^T = A^2 \right\} \\ (C) & \mathcal{C} = \left\{ A \in M_n(\mathbb{R}) : A^T = -A^2 \right\} \\ (D) & \mathcal{D} = \left\{ A \in M_n(\mathbb{R}) : A^T = A^4 \right\} \end{array}$$

2. Let $\alpha \neq 0$, $\beta \neq 0$ and $\lambda \neq 0$ be real constants, such that

$$\alpha + \beta \neq 0, \qquad \alpha - 2\beta \neq 0, \qquad \lambda \neq \alpha + \beta, \qquad \lambda \neq \alpha - 2\beta.$$

Which of the following vector sets is NOT a spanning set of $\mathbb{R}^3$?

$$(A) \quad \mathcal{A} = \left\{ \begin{pmatrix} +22\alpha \\ -11\alpha \\ -11\alpha \end{pmatrix}, \quad \begin{pmatrix} -11\alpha \\ +22\alpha \\ -11\alpha \end{pmatrix}, \quad \begin{pmatrix} -11\alpha \\ -11\alpha \\ +22\alpha \end{pmatrix} \right\}$$

$$(B) \quad \mathcal{B} = \left\{ \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \end{pmatrix}, \quad \begin{pmatrix} -\alpha \\ \alpha \\ \alpha \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} \right\}$$

$$(C) \quad \mathcal{C} = \left\{ \begin{pmatrix} \alpha \\ -\beta \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ \alpha \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

$$(D) \quad \mathcal{D} = \left\{ \begin{pmatrix} \lambda - \alpha \\ \beta \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \lambda - \alpha \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \beta \\ \lambda - \alpha \end{pmatrix} \right\}$$

3. Let α, β and λ be real constants. Define the vector set

$$\mathcal{B} = \left\{ \begin{pmatrix} \lambda - \alpha \\ \beta \\ \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \lambda - \alpha \\ \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \\ \lambda - \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \\ \beta \\ \lambda - \alpha \end{pmatrix} \right\}.$$

Given that the set $\mathcal{B}$ is a spanning set of the vector space $\mathbb{R}^4$. Which of the following conditions must be true?

$$\begin{array}{lll} (A) & \lambda = \alpha + \beta, & \lambda = \alpha - 3\beta \\ (B) & \lambda = \alpha + \beta, & \lambda \neq \alpha - 3\beta \\ (C) & \lambda \neq \alpha + \beta, & \lambda \neq \alpha - 3\beta \\ (D) & \lambda \neq \alpha + \beta, & \lambda = \alpha - 3\beta \end{array}$$

4. Let α and β be real constants, such that $\alpha + \beta > 0$ and $\alpha - 3\beta > 0$. Define the vector set

$$\mathcal{B} = \left\{ \begin{pmatrix} \alpha \\ -\beta \\ -\beta \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ -\beta \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

Which of the following statements is NOT true?

(A) The set $\mathcal{B}$ is linearly independent.

(B) The set $\mathcal{B}$ is linearly dependent and is a spanning set.

(C) The set $\mathcal{B}$ is a spanning set.

(D) The set $\mathcal{B}$ is a basis.

5. Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants, such that $\alpha^2 > \beta^2$. Define the vector set

$$\mathcal{S} = \left\{ \begin{pmatrix} \alpha \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ \alpha \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

Which of the following statements is true?

(A) The set $\mathcal{S}$ is a basis of $\mathbb{R}^4$.

(B) The set $\mathcal{S}$ is a spanning set of $\mathbb{R}^4$.

(C) The set $\mathcal{S}$ is linearly independent in $\mathbb{R}^4$.

(D) The set $\mathcal{S}$ is linearly dependent in $\mathbb{R}^4$.

6. Let

$$\mathcal{S} = \left\{\mathbf{v}_1, \qquad \mathbf{v}_2, \qquad \mathbf{v}_3, \qquad \cdots\cdots, \qquad \mathbf{v}_n\right\}$$

be a vector set of linearly dependent vectors in the vector space $\mathbb{R}^n$. Let $A = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \cdots\ \mathbf{v}_n)$ be the matrix made by using all the vectors in $\mathcal{S}$. Which of the following statements is true?

(A) There exist infinitely many solutions to the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$.

(B) The inverse matrix A^{-1} exists.

(C) The rank of the matrix A is n.

(D) The determinant $\det(A) \neq 0$.

7. Which of the following statements is NOT true?

$$(A) \quad \mathcal{A} = \{1, x, x^2, x^3, \quad \cdots\cdots, \quad x^n, \cdots\cdots\}$$

is a basis of $C(\mathbb{R})$ and the dimension of $C(\mathbb{R})$ is ∞.

$$(B) \quad \mathcal{B} = \{1, x, x^2, x^3, \quad \cdots\cdots, \quad x^{100}\}$$

is a basis of $P_{100}(\mathbb{R})$ and the dimension of $P_{100}(\mathbb{R})$ is 100.

$$(C) \quad \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis of $\mathbb{R}^4$ and the dimension of $\mathbb{R}^4$ is 4.

$$(D) \quad \mathcal{D} = \left\{ \begin{pmatrix} 100 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 100 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 100 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 100 \end{pmatrix} \right\}$$

is a basis of $M_2(\mathbb{R})$ and the dimension of $M_2(\mathbb{R})$ is 4.

8. Given that

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

is a basis of the vector space $\mathbb{R}^4$. Let $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4$. Which of the following is the component vector of $\mathbf{v}$ relative to the basis $\mathcal{B}$?

$$(A) \quad -\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$(B) \quad \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$(C) \quad \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$(D) \quad \frac{1}{8} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

9. Given the following two bases of $\mathbb{R}^4$:

$$\mathcal{B} = \left\{ \begin{pmatrix} \alpha \\ -\alpha \\ -\alpha \\ -\alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ -\alpha \\ -\alpha \\ -\alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ -\alpha \\ -\alpha \\ -\alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ -\alpha \\ -\alpha \\ -\alpha \end{pmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

where $\alpha > 0$ is a positive constant. Which of the following is the change-of-basis-matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ from the basis $\mathcal{B}$ to the basis $\mathcal{C}$?

$$(A)\quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} +\alpha & -\alpha & -\alpha & -\alpha \\ -\alpha & +\alpha & -\alpha & -\alpha \\ -\alpha & -\alpha & +\alpha & -\alpha \\ -\alpha & -\alpha & -\alpha & +\alpha \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}^{-1}$$

$$(B)\quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} +\alpha & -\alpha & -\alpha & -\alpha \\ -\alpha & +\alpha & -\alpha & -\alpha \\ -\alpha & -\alpha & +\alpha & -\alpha \\ -\alpha & -\alpha & -\alpha & +\alpha \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$(C)\quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} +\alpha & -\alpha & -\alpha & -\alpha \\ -\alpha & +\alpha & -\alpha & -\alpha \\ -\alpha & -\alpha & +\alpha & -\alpha \\ -\alpha & -\alpha & -\alpha & +\alpha \end{pmatrix}^{-1}$$

$$(D)\quad P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} +\alpha & -\alpha & -\alpha & -\alpha \\ -\alpha & +\alpha & -\alpha & -\alpha \\ -\alpha & -\alpha & +\alpha & -\alpha \\ -\alpha & -\alpha & -\alpha & +\alpha \end{pmatrix}$$

10. Which of the following statements is true?

$$\begin{aligned} &(A) \quad \dim[P_n(\mathbb{R})] = n+1 \\ &(B) \quad \dim(\mathbb{R}^n) = n+1 \\ &(C) \quad \dim[M_n(\mathbb{R})] = n+1 \\ &(D) \quad \dim[C(\mathbb{R})] = n+1 \end{aligned}$$

Chapter 4: Linear Transformation. There are 7 problems in Chapter 4.

11. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation, given by

$$P(a + bx + cx^2 + dx^3) = 10a + 20bx + 30cx^2 + 40dx^3.$$

Which of the following statements is NOT true?

(A) The inverse linear transformation T^{-1} does exists.

(B) The inverse linear transformation T^{-1} does not exist.

(C) The linear transformation is one-to-one because the kernel is $\mathrm{Ker}(T) = \{0\}$.

(D) The linear transformation is onto because the range is $\mathrm{Rng}(T) = P_3(\mathbb{R})$.

12. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, given by

$$T(\mathbf{x}) = A\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where A is a real symmetric matrix. All eigenvalues of A are positive. Which of the following statements is NOT true?

(A) The determinant $\det(A) < 0$.

(B) The linear transformation T is one-to-one and onto.

(C) The linear transformation T is one-to-one because the kernel is Ker $(T) = \{\mathbf{0}\}$.

(D) The linear transformation T is onto because the range is Rng $(T) = \mathbb{R}^n$.

13. Let $T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$ be a linear transformation, given by

$$T(A) = A^T - A, \qquad A \in M_3(\mathbb{R}).$$

Which of the following vector sets is a basis of the range of the linear transformation?

$$(A) \quad \mathcal{A} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

$$(B) \quad \mathcal{B} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

$$(C) \quad \mathcal{C} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

$$(D) \quad \mathcal{D} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

14. Which of the following linear transformations is NOT an isomorphism?

$$(A)\quad T:\mathbb{R}^4\to P_3(\mathbb{R}),$$

$$T\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} = a+2bx+3cx^2+4dx^3.$$

$$(B)\quad T:\mathbb{R}^4\to \mathbb{R}^4,$$

$$T\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2\\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2\\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2\\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}.$$

$$(C)\quad T:\mathbb{R}^4\to M_2(\mathbb{R}),$$

$$T\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix} 10a & 10b\\ 10c & 10d\end{pmatrix}.$$

$$(D)\quad T:M_2(\mathbb{R})\to P_3(\mathbb{R}),$$

$$T\begin{pmatrix} a & b\\ c & d\end{pmatrix} = 10a+20bx+30cx^2+40dx^3.$$

15. Let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation, where $\mathbf{V}$ and $\mathbf{W}$ are vector spaces. Which of the following statements is NOT true?

(A) The kernel Ker(T) is a subspace of the vector space $\mathbf{V}$.

(B) The range Rng(T) is a subspace of the vector space $\mathbf{W}$.

(C) The inverse linear transformation T^{-1} exists if T is one-to-one and onto.

(D) The sum of the dimension of the kernel and the dimension of the range is equal to the dimension of the vector space $\mathbf{W}$. That is

$$\dim[\,\mathrm{Ker}\,(T)] + \dim[\,\mathrm{Rng}\,(T)] = \dim \mathbf{W}.$$

16. Let $T : \mathbb{R}^{10} \to \mathbb{R}^5$ be a linear transformation. Given that the dimension of the kernel is $\dim[\,\text{Ker}\,(T)] = 5$. Which of the following statements is true?

(A) The linear transformation T is onto.

(B) The linear transformation T is one-to-one.

(C) The inverse linear transformation T^{-1} exists.

(D) The linear transformation T is one-to-one and onto.

17. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation, given by

$$T(a + bx + cx^2 + dx^3) = (a + b) + (b + c)x + (c + d)x^2 + (d + a)x^3.$$

Which of the following sets is the basis of the kernel $\mathrm{Ker}(T)$?

(A) $\{1 + x + x^2 + x^3\}$
(B) $\{1 - x + x^2 - x^3\}$
(C) $\{1 - x, \quad x^2 - x^3\}$
(D) $\{1 + x^2, \quad x + x^3\}$

Chapter 5: Eigenvalues and Eigenvectors. There are 8 problems in Chapter 5.

18. Let α and β be real nonzero constants, such that $\alpha^2 > \beta^2$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

There are three real eigenvalues to the matrix A. Which of the following choices are the eigenvalues?

$$\begin{array}{llll} (A) & \lambda = 0, & \lambda = \alpha^2 + \beta^2, & \lambda = \alpha^2 - \beta^2 \\ (B) & \lambda = 0, & \lambda = \alpha^2 + \beta^2, & \lambda = \alpha^2 - 2\beta^2 \\ (C) & \lambda = 0, & \lambda = 2\alpha^2 + 2\beta^2, & \lambda = 2\alpha^2 - 2\beta^2 \\ (D) & \lambda = 0, & \lambda = 2\alpha^2 + 2\beta^2, & \lambda = 2\alpha^2 - 3\beta^2 \end{array}$$

19. Let α and β be real nonzero constants, such that $\alpha^2 > 3\beta^2$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

There are two eigenvalues to the matrix A. Which of the following are the eigenvalues of the matrix A?

(A) $\lambda_1 = 2\alpha^2 + 2\beta^2, \qquad \lambda_2 = \alpha^2 + 2\beta^2$

(B) $\lambda_1 = 2\alpha^2 + 2\beta^2, \qquad \lambda_2 = \alpha^2 + 3\beta^2$

(C) $\lambda_1 = \alpha^2 + \beta^2, \qquad \lambda_2 = \alpha^2 - 2\beta^2$

(D) $\lambda_1 = \alpha^2 + \beta^2, \qquad \lambda_2 = \alpha^2 - 3\beta^2$

20. Let A be a real $n \times n$ constant matrix. Let

$$\lambda_1, \qquad \lambda_2, \qquad \lambda_3, \qquad \cdots\cdots \qquad \lambda_n,$$

represent all the eigenvalues of A. Which of the following statements about the determinant and the trace of the matrix A^2 is true?

$$\begin{aligned}
(A)\quad & \det(A^2) = +(\lambda_1\lambda_2\lambda_3\cdots\lambda_n)^2, \\
& \text{Tr }(A^2) = -(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_n^2) \\
(B)\quad & \det(A^2) = +(\lambda_1\lambda_2\lambda_3\cdots\lambda_n)^2, \\
& \text{Tr }(A^2) = +(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_n^2) \\
(C)\quad & \det(A^2) = +(\lambda_1\lambda_2\lambda_3\cdots\lambda_n)^2, \\
& \text{Tr }(A^2) = -(\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n)^2, \\
(D)\quad & \det(A^2) = +(\lambda_1\lambda_2\lambda_3\cdots\lambda_n)^2, \\
& \text{Tr }(A^2) = +(\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n)^2,
\end{aligned}$$

21. Let A be a real 4×4 matrix. Two complex eigenvalues are given

$$\lambda_1 = 10 - 5\mathrm{i}, \qquad \lambda_2 = 20\mathrm{i}$$

What are the other complex eigenvalues of A?

$$\begin{array}{lll} (A) & \lambda_3 = 10 - 5\mathrm{i}, & \lambda_4 = +20\mathrm{i} \\ (B) & \lambda_3 = 10 - 5\mathrm{i}, & \lambda_4 = -20\mathrm{i} \\ (C) & \lambda_3 = 10 + 5\mathrm{i}, & \lambda_4 = -20\mathrm{i} \\ (D) & \lambda_3 = 10 + 5\mathrm{i}, & \lambda_4 = +20\mathrm{i} \end{array}$$

22. Let A be a real $n \times n$ constants diagonalizable matrix. Which of the following statements is true?

(A) The algebraic multiplicity is equal to the geometric multiplicity, for every eigenvalue λ_i of A.

(B) The algebraic multiplicity is equal to the geometric multiplicity, for one eigenvalue λ_i of A.

(C) The algebraic multiplicity is larger than the geometric multiplicity, for each eigenvalue λ_i of A.

(D) The algebraic multiplicity is less than the geometric multiplicity, for some eigenvalue λ_i of A.

23. Let A be a real $n \times n$ constant matrix, let

$$\lambda_1, \qquad \lambda_2, \qquad \lambda_3, \qquad \cdots, \qquad \lambda_n$$

be the eigenvalues of the matrix A. Define

$$B = (\lambda_n I + A)(\lambda_n I - A).$$

Which of the following statements about the matrix B is true?

(A) The eigenvalues of A are the same as the eigenvalues of B.

(B) The determinant $\det(B) = \lambda_1^2 \lambda_2^2 \lambda_3^2 \cdots \lambda_n^2$.

(C) There exists at least one zero eigenvalue $\lambda = 0$ for the matrix B.

(D) The trace $\mathrm{Tr}\,(B) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_n^2$.

24. Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants, such that $\alpha^2 > 3\beta^2$. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix} \mathbf{u}$$

Which of the following is the solution of the system of differential equations?

$$(A) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \ C_3 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 \exp[(\alpha^2-3\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(B) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2-3\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \ C_3 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(C) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \exp[(\alpha^2-3\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \; C_3 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(D) \qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \; C_3 \exp[(\alpha^2-3\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

25. Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants, such that $\alpha^2 > \beta^2$. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix} \mathbf{u}$$

Which of the following is the solution of the system of differential equations?

$$(A) \qquad \mathbf{u}(t) = C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \exp[(2\alpha^2 + 2\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$+ \; C_3 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + C_4 \exp[(2\alpha^2 - 2\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(B) \qquad \mathbf{u}(t) = C_1 \exp[(2\alpha^2 + 2\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$+ \; C_3 \exp[(2\alpha^2 - 2\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} + C_4 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(C) \qquad \mathbf{u}(t) = C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \exp[(2\alpha^2 - 2\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \ C_3 \exp[(2\alpha^2 + 2\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$(D) \qquad \mathbf{u}(t) = C_1 \exp[(2\alpha^2 - 2\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$+ \ C_3 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + C_4 \exp[(2\alpha^2 + 2\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

26. Which of the following vector sets is NOT a basis of $M_2(\mathbb{R})$?

(A) $\quad \mathcal{A} = \left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \right\}$

(B) $\quad \mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

(C) $\quad \mathcal{C} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$

(D) $\quad \mathcal{D} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \right\}$

27. Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$ and $\lambda > 0$ be positive constants, such that $\alpha > \beta > \gamma > \delta$, $\alpha > 3\beta$, $\lambda > \alpha + \beta$ and $\lambda > \alpha - 3\beta$. Which of the following vector sets is NOT a basis of the vector space $\mathbb{R}^4$?

$$(A) \quad \mathcal{B} = \left\{ \begin{pmatrix} -\alpha \\ \alpha \\ \alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ -\alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ -\alpha \end{pmatrix} \right\}$$

$$(B) \quad \mathcal{C} = \left\{ \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \\ -\alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} \alpha \\ \alpha \\ -\alpha \\ \alpha \end{pmatrix}, \begin{pmatrix} -\alpha \\ \alpha \\ -\alpha \\ \alpha \end{pmatrix} \right\}$$

$$(C) \quad \mathcal{C} = \left\{ \begin{pmatrix} \alpha \\ -\beta \\ -\beta \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ -\beta \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

$$(D) \quad \mathcal{D} = \left\{ \begin{pmatrix} \lambda - \alpha \\ \beta \\ \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \lambda - \alpha \\ \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \\ \lambda - \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \\ \beta \\ \lambda - \alpha \end{pmatrix} \right\}$$

28. Let $\alpha > 0$ be a positive constant. Define

$$A = \begin{pmatrix} \alpha & -\alpha & \alpha & -\alpha \\ -\alpha & \alpha & -\alpha & \alpha \\ \alpha & -\alpha & \alpha & -\alpha \\ -\alpha & \alpha & -\alpha & \alpha \end{pmatrix}$$

Which of the following is a basis of the nullspace of A?

$$(A) \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

$$(B) \quad \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$(C) \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$(D) \quad \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$$

29. Let $\mathbf{V}$ and $\mathbf{W}$ be vector spaces and let $T : \mathbf{V} \to \mathbf{W}$ be a linear transformation. Let

$$\mathcal{C} = \Big\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots\cdots, \mathbf{v}_k\Big\}$$

be a basis of the kernel Ker (T), let

$$\mathcal{B} = \Big\{\mathbf{v}_1, \cdots\cdots, \mathbf{v}_k, \mathbf{v}_{k+1}, \cdots\cdots, \mathbf{v}_n\Big\}$$

be a basis of the vector space $\mathbf{V}$, where k and n are positive integers, $1 < k < n$. Define the vector set

$$\mathcal{A} = \Big\{T(\mathbf{v}_{k+1}), T(\mathbf{v}_{k+2}), T(\mathbf{v}_{k+3}), \cdots\cdots, T(\mathbf{v}_n)\Big\}.$$

Which of the following statements is true?

(A) $\mathcal{A}$ is a spanning set of the space $\mathbf{W}$.

(B) $\mathcal{A}$ is a basis of the vector space $\mathbf{W}$.

(C) $\mathcal{A}$ is linearly dependent.

(D) $\mathcal{A}$ is linearly independent.

30. Let A be a real $n \times n$ constant matrix, let

$$\lambda_1, \qquad \lambda_2, \qquad \lambda_3, \qquad \cdots\cdots, \qquad \lambda_n$$

represent all the eigenvalues of the matrix A. Consider the matrix

$$B \overset{\text{def}}{=} I + 2A + A^2 - (\lambda_1 + 1)^2 I$$

Which of the following statements about the matrix B is true?

(A) Tr $(B) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_n^2$

(B) $\det(B) = (\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n)^2$

(C) The eigenvectors of the matrix B
are exactly the same as the eigenvectors of A

(D) The eigenvalues of the matrix B are
$(\lambda_1 - 1)^2, \qquad (\lambda_2 - 1)^2, \qquad (\lambda_3 - 1)^2, \qquad \cdots, \qquad (\lambda_n - 1)^2$

Mathematics 43 - The Final Examination - 2023

There are 50 multiple choice problems in the Final Examination. Each problem is worth 4 points. The Final Exam is 200 points and it is 3 hours. In each problem, there are four choices, marked with (A), (B), (C) and (D). Only one choice is correct in every problem. On the answer sheet, write down the answer you think is correct. If you do not know the answer to a problem, you may make the possible, best guess. No supporting work is necessary for any problem.

There are eight bonus problems at the end of the Final Exam. Each problem is worth 4 points as well.

In this exam, A, B, C, M represent real matrices; A^T represents the transposed matrix of A; $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$, $\lambda \neq 0$, $\mu \neq 0$ represent real nonzero constants; $m \geq 1$ and $n \geq 1$ represent positive integers; $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, ξ and η represent real vectors. $\mathbb{R}^n$, $M_n(\mathbb{R})$, $P_n(\mathbb{R})$ and $C(\mathbb{R})$ represent the popular vector spaces, $\mathbf{V}$ and $\mathbf{W}$ represent any vector spaces, $T : \mathbf{V} \to \mathbf{W}$ represents a linear transformation.

Hint 1: Let

$$T = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Then

$$T^2 = I.$$

Hint 2: Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}$$

$$B = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix}$$

Then

$$\begin{aligned} \det(\lambda I - A) &= (\lambda - \alpha - \beta)^3(\lambda - \alpha + 3\beta), \\ \det(\lambda I - B) &= \lambda^2(\lambda - 2\alpha + 2\beta)(\lambda - 2\alpha - 2\beta). \end{aligned}$$

Hint 3: Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix},$$

$$B = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix},$$

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Then

$$T^{-1}AT = \begin{pmatrix} \alpha - 3\beta & 0 & 0 & 0 \\ 0 & \alpha + \beta & 0 & 0 \\ 0 & 0 & \alpha + \beta & 0 \\ 0 & 0 & 0 & \alpha + \beta \end{pmatrix},$$

$$T^{-1}BT = \begin{pmatrix} 2\alpha - 2\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha + 2\beta \end{pmatrix}.$$

Chapter 1: System of Linear Equations and Inverse Matrices. There are 5 problems in Chapter 1.

1. The reduced row echelon form of the augmented matrix of the system of equations $A\mathbf{x} = \mathbf{b}$ is a zero matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

How many free variables are there in the solutions?
(A) 6. (B) 4. (C) 2. (D) 0.

2. Let $\alpha > 0$ be a positive constant. Let

$$A = \frac{1}{2}\alpha \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Which of the following statements is true?

$$(A) \quad A^{-1} = \frac{1}{2}\alpha \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$(B) \quad A^{-1} = \frac{1}{2\alpha} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$(C) \quad A^{-1} = \frac{1}{4}\alpha \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$(D) \quad A^{-1} = \frac{1}{4\alpha} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

3. Let A be a real $m \times n$ constant matrix, let $\mathbf{b} \in \mathbb{R}^m$ be a real constant vector and let $A^\#$ represent the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$. Which of the following statements is NOT true?

(A) There exists a unique solution to the system $A\mathbf{x} = \mathbf{b}$, if both the rank of the coefficient matrix A and the rank of the augmented matrix $A^\#$ are equal to n.

(B) There exist infinitely many solutions to the system $A\mathbf{x} = \mathbf{b}$, if both the rank of the coefficient matrix A and the rank of the augmented matrix $A^\#$ are equal to r, and $r < n$.

(C) There exists one solution to the system $A\mathbf{x} = \mathbf{b}$, if the rank of the augmented matrix $A^\#$ is larger than the rank of the coefficient matrix A.

(D) There exists a unique solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$, if the rank of the coefficient matrix A is equal to n.

4. Let the positive constants $\alpha > 0$ and $\beta > 0$ satisfy the condition $\alpha^2 > 3\beta^2$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

Which of the following statements is NOT true?

(A) There is a zero eigenvalue $\lambda = 0$.

(B) The rank of A is 4.

(C) The inverse matrix A^{-1} exists.

(D) There exists a unique solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for each vector $\mathbf{b} \in \mathbb{R}^4$.

5. Let $\alpha > 0$ be a positive constant. Which of the following statements is true?

$$(A)\quad \begin{pmatrix} -\alpha & \alpha & \alpha & \alpha \\ \alpha & -\alpha & \alpha & \alpha \\ \alpha & \alpha & -\alpha & \alpha \\ \alpha & \alpha & \alpha & -\alpha \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} -\alpha & \alpha & \alpha & \alpha \\ \alpha & -\alpha & \alpha & \alpha \\ \alpha & \alpha & -\alpha & \alpha \\ \alpha & \alpha & \alpha & -\alpha \end{pmatrix}$$

$$(B)\quad \begin{pmatrix} -\alpha & \alpha & \alpha & \alpha \\ \alpha & -\alpha & \alpha & \alpha \\ \alpha & \alpha & -\alpha & \alpha \\ \alpha & \alpha & \alpha & -\alpha \end{pmatrix}^{-1} = \frac{1}{4\alpha^2}\begin{pmatrix} -\alpha & \alpha & \alpha & \alpha \\ \alpha & -\alpha & \alpha & \alpha \\ \alpha & \alpha & -\alpha & \alpha \\ \alpha & \alpha & \alpha & -\alpha \end{pmatrix}$$

$$(C)\quad \begin{pmatrix} -\alpha & \alpha & \alpha & \alpha \\ \alpha & -\alpha & \alpha & \alpha \\ \alpha & \alpha & -\alpha & \alpha \\ \alpha & \alpha & \alpha & -\alpha \end{pmatrix}^{-1} = \frac{1}{16}\begin{pmatrix} -\alpha & \alpha & \alpha & \alpha \\ \alpha & -\alpha & \alpha & \alpha \\ \alpha & \alpha & -\alpha & \alpha \\ \alpha & \alpha & \alpha & -\alpha \end{pmatrix}$$

$$(D)\quad \begin{pmatrix} -\alpha & \alpha & \alpha & \alpha \\ \alpha & -\alpha & \alpha & \alpha \\ \alpha & \alpha & -\alpha & \alpha \\ \alpha & \alpha & \alpha & -\alpha \end{pmatrix}^{-1} = \frac{1}{16\alpha^2}\begin{pmatrix} -\alpha & \alpha & \alpha & \alpha \\ \alpha & -\alpha & \alpha & \alpha \\ \alpha & \alpha & -\alpha & \alpha \\ \alpha & \alpha & \alpha & -\alpha \end{pmatrix}$$

Chapter 2: Determinant. There are 5 problems in Chapter 2.

6. Let α, β, γ be real constants. Which of the following statements is true?

$$(A) \quad \det\begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^3 + (\alpha^2 + \beta^2 + \gamma^2)$$

$$(B) \quad \det\begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^3 - (\alpha^2 + \beta^2 + \gamma^2)$$

$$(C) \quad \det\begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^3 + (\alpha^2 + \beta^2 + \gamma^2)\lambda$$

$$(D) \quad \det\begin{pmatrix} \lambda & \alpha & \beta \\ -\alpha & \lambda & \gamma \\ -\beta & -\gamma & \lambda \end{pmatrix} = \lambda^3 - (\alpha^2 + \beta^2 + \gamma^2)\lambda$$

7. Let A be a real $n \times n$ constant matrix, $\det(A) > 0$. Which of the following statements is NOT true?

$$\begin{aligned} &(A) \quad \det(A^{-1}) = \frac{1}{\det(A)} \\ &(B) \quad \det(A^2) = [\det(A)]^2 > 0 \\ &(C) \quad \det(\alpha A) = \alpha^n \det(A) \\ &(D) \quad \det(A^T A) = [\det(A)]^n \end{aligned}$$

where α represents any real constant.

8. Let A and B be real $n \times n$ constant matrices, and B is the adjoint matrix of A. Which of the following statements is NOT true?

$$
\begin{array}{ll}
(A) & A^2B^2 = B^2A^2 = [\det(AB)]I \\
(B) & A^{10}B^{20} = B^{20}A^{10} = [\det(A)]^{10}B^{10} \\
(C) & A^{-1} = \dfrac{1}{\det(A)}B, \text{ if } \det(A) \neq 0 \\
(D) & \det(A) \neq 0, \text{ if and only if } \det(B) \neq 0
\end{array}
$$

9. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

Which of the following statements is true?

(A) $\det(\lambda I - A) = (\lambda - \alpha^2 - \beta^2)^2(\lambda - \alpha^2 - 3\beta^2)$
(B) $\det(\lambda I - A) = (\lambda - \alpha^2 - \beta^2)^2(\lambda - \alpha^2 + 3\beta^2)$
(C) $\det(\lambda I - A) = (\lambda + \alpha^2 + \beta^2)^2(\lambda + \alpha^2 + 3\beta^2)$
(D) $\det(\lambda I - A) = (\lambda + \alpha^2 + \beta^2)^2(\lambda + \alpha^2 - 3\beta^2)$

10. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

Which of the following statements is true?

(A) $\det(\lambda I - A) = \lambda(\lambda + \alpha^2 - \beta^2)(\lambda - \alpha^2 + \beta^2)$

(B) $\det(\lambda I - A) = \lambda(\lambda + \alpha^2 - \beta^2)(\lambda - \alpha^2 - \beta^2)$

(C) $\det(\lambda I - A) = \lambda^2(\lambda - 2\alpha^2 - 2\beta^2)(\lambda - 2\alpha^2 + 2\beta^2)$

(D) $\det(\lambda I - A) = \lambda^2(\lambda + 2\alpha^2 - 2\beta^2)(\lambda - 2\alpha^2 + 2\beta^2)$

Chapter 3: Vector Spaces. There are 10 problems in Chapter 3.

11. Let

$$\mathcal{S} = \{x + x^2 + x^3 + x^4, \quad 1 + x^2 + x^3 + x^4, \quad 1 + x + x^3 + x^4, \quad 1 + x + x^2 + x^4, \quad 1 + x + x^2 + x^3\} \subset P_5(\mathbb{R}).$$

Which of the following statements is true about the set $\mathcal{S}$?

(A) $\mathcal{S}$ is linearly independent.

(B) $\mathcal{S}$ is a basis of $C(\mathbb{R})$.

(C) $\mathcal{S}$ is a spanning set of $P_5(\mathbb{R})$.

(D) $\mathcal{S}$ is a basis of $P_5(\mathbb{R})$.

12. Let the real nonzero constants α and β satisfy the condition $\alpha = 3\beta$. Which of the following vector sets is NOT a basis of the vector space $\mathbb{R}^4$?

$$(A) \quad \mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

$$(B) \quad \mathcal{B} = \left\{ \begin{pmatrix} \alpha \\ -\beta \\ -\beta \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ -\beta \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

$$(C) \quad \mathcal{C} = \left\{ \begin{pmatrix} +2 \\ -2 \\ -2 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ +2 \\ -2 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ -2 \\ +2 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ -2 \\ -2 \\ +2 \end{pmatrix} \right\}$$

$$(D) \quad \mathcal{D} = \left\{ \begin{pmatrix} -\alpha \\ +\alpha \\ +\alpha \\ +\alpha \end{pmatrix}, \quad \begin{pmatrix} +\alpha \\ -\alpha \\ +\alpha \\ +\alpha \end{pmatrix}, \quad \begin{pmatrix} +\alpha \\ +\alpha \\ -\alpha \\ +\alpha \end{pmatrix}, \quad \begin{pmatrix} +\alpha \\ +\alpha \\ +\alpha \\ -\alpha \end{pmatrix} \right\}$$

13. Let the positive constants $\alpha > 0$, $\beta > 0$ and $\lambda > 0$ satisfy the following conditions

$$\alpha + \beta > 0, \qquad \alpha > 2\beta, \qquad \lambda > \alpha + \beta, \qquad \lambda > \alpha - 2\beta.$$

Which of the following vector sets is NOT a spanning set of $\mathbb{R}^3$?

$$(A) \quad \mathcal{A} = \left\{ \begin{pmatrix} \alpha \\ -\alpha \\ \alpha \end{pmatrix}, \quad \begin{pmatrix} -\alpha \\ \alpha \\ \alpha \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} \right\}$$

$$(B) \quad \mathcal{B} = \left\{ \begin{pmatrix} \alpha \\ -\beta \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ \alpha \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

$$(C) \quad \mathcal{C} = \left\{ \begin{pmatrix} +\alpha \\ -\alpha \\ +\alpha \end{pmatrix}, \quad \begin{pmatrix} -\alpha \\ +\alpha \\ -\alpha \end{pmatrix}, \quad \begin{pmatrix} +\alpha \\ -\alpha \\ +\alpha \end{pmatrix} \right\}$$

$$(D) \quad \mathcal{D} = \left\{ \begin{pmatrix} \lambda - \alpha \\ \beta \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \lambda - \alpha \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \beta \\ \lambda - \alpha \end{pmatrix} \right\}$$

14. Let the positive constants $\alpha > 0$, $\beta > 0$, $\gamma > 0$ satisfy the conditions $\alpha > \beta > \gamma$. Let the positive integer $n > 10$. Which of the following sets is NOT linearly independent?

(A) $\mathcal{A} = \{\cos x, \quad \sin x\}$

(B) $\mathcal{B} = \{1, \quad x, \quad x^2, \quad x^3, \quad \cdots\cdots, \quad x^n\}$

(C) $\mathcal{C} = \{\exp(\alpha x), \quad \exp(\beta x), \quad \exp(\gamma x)\}$

(D) $\mathcal{D} = \{1 + x, \quad x + x^2, \quad 1 + 2x + x^2\}$

15. Let the positive constants $\alpha > 0$, $\beta > 0$, $\gamma > 0$ satisfy the conditions $\alpha > \beta > \gamma > 0$. Let $W = W[f_1, f_2, f_3, \cdots, f_n]$ represent the Wronskian of the functions inside the parenthesis. Which of the following computations is wrong?

$$(A) \qquad \begin{aligned} & W[1+x, x+x^2, 1+2x+x^2] \\ = \ & \det\begin{pmatrix} 1+x & x+x^2 & 1+2x+x^2 \\ 1 & 1+2x & 2+2x \\ 0 & 2 & 2 \end{pmatrix} = 1 > 0 \end{aligned}$$

$$(B) \qquad W[\cos x, \sin x] = \det\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = 1$$

$$(C) \qquad \begin{aligned} & W[\exp(\alpha x), \exp(\beta x), \exp(\gamma x)] \\ = \ & \det\begin{pmatrix} \exp(\alpha x) & \exp(\beta x) & \exp(\gamma x) \\ \alpha\exp(\alpha x) & \beta\exp(\beta x) & \gamma\exp(\gamma x) \\ \alpha^2\exp(\alpha x) & \beta^2\exp(\beta x) & \gamma^2\exp(\gamma x) \end{pmatrix} \\ = \ & (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\exp[(\alpha+\beta+\gamma)x] \neq 0 \end{aligned}$$

$$(D) \qquad W[1, x, x^2, x^3, x^4] = \det\begin{pmatrix} 1 & x & x^2 & x^3 & x^4 \\ 0 & 1 & 2x & 3x^2 & 4x^3 \\ 0 & 0 & 2 & 6x & 12x^2 \\ 0 & 0 & 0 & 6 & 24x \\ 0 & 0 & 0 & 0 & 24 \end{pmatrix} > 0$$

16. Define the following vector spaces

$$\mathbf{V} = \left\{A \in M_2(\mathbb{R}) : \qquad A^T = A\right\}$$
$$\mathbf{W} = \left\{A \in M_2(\mathbb{R}) : \qquad A^T = -A\right\}.$$

Let us use $\mathcal{B}$ and $\mathcal{C}$ to represent the possible basis for $\mathbf{V}$ and $\mathbf{W}$, respectively. Which of the following sets are the bases of the vector spaces?

(A) $\quad \mathcal{B} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

$\mathcal{C} = \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$

(B) $\quad \mathcal{B} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\},$

$\mathcal{C} = \left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$

(C) $\quad \mathcal{B} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$

$\mathcal{C} = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$

(D) $\quad \mathcal{B} = \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

$\mathcal{C} = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$

17. Let A be a real 100×100 constant matrix. The rank of A is equal 50. What is the dimension of the nullspace $NS(A)$?
(A) 30. (B) 40. (C) 50. (D) 60.

18. Let A and B be real $n \times n$ constant matrices. Suppose that the rank of A is n and the rank of B is 0. Which of the following statements is true?

$$
\begin{array}{lll}
(A) & NS(A) = \mathbb{R}^n, & NS(B) = \mathbb{R}^n \\
(B) & NS(A) = \mathbb{R}^n, & NS(B) = \{\mathbf{0}\} \\
(C) & CS(A) = \mathbb{R}^n, & CS(B) = \mathbb{R}^n \\
(D) & CS(A) = \mathbb{R}^n, & CS(B) = \{\mathbf{0}\}
\end{array}
$$

19. Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of the vector space $\mathbb{R}^n$. Which of the following statements is NOT true?

$$
\begin{array}{ll}
(A) & P_{\mathcal{C}\leftarrow\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}} = \mathcal{I} \\
(B) & P_{\mathcal{B}\leftarrow\mathcal{C}}P_{\mathcal{C}\leftarrow\mathcal{B}} = \mathcal{I} \\
(C) & [P_{\mathcal{C}\leftarrow\mathcal{B}}]^{-1} = +P_{\mathcal{B}\leftarrow\mathcal{C}} \\
(D) & [P_{\mathcal{B}\leftarrow\mathcal{C}}]^{-1} = -P_{\mathcal{C}\leftarrow\mathcal{B}}
\end{array}
$$

20. Let the positive constants $\alpha > 0$ and $\beta > 0$ satisfy the condition $\alpha^2 > 2\beta^2$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

Which of the following statements is NOT true?

(A) The determinant $\det(A) = 0$.

(B) There exists a zero eigenvalue $\lambda = 0$.

(C) The vector set

$$\mathcal{S} = \left\{ \begin{pmatrix} \alpha^2 \\ -\beta^2 \\ \alpha^2 \\ -\beta^2 \end{pmatrix}, \quad \begin{pmatrix} -\beta^2 \\ \alpha^2 \\ -\beta^2 \\ \alpha^2 \end{pmatrix} \right\}$$

is a basis of the column space $CS(A)$.

(D) The trace of the matrix A is $4\beta^2$.

Chapter 4: Linear Transformation. There are 5 problems in Chapter 5.

21. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation, given by

$$\begin{aligned} & T(a + bx + cx^2 + dx^3) \\ = \ & (b + c + d) + (a + c + d)x + (a + b + d)x^2 + (a + b + c)x^3. \end{aligned}$$

Which of the following statements is true?

$$\begin{array}{llll} (A) & \mathrm{Ker}(T) = \{0\}, & \mathrm{Rng}(T) = P_3(\mathbb{R}) \\ (B) & \mathrm{Ker}(T) = \{0\}, & \mathrm{Rng}(T) = \{0\} \\ (C) & \mathrm{Ker}(T) = P_3(\mathbb{R}), & \mathrm{Rng}(T) = \{0\} \\ (D) & \mathrm{Ker}(T) = P_3(\mathbb{R}), & \mathrm{Rng}(T) = P_3(\mathbb{R}) \end{array}$$

22. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation, given by

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \\ a+c \\ b+d \end{pmatrix}.$$

Which of the following statements is true?

(A) The linear transformation T is not one-to-one but is onto, because the dimension of the kernel is $\dim[\,\mathrm{Ker}\,(T)] = 1 > 0$ and the range $\mathrm{Rng}(T) = \mathbb{R}^4$.

(B) The linear transformation T is neither one-to-one nor onto, because the dimension of the kernel is $\dim[\,\mathrm{Ker}\,(T)] = 1 > 0$ and the dimension of the range is $\dim[\,\mathrm{Rng}\,(T)] = 3 < 4$.

(C) The linear transformation T is one-to-one but not onto, because the kernel $\mathrm{Ker}(T) = \{\mathbf{0}\}$ and the dimension of the range is $\dim[\,\mathrm{Rng}\,(T)] = 3 < 4$.

(D) The linear transformation T is one-to-one and onto, because the kernel $\mathrm{Ker}(T) = \{\mathbf{0}\}$ and the range $\mathrm{Rng}(T) = \mathbb{R}^4$.

23. Let $T : \mathbf{V} \to \mathbf{V}$ be a linear transformation, where $\mathbf{V}$ is an n-dimensional real vector space. Suppose that the kernel $\text{Ker}(T) = \{\mathbf{0}\}$. Which of the following statements is NOT true?

(A) The linear transformation T is one-to-one.

(B) The linear transformation T is onto.

(C) The inverse linear transformation T^{-1} does not exist.

(D) The linear transformation T is an isomorphism.

24. Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$ be positive constants. Which of the following linear transformations is NOT an isomorphism?

(A) $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

(B) $T : \mathbb{R}^4 \to M_2(\mathbb{R})$,

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(C) $T : \mathbb{R}^4 \to P_3(\mathbb{R})$,

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a + bx + cx^2 + dx^3$$

(D) $T : M_2(\mathbb{R}) \to P_3(\mathbb{R})$,

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + bx + cx^2 + cx^3$$

25. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation, given by

$$T(\mathbf{x}) = \begin{pmatrix} -\alpha & +\alpha & +\alpha & +\alpha \\ +\alpha & -\alpha & +\alpha & +\alpha \\ +\alpha & +\alpha & -\alpha & +\alpha \\ +\alpha & +\alpha & +\alpha & -\alpha \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^4,$$

where $\alpha > 0$ is a positive constant. Which of the following statements is NOT true?

(A) The inverse linear transformation T^{-1} exist.

(B) The inverse linear transformation T^{-1} does not exist.

(C) The linear transformation T is one-to-one.

(D) The linear transformation T is onto.

Chapter 5: Eigenvalues and Eigenvectors. There are 5 problems in Chapter 5.

26. Let A, B and T be real $n \times n$ constant matrices, and the inverse matrix T^{-1} exists, such that $B = T^{-1}AT$. Which of the following statements is NOT true?

(A) $\det(\lambda I - A) = \det(\lambda I - B)$.

(B) $\det(\lambda I - AB) = \det(\lambda I - BA)$.

(C) The eigenvectors of A and the eigenvectors of B are the same.

(D) The eigenvalues of A and the eigenvalues of B are the same.

27. Let A be a real 4×4 constant matrix. There are two complex eigenvalues, given by

$$\lambda_1 = \alpha + \beta \mathrm{i}, \qquad \lambda_2 = \alpha^2 + \beta^2 \mathrm{i},$$

where $\alpha > 0$ and $\beta > 0$ are positive constants. What are the other complex eigenvalues of A?

$$\begin{array}{lll} (A) & \lambda_3 = \alpha + \beta \mathrm{i}, & \lambda_4 = \alpha^2 + \beta^2 \mathrm{i} \\ (B) & \lambda_3 = \alpha + \beta \mathrm{i}, & \lambda_4 = \alpha^2 - \beta^2 \mathrm{i} \\ (C) & \lambda_3 = \alpha - \beta \mathrm{i}, & \lambda_4 = \alpha^2 + \beta^2 \mathrm{i} \\ (D) & \lambda_3 = \alpha - \beta \mathrm{i}, & \lambda_4 = \alpha^2 - \beta^2 \mathrm{i} \end{array}$$

28. Let $\alpha > 0$ be a positive constant. Let

$$A = \begin{pmatrix} \alpha^2 & -\alpha^2 & \alpha^2 \\ -\alpha^2 & \alpha^2 & \alpha^2 \\ \alpha^2 & \alpha^2 & \alpha^2 \end{pmatrix}$$

There are two eigenvalues $\lambda_1 = 2\alpha^2$ and $\lambda_2 = -\alpha^2$ to the matrix A. Which of the following statements is true?

(A) The algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = 2\alpha^2$ is equal to 2.

(B) The algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = 2\alpha^2$ is equal to 1.

(C) The algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_2 = -\alpha^2$ is equal to 2.

(D) The algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_2 = -\alpha^2$ is equal to 0.

29. Let A be a real $n \times n$ constant matrix. Let

$$\lambda_1 > 0, \qquad \lambda_2 > 0, \qquad \lambda_3 > 0, \qquad \cdots\cdots, \qquad \lambda_n > 0$$

represent all the positive eigenvalues of A. Which of the following statements is NOT true?

(A) $\det(A^{2024}) = (\lambda_1\lambda_2\lambda_3 \cdots\cdots \lambda_n)^{2024}$

(B) $\det[(A^{-1})^{2024}] = [\det(A)]^{-2024} = (\lambda_1\lambda_2\lambda_3 \cdots \lambda_n)^{2024}$

(C) $(A^{2024})^{-1} = (A^{-1})^{2024}$

(D) $\mathrm{Tr}\ (A^{2024}) = \lambda_1^{2024} + \lambda_2^{2024} + \lambda_3^{2024} + \cdots + \lambda_n^{2024}$

30. Let the positive constants $\alpha > 0$ and $\beta > 0$ satisfy the condition $\alpha^2 > 2\beta^2$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

Which of the following is the solution of the system of differential equations $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = A\mathbf{u}$?

$$(A)\qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$+\ C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_3 \exp[(\alpha^2-2\beta^2)t] \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$(B)\qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$+\ C_2 \exp[(\alpha^2-2\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_3 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$(C)\qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2-3\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$+\ C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_3 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$(D)\qquad \mathbf{u}(t) = C_1 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$+\ C_2 \exp[(\alpha^2+\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 \exp[(\alpha^2-2\beta^2)t] \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Chapter 6: Scalar Product Spaces. There are 10 problems in Chapter 6.

31. Let the positive constants $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ satisfy the condition $\alpha^2 + \beta^2 = \gamma^2$. Define

$$\mathbf{a} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix}$$

What is the scalar product of the vectors **a** and **b**?

(A) $\mathbf{a} \cdot \mathbf{b} = 0$.

(B) $\mathbf{a} \cdot \mathbf{b} = \alpha^2$.

(C) $\mathbf{a} \cdot \mathbf{b} = \beta^2$.

(D) $\mathbf{a} \cdot \mathbf{b} = \gamma^2$.

32. Let the positive constants $m > 0$, $n > 0$, $\alpha > 0$ and $\beta > 0$. Define the vectors

$$\mathbf{a} = \begin{pmatrix} +\alpha^m \\ +\alpha^m \\ -\alpha^m \\ -\alpha^m \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -\beta^n \\ -\beta^n \\ +\beta^n \\ +\beta^n \end{pmatrix}.$$

What is the angle between the two vectors?
(A) $\theta = 2\pi$. (B) $\theta = \pi$. (C) $\theta = \frac{1}{2}\pi$. (D) $\theta = \frac{1}{4}\pi$.

33. Let the positive integers m and p satisfy $m > p > 1$. Let

$$\mathcal{S} = \left\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_p\right\} \subset \mathbb{R}^m,$$

be a nontrivial orthonormal set. Define the $m \times m$ matrix

$$A = \left(\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \cdots \mathbf{a}_p\right)\left(\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \cdots \mathbf{a}_p\right)^T.$$

Which of the following statements is NOT true?

(A) The rank of A is less than m.

(B) The matrix A is a projection matrix, i.e. $A^2 = A$.

(C) The inverse matrix A^{-1} exists.

(D) There exists a zero eigenvalue: $\lambda = 0$.

34. Let the positive constants $\alpha > 0$ and $\beta > 0$ satisfy the condition $\alpha > 3\beta$. Which of the following sets is NOT an orthogonal basis of the vector space $\mathbb{R}^4$?

$$(A) \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$(B) \quad \left\{ \begin{pmatrix} +2 \\ -2 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ +2 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ +2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ -2 \\ +2 \end{pmatrix} \right\}$$

$$(C) \quad \left\{ \begin{pmatrix} -\alpha \\ +\alpha \\ +\alpha \\ +\alpha \end{pmatrix}, \begin{pmatrix} +\alpha \\ -\alpha \\ +\alpha \\ +\alpha \end{pmatrix}, \begin{pmatrix} +\alpha \\ +\alpha \\ -\alpha \\ +\alpha \end{pmatrix}, \begin{pmatrix} +\alpha \\ +\alpha \\ +\alpha \\ -\alpha \end{pmatrix} \right\}$$

$$(D) \quad \left\{ \begin{pmatrix} \alpha \\ -\beta \\ -\beta \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ -\beta \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

35. Given that

$$\mathcal{B} = \left\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_n\right\}$$

is an orthonormal basis of $\mathbb{R}^n$. Let $\mathbf{a} \in \mathbb{R}^n$. Which of the following is the projection of the vector $\mathbf{a}$ onto the vector space $\mathbb{R}^n$?

$$\begin{array}{ll} (A) & +\mathbf{a} \\ (B) & -\mathbf{a} \\ (C) & +\dfrac{1}{2}\mathbf{a} \\ (D) & -\dfrac{1}{2}\mathbf{a} \end{array}$$

36. Let $\mathcal{B} = \left\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_p\right\}$ be an orthogonal basis of the subspace $\mathcal{E}$ of the vector space $\mathbb{R}^n$. Let the vector $\mathbf{a} \in \mathbb{R}^n$, but $\mathbf{a} \notin \mathcal{E}$. The projection of $\mathbf{a}$ onto $\mathcal{E}$ is given by

$$\mathbf{a}_0 = \frac{\mathbf{a} \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{a} \cdot \mathbf{a}_2}{\mathbf{a}_2 \cdot \mathbf{a}_2}\mathbf{a}_2 + \frac{\mathbf{a} \cdot \mathbf{a}_3}{\mathbf{a}_3 \cdot \mathbf{a}_3}\mathbf{a}_3 + \cdots + \frac{\mathbf{a} \cdot \mathbf{a}_p}{\mathbf{a}_p \cdot \mathbf{a}_p}\mathbf{a}_p$$

Which of the following statements is NOT true?

(A) $\mathbf{a}_0 \in \text{ span } \left\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_p\right\}$

(B) $\mathbf{a} - \mathbf{a}_0 \in \text{ span } \left\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \cdots, \mathbf{a}_p\right\}$

(C) $\mathbf{a}_0 \perp (\mathbf{a} - \mathbf{a}_0)$

(D) $\|\mathbf{a}\|^2 = \|\mathbf{a}_0\|^2 + \|\mathbf{a} - \mathbf{a}_0\|^2$

37. Let the positive constants $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ satisfy the conditions

$$\alpha^2 + \beta^2 + \gamma^2 = 1, \qquad \alpha^2 + \beta^2 = \gamma^2.$$

Consider the system of linear equations

$$\begin{pmatrix} \alpha & \alpha \\ \beta & \beta \\ \gamma & -\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Which of the following is the least square solution of the system?

$$(A) \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(B) \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(C) \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(D) \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

38. Let $\mathbf{x}_0 \in \mathbb{R}^n$ be a least square solution of the system of equations

$$A\mathbf{x} = \mathbf{b}.$$

Let $\mathbf{b}_0$ represent the projection of $\mathbf{b}$ onto the column space of A. Which of the following statements is NOT true?

(A) $A^T A\mathbf{x}_0 = A^T\mathbf{b}$

(B) $\|A\mathbf{x}_0 - \mathbf{b}\| = \min_{\mathbf{x}\in\mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\|$

(C) $A\mathbf{x}_0 = \mathbf{b}_0$

(D) $A\mathbf{x}_0 = \mathbf{b}$

39. Which of the following matrices is NOT a projection matrix?

$$(A) \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(B) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(C) \quad \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$(D) \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

40. Let the positive constants $\alpha > 0$ and $\beta > 0$. Define the real vectors

$$\mathbf{a} = \begin{pmatrix} +\alpha^{10} \\ +\alpha^{10} \\ -\alpha^{10} \\ -\alpha^{10} \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -\beta^{5} \\ -\beta^{5} \\ +\beta^{5} \\ \beta^{5} \end{pmatrix}.$$

Which of the following is the projection of $\mathbf{b}$ onto $\mathbf{a}$?
(A) $-\mathbf{a}$. (B) $+\mathbf{a}$. (C) $-\mathbf{b}$. (D) $+\mathbf{b}$.

Chapter 7: Real Symmetric Matrices. There are 10 problems in Chapter 7.

41. Let A be a real $n \times n$ constant matrix, such that

$$A^T A = AA^T = I.$$

Which of the following statements is NOT true?

$$\begin{array}{ll}
(A) & A\mathbf{x} = \mathbf{x}, \qquad \text{for all } \mathbf{x} \in \mathbb{R}^n \\
(B) & \|A\mathbf{x}\|^2 = \|\mathbf{x}\|^2, \text{ for all } \mathbf{x} \in \mathbb{R}^n \\
(C) & (A\mathbf{x}) \cdot (A\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}, \text{ for all } \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^n \\
(D) & (A\mathbf{x}) \cdot (A\mathbf{y}) = 0 \text{ if } \mathbf{x} \cdot \mathbf{y} = 0, \\
 & \text{for all } \mathbf{x} \in \mathbb{R}^n, \qquad \mathbf{y} \in \mathbb{R}^n
\end{array}$$

42. Let A be a real $n \times n$ constant matrix, not necessarily symmetric. Which of the following statements about the matrix $AA^T - A^TA$ is NOT true?

(A) The determinant $\det(AA^T - A^TA) = 0$.

(B) There exists no zero eigenvalue to $AA^T - A^TA$.

(C) The matrix $AA^T - A^TA$ is symmetric.

(D) The trace of $AA^T - A^TA$ is 0.

43. Let the positive constants $\alpha > 0$ and $\beta > 0$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

There are two eigenvalues $\lambda_1 = \alpha^2 + \beta^2$ and $\lambda_2 = \alpha^2 - 3\beta^2$. Which of the following statements about the eigenspaces is NOT true?

(A) $\dim(\mathbf{E}_{\lambda_1=\alpha^2+\beta^2}) = 3$.

(B) $\dim(\mathbf{E}_{\lambda_2=\alpha^2-3\beta^2}) = 1$.

(C) $\mathbf{E}_{\lambda_1=\alpha^2+\beta^2} \cap \mathbf{E}_{\lambda_2=\alpha^2-3\beta^2} = \emptyset$, the empty set.

(D) $\mathbf{E}_{\lambda_1=\alpha^2+\beta^2} \perp \mathbf{E}_{\lambda_2=\alpha^2-3\beta^2}$.

44. Let the positive constants $\alpha > 0$ and $\beta > 0$ satisfy the condition $\alpha^2 > \beta^2$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

There are three eigenvalues to this matrix: $\lambda_0 = 0$, $\lambda_1 = 2\alpha^2 - 2\beta^2$ and $\lambda_2 = 2\alpha^2 + 2\beta^2$. Which of the following statements about the eigenspaces of A is NOT true?

(A) $\mathbf{E}_{\lambda_1=2\alpha^2-2\beta^2} \cap \mathbf{E}_{\lambda_2=2\alpha^2+2\beta^2} = \emptyset$, the empty set.

(B) $\mathbf{E}_{\lambda_1=2\alpha^2-2\beta^2} \perp \mathbf{E}_{\lambda_2=2\alpha^2+2\beta^2}$.

(C) $\mathbf{E}_{\lambda_0=0} \perp \mathbf{E}_{\lambda_2=2\alpha^2+2\beta^2}$ and $\dim(\mathbf{E}_{\lambda_2=2\alpha^2+2\beta^2}) = 1$.

(D) $\mathbf{E}_{\lambda_0=0} \perp \mathbf{E}_{\lambda_1=2\alpha^2-2\beta^2}$ and $\dim(\mathbf{E}_{\lambda_1=2\alpha^2-2\beta^2}) = 1$.

45. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Which of the following statements is true?

$$(A) \quad T^{-1}AT = \begin{pmatrix} \alpha^2+\beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2+\beta^2 & 0 & 0 \\ 0 & 0 & \alpha^2+\beta^2 & 0 \\ 0 & 0 & 0 & \alpha^2-3\beta^2 \end{pmatrix}$$

$$(B) \quad T^{-1}AT = \begin{pmatrix} \alpha^2-3\beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2+\beta^2 & 0 & 0 \\ 0 & 0 & \alpha^2+\beta^2 & 0 \\ 0 & 0 & 0 & \alpha^2+\beta^2 \end{pmatrix}$$

$$(C) \quad T^{-1}AT = \begin{pmatrix} \alpha^2+\beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2-3\beta^2 & 0 & 0 \\ 0 & 0 & \alpha^2+\beta^2 & 0 \\ 0 & 0 & 0 & \alpha^2+\beta^2 \end{pmatrix}$$

$$(D) \quad T^{-1}AT = \begin{pmatrix} \alpha^2+\beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2+\beta^2 & 0 & 0 \\ 0 & 0 & \alpha^2-3\beta^2 & 0 \\ 0 & 0 & 0 & \alpha^2+\beta^2 \end{pmatrix}$$

46. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Which of the following statements is true?

$$(A) \quad T^{-1}AT = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\alpha^2 + 2\beta^2 & 0 & 0 \\ 0 & 0 & 2\alpha^2 - 2\beta^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(B) \quad T^{-1}AT = \begin{pmatrix} 2\alpha^2 + 2\beta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha^2 - 2\beta^2 \end{pmatrix}$$

$$(C) \quad T^{-1}AT = \begin{pmatrix} 2\alpha^2 - 2\beta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha^2 + 2\beta^2 \end{pmatrix}$$

$$(D) \quad T^{-1}AT = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2\alpha^2 - 2\beta^2 & 0 & 0 \\ 0 & 0 & 2\alpha^2 + 2\beta^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

47. Let $\alpha > 0$ and $\beta > 0$ be positive constants. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

$$B = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

Suppose that the matrix $A + B$ is positively definite. Which of the following statements is true?

(A) $3\alpha^2 + 5\beta^2 > 0$

(B) $3\alpha^2 + 5\beta^2 < 0$

(C) $3\alpha^2 - 5\beta^2 < 0$

(D) $3\alpha^2 - 5\beta^2 > 0$

48. Let the positive constants $\alpha > 0$ and $\beta > 0$. Define

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

$$B = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

Which of the following statements about the matrix $A - B$ is true?

(A) The matrix is indefinite.

(B) The matrix is negatively definite.

(C) The matrix is positively definite.

(D) The matrix is skew-symmetric.

49. Let the positive constants $\alpha > 0$ and $\beta > 0$ satisfy the condition $\alpha^2 < 3\beta^2$. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

$$B = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

Let

$$Q(\mathbf{x}) = \mathbf{x}^T(A+B)\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n.$$

Which of the following statements is true?

(A) $\max_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 3\alpha^2 + 3\beta^2, \qquad \min_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = -\alpha^2 - \beta^2$

(B) $\max_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 3\alpha^2 + 3\beta^2, \qquad \min_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 3\alpha^2 - 5\beta^2$

(C) $\max_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 + 2\beta^2, \qquad \min_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 3\alpha^2 - 5\beta^2$

(D) $\max_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 + 2\beta^2, \qquad \min_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = -\alpha^2 - \beta^2$

50. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

$$B = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

Let

$$Q(\mathbf{x}) = \mathbf{x}^T(A - B)\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n.$$

Which of the following statements is true?

(A) $\max\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 + 2\beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = -2\alpha^2 - 2\beta^2$

(B) $\max\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + \beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 - \beta^2$

(C) $\max\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = \alpha^2 + \beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = -\alpha^2 - \beta^2$

(D) $\max\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 + 2\beta^2, \qquad \min\limits_{\mathbf{x}\in\mathbb{R}^4, \|\mathbf{x}\|=1} Q(\mathbf{x}) = 2\alpha^2 - 2\beta^2$

Bonus Problems: There are 8 problems.

51. Let

$$\mathcal{S} = \left\{f_1, f_2, f_3, \cdots\cdots, f_n\right\} \subset P_n(\mathbb{R}).$$

Suppose that the Wronskian $W[f_1, f_2, f_3, \cdots, f_n] = 1$. Which of the following statements is true?

(A) $\mathcal{S}$ is linearly independent.

(B) $\mathcal{S}$ is a spanning set of $P_n(\mathbb{R})$.

(C) $\mathcal{S}$ is a basis of $P_n(\mathbb{R})$.

(D) $\mathcal{S}$ is linearly dependent.

52. Let A be a real $m \times n$ constant matrix. Suppose that the rank of A is n, where $m > n$. Which of the following statements about the matrix AA^TA is true?

(A) The rank of AA^TA is m.

(B) The rank of AA^TA is n.

(C) The rank of AA^TA is $2m^2$.

(D) The rank of AA^TA is $2n^2$.

53. Let A be a real $n \times n$ constant matrix. Suppose that all eigenvalues are positive, that is,

$$\lambda_1 > 0, \qquad \lambda_2 > 0, \qquad \lambda_3 > 0, \qquad \cdots\cdots, \qquad \lambda_n > 0.$$

Let

$$M = (I + 2A + A^2)^{-1}.$$

Which of the following statements about the matrix M is NOT true?

(A) The rank of M is n.

(B) The determinant $\det(M) > 0$.

(C) The trace of M is $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \cdots + \lambda_n^2$.

(D) The eigenvalues of M are

$$(\lambda_1 + 1)^{-2}, \qquad (\lambda_2 + 1)^{-2}, \qquad (\lambda_3 + 1)^{-2}, \qquad \cdots, \qquad (\lambda_n + 1)^{-2}$$

54. Let A and B be any real $n \times n$ constant matrices. Which of the following statements is NOT true?

(A) The eigenvalues of AB coincide with the eigenvalues of BA.

(B) The determinants $\det(AB) = \det(BA)$.

(C) The trace of AB is equal to the trace of BA.

(D) The eigenvectors of AB coincide with the eigenvectors of BA.

55. Let A be a real $(2n+1) \times (2n+1)$ skew-symmetric matrix, that is, $A^T = -A$, where $n > 1$ is a positive integer. Which of the following statements is NOT true?

- (A) There exists one positive eigenvalue $\lambda > 0$.
- (B) All nonzero eigenvalues are pure imaginary.
- (C) $\mathbf{x}^T A \mathbf{x} = 0,$ for all real vectors $\mathbf{x} \in \mathbb{R}^{2n+1}$
- (D) $\det(A) = 0$

56. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

$$B = \begin{pmatrix} \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \\ \alpha^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & \alpha^2 \end{pmatrix}$$

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Which of the following statements is true?

(A) $T^{-1}(A-B)^{2024}T$

$$= \begin{pmatrix} (\alpha^2-\beta^2)^{2024} & 0 & 0 & 0 \\ 0 & (\alpha^2+\beta^2)^{2024} & 0 & 0 \\ 0 & 0 & (\alpha^2+\beta^2)^{2024} & 0 \\ 0 & 0 & 0 & (\alpha^2-\beta^2)^{2024} \end{pmatrix}$$

(B) $T^{-1}(A-B)^{2024}T$

$$= \begin{pmatrix} (\alpha^2+\beta^2)^{2024} & 0 & 0 & 0 \\ 0 & (\alpha^2+\beta^2)^{2024} & 0 & 0 \\ 0 & 0 & (\alpha^2+\beta^2)^{2024} & 0 \\ 0 & 0 & 0 & (\alpha^2+\beta^2)^{2024} \end{pmatrix}$$

(C) $[T^{-1}(A-B)T]^{2024} = T^{-1}(A+B)^{2024}T$

(D) $[T^{-1}AT - T^{-1}BT]^{2024} = T^{-1}(A+B)^{2024}T$

57. Let A be a real $n \times n$ constant matrix, not necessarily symmetric, with the rank being equal to n. Let

$$B = A^T A.$$

Which of the following statements is NOT true?

(A) The determinant $\det(A^T A) > 0$.

(B) All eigenvalues of $A^T A$ are positive.

(C) There exists a zero eigenvalue $\lambda = 0$ to the matrix $A^T A$.

(D) The matrix $A^T A$ is positively definite.

58. Let A be a real constant matrix, not necessarily symmetric, with the rank being smaller than n. Let

$$B = \varepsilon I + A^T A,$$

where $0 < \varepsilon \ll 1$ is a sufficiently small positive constant. Which of the following statements is NOT true?

(A) The matrix B is positively definite.

(B) All eigenvalues of B are positive.

(C) There exists a sufficiently small positive eigenvalue $\lambda = \varepsilon$ to the matrix B.

(D) There exists a zero eigenvalue $\lambda = 0$ to B.

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31. Let

$$A = 2\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

Which of the following statements is true?

$$\begin{array}{ll} (A) & (AB)^{2024} = 4^{2024} I \\ (B) & (AB)^{2024} = (16)^{2024} I \\ (C) & (AB)^{2024} = (32)^{2024} A \\ (D) & (AB)^{2024} = (64)^{2024} B \end{array}$$

(A) 1.

(B) 2.

(C) 3.

(D) 4.

(A) 0.

(B) 0.

(C) 0.

(D) 0.

Let

$$A = \begin{pmatrix} \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{pmatrix}.$$

Let $\alpha > 0$ and $\beta > 0$ be positive constants. Let

$$A = \begin{pmatrix} 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Which of the following integers is equal to the rank of the matrix A?

0

Mathematics 22 - Calculus Two - 2022

Evaluate the following integrals

$$(1) \qquad \int \arcsin x \mathrm{d}x$$

$$(2) \qquad \int e^{12x} \cos(5x) \mathrm{d}x$$

Evaluate the following integrals

$$(A) \qquad \int (\arcsin x)^2 \mathrm{d}x$$

$$(B) \qquad \int (2x+2) \arctan x \mathrm{d}x$$

$$(C) \qquad \int (\cos x)^4 \mathrm{d}x$$

The integral

$$\int \ln(x + \sqrt{x^2 + \alpha^2}) \mathrm{d}x$$

is equal to which of the following functions?

$$(A) \qquad x \ln(x + \sqrt{x^2+\alpha^2}) + x\sqrt{x^2+\alpha^2} + C.$$

$$(B) \qquad x \ln(x + \sqrt{x^2+\alpha^2}) - x\sqrt{x^2+\alpha^2} + C.$$

$$(C) \qquad x \ln(x + \sqrt{x^2+\alpha^2}) + \sqrt{x^2+\alpha^2} + C.$$

$$(D) \qquad x \ln(x + \sqrt{x^2+\alpha^2}) - \sqrt{x^2+\alpha^2} + C.$$

Determine if the improper integral

$$\int_0^\infty \frac{30}{(x^2+15x+50)(x^2+15x+56)} \mathrm{d}x$$

is convergent. Find the precise value if it is convergent.

Evaluate the following integrals

$$(A) \quad \int \frac{2\tan x(\sec x)^2}{(\tan x)^2[1+(\tan x)^4]}\mathrm{d}x$$

$$(B) \quad \int (\sin x)^6(\cos x)^3\mathrm{d}x$$

$$(C) \quad \int \sqrt{x^2+\alpha^2}\mathrm{d}x$$

Find the volume of the solid generated by rotating about the x-axis the region below the following curve

$$f(x) = \arcsin x, \qquad 0 \le x \le 1.$$

Let $R > 0$ be a positive constant. Find the arclength of the smooth curve

$$f(x) = \frac{1}{2}(e^x + e^{-x}), \qquad -R \le x \le R.$$

Evaluate the following integrals

(1) $\int \frac{2\arctan x}{x^3}\mathrm{d}x.$

(2) $\int \frac{200}{x(100+x^4)}\mathrm{d}x.$

(3) $\int \frac{x^2}{(x^2+\alpha^2)(x^2+\beta^2)}\mathrm{d}x$, where the constants satisfy$\alpha^2 > \beta^2 > 0.$

(4) $\int \frac{1}{x^3\sqrt{x^2-25}}\mathrm{d}x.$

Below Are the Multiple Choice Problems

Chapter 7:

The integral

$$\int \ln x \mathrm{d}x$$

is equal to which of the following functions

$$\begin{array}{ll} (A) & x\ln x + x + C. \\ (B) & x\ln x - x + C. \\ (C) & x^2\ln x + x^2 + C. \\ (D) & x^2\ln x - x^2 + C. \end{array}$$

The integral

$$\int (\arcsin x)^2 \mathrm{d}x$$

is equal to which of the following functions?

(A) $x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x + 2x + C.$

(B) $x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C.$

(C) $x(\arcsin x)^2 - 2\sqrt{1-x^2}\arcsin x + 2x + C.$

(D) $x(\arcsin x)^2 - 2\sqrt{1-x^2}\arcsin x - 2x + C.$

The integral

$$\int (\arccos x)^2 \mathrm{d}x$$

is equal to which of the following functions

(A) $x(\arccos x)^2 + 2\sqrt{1-x^2}\arccos x + 2x + C.$

(B) $x(\arccos x)^2 + 2\sqrt{1-x^2}\arccos x - 2x + C.$

(C) $x(\arccos x)^2 - 2\sqrt{1-x^2}\arccos x + 2x + C.$

(D) $x(\arccos x)^2 - 2\sqrt{1-x^2}\arccos x - 2x + C.$

The integral

$$\int 4(\sin x)^3 \cos x \mathrm{d}x$$

is equal to which of the following functions

$$\begin{array}{ll} (A) & (\sin x)^4 + C. \\ (B) & (\cos x)^4 + C. \\ (C) & 4(\sin x)^4 + C. \\ (D) & 4(\cos x)^4 + C. \end{array}$$

The integral

$$\int 15(\tan x)^3(\sec x)^3 \mathrm{d}x$$

is equal to which of the following functions

$$\begin{array}{ll} (A) & 5(\sec x)^5 + 3(\sec x)^3 + C. \\ (B) & 5(\sec x)^5 - 3(\sec x)^3 + C. \\ (C) & 3(\sec x)^5 + 5(\sec x)^3 + C. \\ (D) & 3(\sec x)^5 - 5(\sec x)^3 + C. \end{array}$$

The integral

$$\int 35(\tan x)^4(\sec x)^4 \mathrm{d}x$$

is equal to which of the following functions

$$\begin{array}{ll}
(A) & 5(\tan x)^5 + 7(\tan x)^7 + C. \\
(B) & 5(\tan x)^5 - 7(\tan x)^7 + C. \\
(C) & 7(\tan x)^5 + 5(\tan x)^7 + C. \\
(D) & 7(\tan x)^5 - 7(\tan x)^7 + C.
\end{array}$$

The integral

$$\int 3[(\tan x)^2 + (\tan x)^4]\mathrm{d}x$$

is equal to which of the following functions

$$\begin{array}{ll} (A) & (\tan x)^3 + C. \\ (B) & 2(\tan x)^3 + C. \\ (C) & 3(\tan x)^3 + C. \\ (D) & 4(\tan x)^3 + C. \end{array}$$

The integral

$$\int 2\sqrt{x^2+\alpha^2}\mathrm{d}x$$

is equal to which of the following functions

$$
\begin{array}{ll}
(A) & x\sqrt{x^2+\alpha^2}+\alpha^2\ln(x+\sqrt{x^2+\alpha^2})+C. \\
(B) & x\sqrt{x^2+\alpha^2}-\alpha^2\ln(x+\sqrt{x^2+\alpha^2})+C. \\
(C) & 2x\sqrt{x^2+\alpha^2}+2\alpha^2\ln(x+\sqrt{x^2+\alpha^2})+C. \\
(D) & 2x\sqrt{x^2+\alpha^2}-2\alpha^2\ln(x+\sqrt{x^2+\alpha^2})+C.
\end{array}
$$

The integral

$$\int 2\sqrt{\alpha^2 - x^2}\mathrm{d}x$$

is equal to which of the following functions

$$(A) \quad x\sqrt{\alpha^2 - x^2} + \alpha^2 \arcsin\left(\frac{x}{\alpha}\right) + C.$$

$$(B) \quad x\sqrt{\alpha^2 - x^2} - \alpha^2 \arcsin\left(\frac{x}{\alpha}\right) + C.$$

$$(C) \quad x\sqrt{\alpha^2 - x^2} + \alpha^2 \arcsin\left(\frac{x}{\alpha}\right) + C.$$

$$(D) \quad x\sqrt{\alpha^2 - x^2} - \alpha^2 \arcsin\left(\frac{x}{\alpha}\right) + C.$$

The integral

$$\int \frac{12}{x(1+x^2)} \mathrm{d}x$$

is equal to which of the following functions

$$(A) \quad -2\ln\left(\frac{x^2}{1+x^2}\right) + C.$$

$$(B) \quad 2\ln\left(\frac{x^2}{1+x^2}\right) + C.$$

$$(C) \quad -6\ln\left(\frac{x^2}{1+x^2}\right) + C.$$

$$(D) \quad 6\ln\left(\frac{x^2}{1+x^2}\right) + C.$$

The integral

$$\int \frac{\alpha\beta(\beta^2-\alpha^2)}{(x^2+\alpha^2)(x^2+\beta^2)}\mathrm{d}x$$

is equal to which of the following functions

$$(A)\quad \arctan\left(\frac{x}{\alpha}\right)+\arctan\left(\frac{x}{\beta}\right)+C.$$

$$(B)\quad \arctan\left(\frac{x}{\alpha}\right)-\arctan\left(\frac{x}{\beta}\right)+C.$$

$$(C)\quad \beta\arctan\left(\frac{x}{\alpha}\right)+\alpha\arctan\left(\frac{x}{\beta}\right)+C.$$

$$(D)\quad \beta\arctan\left(\frac{x}{\alpha}\right)-\alpha\arctan\left(\frac{x}{\beta}\right)+C.$$

Which of the following improper integrals is divergent?

$$(A)\quad \int_0^{\infty} \frac{2\arctan x}{1+x^2}\mathrm{d}x.$$

$$(B)\quad \int_1^{\infty} \frac{\ln x}{x}\mathrm{d}x.$$

$$(C)\quad \int_{-1}^{1} \frac{2\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x.$$

$$(D)\quad \int_{-1}^{1} \frac{2\arccos x}{\sqrt{1-x^2}}\mathrm{d}x.$$

Which of the following improper integrals is not only convergent but also has exact value 1?

$$(A)\quad \int_e^{\infty} \frac{2}{x\ln x[1+(\ln x)^2]}\mathrm{d}x.$$

$$(B)\quad \int_{-\infty}^{\infty} \frac{3x^2}{100+x^6}\mathrm{d}x.$$

$$(C)\quad \int_0^{\infty} 2x^3e^{-x^2}\mathrm{d}x$$

$$(D)\quad \int_0^{1} \ln x\mathrm{d}x.$$

The integral

$$\int 3x(\cos x)^2 \sin x \mathrm{d}x$$

is equal to which of the following functions

$$(A) \quad x(\cos x)^3 + \sin x + \frac{1}{3}(\sin x)^3 + C.$$

$$(B) \quad x(\cos x)^3 + \sin x - \frac{1}{3}(\sin x)^3 + C.$$

$$(C) \quad -x(\cos x)^3 - \sin x - \frac{1}{3}(\sin x)^3 + C.$$

$$(D) \quad -x(\cos x)^3 + \sin x - \frac{1}{3}(\sin x)^3 + C.$$

The integral

$$\int \frac{4x\arctan x}{(1+x^2)^2}\mathrm{d}x$$

is equal to which of the following functions

$$\begin{array}{ll}
(A) & \arctan x + \dfrac{x}{1+x^2} + C. \\
(B) & -\dfrac{2}{1+x^2}\arctan x + \dfrac{x}{1+x^2} + C. \\
(C) & -\dfrac{2}{1+x^2}\arctan x + \arctan x + C. \\
(D) & -\dfrac{2}{1+x^2}\arctan x + \arctan x + \dfrac{x}{1+x^2} + C.
\end{array}$$

The integral

$$\int e^{\alpha x}\cos(\beta x)\mathrm{d}x$$

is equal to which of the following functions

$$(A)\quad e^{\alpha x}[\cos(\beta x)+\sin(\beta x)]+C.$$

$$(B)\quad e^{\alpha x}[\alpha\cos(\beta x)+\beta\sin(\beta x)]+C.$$

$$(C)\quad \frac{1}{\alpha^2+\beta^2}e^{\alpha x}[\cos(\beta x)+\sin(\beta x)]+C.$$

$$(D)\quad \frac{1}{\alpha^2+\beta^2}e^{\alpha x}[\alpha\cos(\beta x)+\beta\sin(\beta x)]+C.$$

Chapter 6:

Chapter 8:

Chapter 9:

Consider the separable differential equation

$$e^{y}\mathrm{d}y = (\arccos x)^{2}\mathrm{d}x.$$

Which of the following functions is equal to the general implicit solution?

$(A)\quad e^{y} = x(\arccos x)^{2} + 2\sqrt{1-x^{2}}\arccos x + 2x + C.$

$(B)\quad e^{y} = x(\arccos x)^{2} + 2\sqrt{1-x^{2}}\arccos x - 2x + C.$

$(C)\quad e^{y} = x(\arccos x)^{2} - 2\sqrt{1-x^{2}}\arccos x + 2x + C.$

$(D)\quad e^{y} = x(\arccos x)^{2} - 2\sqrt{1-x^{2}}\arccos x - 2x + C.$

Consider the separable differential equation

$$-\sin y \mathrm{d}y = \ln(x + \sqrt{x^2 + \alpha^2})\mathrm{d}x.$$

Which of the following is the general implicit solution?

(A) $\cos y = \sqrt{x^2 + \alpha^2} + C.$

(B) $\cos y = x\ln(x + \sqrt{x^2 + \alpha^2}) + C.$

(C) $\cos y = x\ln(x + \sqrt{x^2 + \alpha^2}) + \sqrt{x^2 + \alpha^2}.$

(D) $\cos y = x\ln(x + \sqrt{x^2 + \alpha^2}) - \sqrt{x^2 + \alpha^2} + C.$

Consider the separable differential equation

$$(2y + e^y)\cos(y^2 + e^y)\mathrm{d}y = 2x[\cos(x^2) - \sin(x^2)]e^{\sin(x^2)+\cos(x^2)}\mathrm{d}x.$$

Which of the following functions is equal to the general implicit solution?

$$\begin{aligned}
&(A) \quad \cos(y^2 + e^y) = e^{\sin(x^2)+\cos(x^2)} + C.\\
&(B) \quad \sin(y^2 + e^y) = e^{\sin(x^2)+\cos(x^2)} + C.\\
&(C) \quad \cos(y + e^y) = e^{\sin(x)+\cos(x)} + C.\\
&(D) \quad \sin(y + e^y) = e^{\sin(x)+\cos(x)} + C.
\end{aligned}$$

The function $\mu(x) = e^{x^2}$ is an integrating factor of which of the following differential equations?

$$\begin{array}{ll} (A) & y' + y = 4x^3 e^{-x}. \\ (B) & y' + 2xy = 2xe^{-x^2}. \\ (C) & y' - 2xy = 2xe^{x^2}. \\ (D) & xy' + y = 2x. \end{array}$$

Chapter 10:

Given the parametric curve

$$f(\theta) = \theta - \sin\theta, g(\theta) = 1 - \cos\theta, 0 \le \theta \le 2\pi.$$

Which of the following gives the area of the surface generated by rotating the parametric curve about the x-aixs?

$$(A) \quad \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\mathrm{d}\theta.$$

$$(B) \quad \int_0^{2\pi} 2\pi\sqrt{2 - 2\cos\theta}\mathrm{d}\theta.$$

$$(C) \quad \int_0^{2\pi} 2\pi(\theta - \sin\theta)\sqrt{2 - 2\cos\theta}\mathrm{d}\theta.$$

$$(D) \quad \int_0^{2\pi} 2\pi(1 - \cos\theta)\sqrt{2 - 2\cos\theta}\mathrm{d}\theta.$$

Chapter 11:

Which of the following series is absolutely convergent?

$$(1) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

$$(2) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

$$(3) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

(A)

(B)

(C)

(D)

$$(1) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

$$(2) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

$$(3) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

(A)

(B)

(C)

(D)

(C)

(D)

$$(1) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

$$(2) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

$$(3) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

(A)

(B)

(C)

(D)

$$(1) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

$$(2) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

$$(3) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

(A)

(B)

(C)

(D)

$$(1) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

$$(2) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

$$(3) \quad \sum_{n=1}^{\infty} \frac{n}{n}$$

(A)

(B)

(C)

(D)

(3) $$\sum_{n=1}^{\infty} \frac{n}{n}$$

(A)

(B)

(C)

(D)

(A)

(B)

(C)

(D)

(1) $$\sum_{n=1}^{\infty} \frac{n}{n}$$

(2) $$\sum_{n=1}^{\infty} \frac{n}{n}$$

(3) $$\sum_{n=1}^{\infty} \frac{n}{n}$$

(A)

(B)

(C)

(D)

(A)

(B)

(C)

(D)

The Final Exam - Mathematics 205 - 2020

There are fifty multiple choice problems in the Final Exam. Each problem is worth four points. The total is 200 points. The exam is three hours long. In each problem, there are four choices marked (A), (B), (C) and (D). Only one choice is correct. Choose the one you think is correct. If you do not know the answer to a problem, you may make a reasonable, best possible guess. No supporting work is necessary. No calculators, computers, cell phones, i-pads, i-touches or any other electronic devices are allowed in the exam. You are not allowed to ask for assistance from anybody else. You are not allowed to use any websites for assistance. Any student who cheats will receive an F as the Final Grade. This is Absolutely Firm! Therefore, be honest and solve all problems by yourself. The
First Midterm Exam 100 points
The Second Midterm Exam 100 points
The Final Exam 200 points
The Quizzes 100 points
Total 500 points

1. The reduced row echelon form of the augmented matrix of the system of equations $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

How many free variables are there in the solutions?

(A) 3. (B) 4. (C) 5. (D) 6.

2. The reduced row echelon form of the augmented matrix of the system of equations $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 5 & 6 & 7 \\ 0 & 1 & 1 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

How many solutions are there to the system of equations?

(A) 0. (B) 1. (C) 2. (D) ∞.

3. Let A be an $n \times n$ real matrix with the rank of A being equal to n. Which of the following statements is true?

(A) There exists infinitely many solutions to the system of equations $A\mathbf{x} = \mathbf{b}$, for every vector $\mathbf{b} \in \mathbb{R}^n$.

(B) There exists a unique solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for every vector $\mathbf{b} \in \mathbb{R}^n$.

(C) There exists a solution to the system of equations $A\mathbf{x} = \mathbf{b}$, only for some vector $\mathbf{b} \in \mathbb{R}^n$.

(D) There exists no solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for every vector $\mathbf{b} \in \mathbb{R}^n$.

4. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 4 \\ 3 & 4 & 4 & 5 \end{pmatrix}$. What is the rank of A?

(A) 1. (B) 2. (C) 3. (D) 4.

5. Let A be an $n \times n$ real constant matrix with $\det(A) \neq 0$. Then which of the following statements is NOT equivalent to the others?

(A) The rank of A is equal to n.

(B) The inverse matrix A^{-1} exists.

(C) There exists a unique solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for every vector $\mathbf{b} \in \mathbb{R}^n$.

(D) There exists no solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for some vector $\mathbf{b} \in \mathbb{R}^n$.

6. Let A be an $n \times n$ real constant matrix with $\det(A) = 0$. Then which of the following statements is NOT equivalent to the others?

(A) The rank of A is less than n.

(B) The inverse matrix A^{-1} does not exist.

(C) There exists infinitely many solutions to the system of equations $A\mathbf{x} = \mathbf{0}$.

(D) There exists a solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for every vector $\mathbf{b} \in \mathbb{R}^n$.

7. Let

$$A = p\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} .p = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}.$$

Which of the following is equal to the inverse matrix of A?

$$(A) \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

$$(B) \quad 2\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

$$(C) \quad \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

$$(D) \quad \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

8. Let A and B be $n \times n$ real matrices. Let α be a real nonzero constant. Which of the following is wrong?

$$(A) \quad \det[(A^{-1})^T] = \det(A^{-1}).$$

$$(B) \quad \det(A^{-1}) = \frac{1}{\det(A)}.$$

$$(C) \quad \det(\alpha A) = \alpha^2 \det(A).$$

$$(D) \quad \det(A^2 B^2) = \det(A)\det(B).$$

9. Let $A = (a_{ij})$ be an $n \times n$ real matrix. Let C_{ij} be the cofactor of the element a_{ij}, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$. Which of the following is wrong?

$$(A) \quad \sum_{k=1}^{n} a_{ik}C_{ik} = \det(A), \qquad \text{for all } i = 1, 2, 3, \cdots, n.$$

$$(B) \quad \sum_{k=1}^{n} a_{kj}C_{kj} = \det(A), \qquad \text{for all } j = 1, 2, 3, \cdots, n.$$

$$(C) \quad \sum_{k=1}^{n} a_{ik}C_{jk} = 0, \qquad \text{for all } i \neq j, i = 1, 2, 3, \cdots, n; j = 1, 2, 3, \cdots, n.$$

$$(D) \quad \sum_{k=1}^{n} a_{ki}C_{kj} = 0, \qquad \text{for all } i \neq j, i = 1, 2, 3, \cdots, n; j = 1, 2, 3, \cdots, n.$$

10. Let $A = (a_{ij})$ and $C = (C_{ij})$ be $n \times n$ real matrices, where C_{ij} represents the cofactor of the element a_{ij}, $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$, let $B = C^T$. Which of the following is wrong?

$$(A) \quad A^2B^2 = [\det(A)]^2I.$$

$$(B) \quad A^3B^3 = [\det(A)]^3I.$$

$$(C) \quad A^4B^4 = [\det(A)]^4I.$$

$$(D) \quad AC = [\det(A)]I.$$

11. Let $A = (a_{ij})$ and $C = (C_{ij})$ be $n \times n$ real matrices with $\det(A) \neq 0$, where C_{ij} represents the cofactor of a_{ij}, $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$. Let $B = C^T$. Which of the following is NOT true?

$$(A) \quad \det(B) = [\det(A)]^{n-1}.$$

$$(B) \quad \det(C) = [\det(A)]^{n-1}.$$

$$(C) \quad A^{-1} = \frac{1}{\det(A)} B.$$

$$(D) \quad A^{-1} = \frac{1}{\det(A)} C.$$

12. Let a_i, b_i, c_i be real constants, where $i = 1, 2, 3$. Which of the following computations is NOT correct?

$$(A) \qquad \det \begin{pmatrix} a_1(\lambda+1) & b_1(\lambda+1) & c_1(\lambda+1) \\ a_2(\lambda+2) & b_2(\lambda+2) & c_2(\lambda+2) \\ a_3(\lambda+3) & b_3(\lambda+3) & c_3(\lambda+3) \end{pmatrix}$$

$$= (\lambda+1)(\lambda+2)(\lambda+3) \det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

$$(B) \qquad \det \begin{pmatrix} a_1(\lambda-1) & a_2(\lambda-2) & a_3(\lambda-3) \\ b_1(\lambda-1) & b_2(\lambda-2) & b_3(\lambda-3) \\ c_1(\lambda-1) & c_2(\lambda-2) & c_3(\lambda-3) \end{pmatrix}$$

$$= (\lambda-1)(\lambda-2)(\lambda-3) \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

$$(C) \qquad \det \begin{pmatrix} \lambda-2 & 2 & -2 \\ 2 & \lambda-2 & -2 \\ -2 & -2 & \lambda-2 \end{pmatrix} = \det \begin{pmatrix} \lambda-2 & 2 & -2 \\ 4-\lambda & \lambda-4 & 0 \\ \lambda-4 & 0 & \lambda-4 \end{pmatrix}$$

$$= (\lambda-4)^k \det \begin{pmatrix} \lambda-2 & 2 & -2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

$$(D) \qquad \det \begin{pmatrix} \lambda-10 & 5 & 5 \\ 5 & \lambda-10 & 5 \\ 5 & 5 & \lambda-10 \end{pmatrix} = \det \begin{pmatrix} \lambda-10 & 5 & 5 \\ 15-\lambda & \lambda-15 & 0 \\ 15-\lambda & 0 & \lambda-15 \end{pmatrix}$$

$$= (\lambda-15)^k \det \begin{pmatrix} \lambda-10 & 5 & 5 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

13. Which of the following computations is NOT correct?

$$(A) \qquad \det \begin{pmatrix} 0 & 0 & \lambda-1 & 0 \\ 0 & \lambda-2 & 0 & 0 \\ 0 & 0 & 0 & \lambda-3 \\ \lambda-4 & 0 & 0 & 0 \end{pmatrix}$$
$$= (\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4).$$

$$(B) \qquad \det \begin{pmatrix} 0 & (\lambda-1)^2 & 0 & 0 \\ 0 & 0 & (\lambda-2)^2 & 0 \\ (\lambda-3)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\lambda-4)^2 \end{pmatrix}$$
$$= (\lambda-1)^2(\lambda-2)^2(\lambda-3)^2(\lambda-4)^2.$$

$$(C) \qquad \det \begin{pmatrix} 0 & 0 & 0 & (\lambda-1)^3 \\ 0 & 0 & (\lambda-2)^3 & 0 \\ 0 & (\lambda-3)^3 & 0 & 0 \\ (\lambda-4)^3 & 0 & 0 & 0 \end{pmatrix}$$
$$= (\lambda-1)^3(\lambda-2)^3(\lambda-3)^3(\lambda-4)^3.$$

$$(D) \qquad \det \begin{pmatrix} 0 & 0 & (\lambda-1)^4 & 0 \\ 0 & 0 & 0 & (\lambda-2)^4 \\ (\lambda-3)^4 & 0 & 0 & 0 \\ 0 & (\lambda-4)^4 & 0 & 0 \end{pmatrix}$$
$$= -(\lambda-1)^4(\lambda-2)^4(\lambda-3)^4(\lambda-4)^4.$$

$(A) \quad ()$

$(B) \quad ()$

$(C) \quad ()$

$(D) \quad ()$

14. Which of the following subsets is NOT a subspace of the vector space $M_n(\mathbb{R})$?

(A) $\{A \in M_n(\mathbb{R}):$ The transposed matrix $A^T = A\}$.

(B) $\{A \in M_n(\mathbb{R}):$ The transposed matrix $A^T = -A\}$.

(C) $\{A \in M_n(\mathbb{R}) : A^2 = A\}$.

(D) $\{A \in M_n(A) : \det(A^T) = \det(A)\}$.

15. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9+k \end{pmatrix} \right\}.$$

Which of the following statements about $\mathcal{B}$ is true?

(A) $\mathcal{B}$ is linearly dependent.

(B) $\mathcal{B}$ is linearly independent.

(C) $\mathcal{B}$ is a spanning set of $\mathbb{R}^3$.

(D) $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

16. Let

$$\mathcal{B} = \{v_1, v_2, v_3, \cdots, v_n\}$$

be a set of vectors in $\mathbb{R}^n$. Let $A = (v_1\ v_2\ v_3\ \cdots\ v_n)$ be the matrix made by using all vectors in $\mathcal{B}$. Given that $\det(A) \neq 0$. Which of the following statements is NOT equivalent to the others?

(A) $\mathcal{B}$ is a spanning set of $\mathbb{R}^n$.

(B) $\mathcal{B}$ is linearly independent.

(C) $\mathcal{B}$ is a basis of $\mathbb{R}^n$.

(D) $\mathcal{B}$ is linearly independent but it is not a spanning set of $\mathbb{R}^n$.

17. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} \alpha \\ -\beta \\ -\beta \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \begin{pmatrix} -\beta \\ -\beta \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

be a basis of $\mathbb{R}^4$. Which of the following conditions must be true?

$$(A) \quad \alpha + \beta = 0, \qquad \alpha - 3\beta \neq 0.$$

$$(B) \quad \alpha + \beta \neq 0, \qquad \alpha - 3\beta = 0.$$

$$(C) \quad \alpha + \beta = 0, \qquad \alpha - 3\beta = 0.$$

$$(D) \quad \alpha + \beta \neq 0, \qquad \alpha - 3\beta \neq 0.$$

18. Which of the following statements is NOT true?

(A) e^x, e^{2x}, e^{3x} and e^{4x} are linearly independent.

(B) $\cos x$, $\sin x$, $\cos(2x)$, $\sin(2x)$ are linearly independent.

(C) 1, x, x^2, x^3, $\cdots$, x^{10} are linearly independent.

(D) $\cos(2x)$, $(\cos x)^2$, $(\sin x)^2$ are linearly independent.

19. Which of the following statements about the vector space $C(\mathbb{R})$ is NOT true?

(A) $C(\mathbb{R})$ is an infinite-dimensional vector space.

(B) $\mathcal{B} = \{1, x, x^2, x^3, \cdots\cdots\}$ is a basis of $C(\mathbb{R})$.

(C) $\mathcal{C} = \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \cdots\cdots\}$ is a basis of $C(\mathbb{R})$.

(D) $C(\mathbb{R})$ is a finite-dimensional vector space.

20. Given the following basis of $\mathbb{R}^4$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}.$$

What is the component vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ relative to the basis $\mathcal{B}$ equal to?

$$(A) \quad [\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

$$(B) \quad [\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

$$(C) \quad [\mathbf{v}]_{\mathcal{B}} = -\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

$$(D) \quad [\mathbf{v}]_{\mathcal{B}} = -\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

21. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\},$$

and

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

be two bases of the vector space $\mathbb{R}^4$. Which of the following products gives the change-of-basis-matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$?

$$(A)\quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$(B)\quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}^{-1},$$

$$(C)\quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix},$$

$$(D)\quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^{-1}.$$

22. Let

$$A = \begin{pmatrix} 1 & -4 & -9 \\ -3 & 12 & 27 \\ -5 & 20 & 45 \end{pmatrix}.$$

Which of the following is a basis of the nullspace of A?

$$(A) \quad \left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$$(B) \quad \left\{ \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ 0 \end{pmatrix} \right\}.$$

$$(C) \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 9 \end{pmatrix} \right\}.$$

$$(D) \quad \left\{ \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

23. Let A be a real $m \times n$ matrix with $m = 9$ and $n = 10$. Given that the rank of A is $k = 4$. What is the dimension of the nullspace of A?

(A) 3. (B) 4. (C) 5. (D) 6.

24. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation, given by

$$\begin{aligned} & T(a + bx + cx^2 + dx^3) \\ = \ & 4[(2a - b) + (4c - d)] + 4[(2a - b) - (4c - d)]x \\ + \ & [(2a - b) + (4c - d)]x^2 + [(2a - b) - (4c - d)]x^3. \end{aligned}$$

Which of the following are the basis of the kernel and the basis of the range of the linear transformation T, respectively?

(A) $\{1 + x, x^2 + 4x^3\}$, $\{1 + x^2, x + x^3\}$,

(B) $\{1 + 2x, x^2 + 4x^3\}$, $\{4 + x^2, 4x + x^3\}$,

(C) $\{1 + 3x, x^2 + 4x^3\}$, $\{2 + x^2, 2x + x^3\}$,

(D) $\{1 + 4x, x^2 + 4x^3\}$, $\{3 + x^2, 3x + x^3\}$.

25. Let $T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$ be a linear transformation, given by $T(A) = A - A^T$. Which of the following is a basis of the range of the linear transformation T?

$$(A) \quad \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\},$$

$$(B) \quad \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\},$$

$$(C) \quad \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\},$$

$$(D) \quad \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

26. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, given by $T(\mathbf{x}) = A\mathbf{x}$, where A is a real $m \times n$ matrix, $m > 2$ and $n > 2$ are positive integers. Given that the rank of A is less than both m and n. Which of the following statements is true?

(A) The linear transformation T is one-to-one.

(B) The linear transformation T is onto.

(C) The inverse linear transformation T^{-1} exists.

(D) The linear transformation T is neither one-to-one nor onto. The inverse linear transformation T^{-1} does not exist.

27. Which of the following linear transformations is NOT an isomorphism?

(A) $T : \mathbb{R}^3 \to P_2(\mathbb{R})$,

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + bx + cx^2,$$

(B) $T : \mathbb{R}^4 \to M_2(\mathbb{R})$,

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

(C) $T : M_2(\mathbb{R}) \to P_3(\mathbb{R})$,

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + bx + cx^2 + dx^3,$$

(D) $T : \mathbb{R}^3 \to \mathbb{R}^3$,

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & -9 & 9 \\ 9 & 0 & -9 \\ -9 & 9 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

28. Let $T : V \to W$ be a linear transformation. Given that

$$\dim[\text{ Ker }(T)] = 5, \qquad \dim[\text{ Rng }(T)] = k = 5.$$

What is the dimension of the space V?

(A) 7. (B) 8. (C) 9. (D) 10.

29. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, give by

$$T(\mathbf{x}) = A\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where A is an $n \times n$ real matrix and $\det(A) \neq 0$. Which of the following statements about the linear transformation T is NOT true?

(A) The linear transformation T is one-to-one.

(B) The linear transformation T is onto.

(C) The inverse linear transformation T^{-1} exists.

(D) The inverse linear transformation T^{-1} does not exist.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

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$$\{\}\,\{\}$$

30. Let A be a 4×4 real matrix. Given two complex eigenvalues $\lambda_1 = 10\mathrm{i}$ and $\lambda_2 = 8 + 5\mathrm{i}$ of the matrix A. What are the other complex eigenvalues of A?

(A) $\lambda_3 = 10\mathrm{i}$, $\lambda_4 = 8 + 5\mathrm{i}$,

(B) $\lambda_3 = 10\mathrm{i}$, $\lambda_4 = 8 - 5\mathrm{i}$,

(C) $\lambda_3 = -10\mathrm{i}$, $\lambda_4 = 8 + 5\mathrm{i}$,

(D) $\lambda_3 = -10\mathrm{i}$, $\lambda_4 = 8 - 5\mathrm{i}$.

31. Let $A = \begin{pmatrix} \alpha^3 & -\beta^3 & -\beta^3 \\ -\beta^3 & \alpha^3 & -\beta^3 \\ -\beta^3 & -\beta^3 & \alpha^3 \end{pmatrix}$, where α and β are positive constants. Which of the following statements is true for the eigenvalues and their algebraic multiplicities?

(A) The first eigenvalue is $\lambda_1 = \alpha^3 + \beta^3$ and the algebraic multiplicity is 1; the second eigenvalue is $\lambda_2 = \alpha^3 - 2\beta^3$ and the algebraic multiplicity is 1.

(B) The first eigenvalue is $\lambda_1 = \alpha^3 + \beta^3$ and the algebraic multiplicity is 1; the second eigenvalue is $\lambda_2 = \alpha^3 - 2\beta^3$ and the algebraic multiplicity is 2.

(C) The first eigenvalue is $\lambda_1 = \alpha^3 + \beta^3$ and the algebraic multiplicity is 2; the second eigenvalue is $\lambda_2 = \alpha^3 - 2\beta^3$ and the algebraic multiplicity is 1.

(D) The first eigenvalue is $\lambda_1 = \alpha^3 + \beta^3$ and the algebraic multiplicity is 2; the second eigenvalue is $\lambda_2 = \alpha^3 - 2\beta^3$ and the algebraic multiplicity is 2.

32. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix},$$

where α and β are positive constants. There are two real eigenvalues $\lambda_1 = \alpha^2 + \beta^2$ and $\lambda_2 = \alpha^2 - 3\beta^2$ to the matrix A. Which of the following statements is true?

(A) Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = \alpha^2 + \beta^2$ are equal to one.

(B) Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = \alpha^2 + \beta^2$ are equal to two.

(C) Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = \alpha^2 + \beta^2$ are equal to three.

(D) Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_2 = \alpha^2 - 3\beta^2$ are equal to three.

33. Let A, B and T be $n \times n$ real matrices, such that $B = T^{-1}AT$. Also let A and B be diagonalizable matrices. Let $T \neq I$, where I represents the identity matrix. Which of the following statements about A and B is correct?

(A) The matrices have the same eigenvalues, but the eigenvectors may be different.

(B) The matrices have the same eigenvectors, but the eigenvalues may be different.

(C) The matrices have the same eigenvalues and the same eigenvectors.

(D) The matrices have different eigenvalues and different eigenvectors.

34. Let A be an $n \times n$ real matrix. Then A is diagonalizable if which of the following conditions is true?

(A) The algebraic multiplicity is larger than the geometric multiplicity, for each fixed eigenvalue of A.

(B) The algebraic multiplicity is less than the geometric multiplicity, for each fixed eigenvalue of A.

(C) The algebraic multiplicity is equal to the geometric multiplicity, for some eigenvalue of A.

(D) The algebraic multiplicity is equal to the geometric multiplicity, for each fixed eigenvalue of A.

35. Let A be a 4×4 real matrix. Let the eigenvalues λ_i and the corresponding eigenvectors ξ_i satisfy the equations

$$\begin{aligned} A\xi_1 &= \lambda_1\xi_1, \\ A\xi_2 &= \lambda_2\xi_2, \\ A\xi_3 &= \lambda_3\xi_3, \\ A\xi_4 &= \lambda_4\xi_4. \end{aligned}$$

To diagonalize the matrix A, a student uses the eigenvectors to define the matrix $T = (\xi_4\ \xi_3\ \xi_2\ \xi_1)$. Then what is the product $T^{-1}AT$ equal to?

$$(A) \quad \begin{pmatrix} \lambda_4 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}.$$

$$(B) \quad \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$

$$(C) \quad \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}.$$

$$(D) \quad \begin{pmatrix} \lambda_4 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}.$$

36. Let A be an $n \times n$ real matrix, let λ and ξ be an eigenvalue and an eigenvector, respectively, that is, $A\xi = \lambda\xi$. Which of the following equations is wrong?

$$(A) \quad A^{20}\xi = \lambda^{20}\xi,$$

$$(B) \quad A^{300}\xi = \lambda^{300}\xi,$$

$$(C) \quad A^{4000}\xi = \lambda^{4000}\xi,$$

$$(D) \quad A^{50000}\xi = \lambda^{50000}\xi^{50000}.$$

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37. Consider the differential equation

$$y'' + y = 2x + 4.$$

Which of the following is the general solution of the differential equation?

$(A)\quad y(x) = C_1 \cos x + C_2 \sin x + 2x.$

$(B)\quad y(x) = C_1 \cos x + C_1 \sin x + 4.$

$(C)\quad y(x) = C_1 \cos x + C_2 \sin x + 2x + 4.$

$(D)\quad y(x) = C_1 \cos x + C_2 \sin x - 2x - 4.$

38. The auxiliary equation of the differential equation

$$y'''' - 32y''' + 384y'' - 2048y' + 4096y = 0$$

is

$$(\lambda - 8)^4 = 0.$$

Which of the following is the solution of the differential equation?

$$(A) \quad y(x) = C_1e^{4x} + C_2xe^{4x} + C_3x^2e^{4x} + C_4x^3e^{4x}.$$

$$(B) \quad y(x) = C_1e^{8x} + C_2xe^{8x} + C_3x^2e^{8x} + C_4x^3e^{8x}.$$

$$(C) \quad y(x) = C_1e^{5x} + C_2e^{6x} + C_3e^{7x} + C_4e^{8x}.$$

$$(D) \quad y(x) = C_1e^{3x} + C_2e^{4x} + C_3e^{5x} + C_4e^{6x}.$$

39. Consider the differential equation

$$y'' + 100y = 36c[\cos(8x) + \sin(8x)].$$
$$c = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{9}. 36c = 4, 9, 12, 18.$$

Which of the following is a particular solution of the differential equation?

$$(A) \quad y_p(x) = \frac{1}{2}[\cos(8x) + \sin(8x)].$$

$$(B) \quad y_p(x) = \frac{1}{3}[\cos(8x) + \sin(8x)].$$

$$(C) \quad y_p(x) = \frac{1}{4}[\cos(8x) + \sin(8x)].$$

$$(D) \quad y_p(x) = \frac{1}{9}[\cos(8x) + \sin(8x)].$$

40. Consider the differential equation

$$y'' - 15y' + 56y = 6ce^{5x}.$$
$$c = \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1.6c = 1, 2, 3, 6.$$

Which of the following is a particular solution of the differential equation?

$$(A) \quad y_p(x) = \frac{1}{6}e^{5x}.$$

$$(B) \quad y_p(x) = \frac{1}{3}e^{5x}.$$

$$(C) \quad y_p(x) = \frac{1}{2}e^{5x}.$$

$$(D) \quad y_p(x) = e^{5x}.$$

41. Consider the differential equation

$$y'' + 25y = 10c[\cos(5x) - \sin(5x)].$$
$$c = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{50}. 10c = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}.$$

Which of the following is a particular solution of the differential equation?

$(A)\quad y_p(x) = \frac{x}{10}[\cos(5x) + \sin(5x)].$

$(B)\quad y_p(x) = \frac{x}{20}[\cos(5x) + \sin(5x)].$

$(C)\quad y_p(x) = \frac{x}{40}[\cos(5x) + \sin(5x)].$

$(D)\quad y_p(x) = \frac{x}{50}[\cos(5x) + \sin(5x)].$

42. Consider the differential equation

$$y'' - 15y' + 56y = 2c(1+x)e^{8x}.$$
$$c = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{32}.2c = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{16}.$$

Which of the following is the general solution of the differential equation?

$$(A) \quad y(x) = C_1 e^{7x} + C_2 e^{8x} + \frac{1}{2} x^2 e^{8x}.$$

$$(B) \quad y(x) = C_1 e^{7x} + C_2 e^{8x} + \frac{1}{4} x^2 e^{8x}.$$

$$(C) \quad y(x) = C_1 e^{7x} + C_2 e^{8x} + \frac{1}{8} x^2 e^{8x}.$$

$$(D) \quad y(x) = C_1 e^{7x} + C_2 e^{8x} + \frac{1}{32} x^2 e^{8x}.$$

43. Consider the differential equation

$$y'' - 20y' + 100y = 6cxe^{10x}.$$
$$c = \frac{1}{6}, \frac{1}{12}, \frac{1}{36}, \frac{1}{60}.6c = 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{10}.$$

Which of the following is a particular solution of the differential equation?

$$(A) \quad y_p(x) = \frac{1}{6}x^3e^{10x}.$$

$$(B) \quad y_p(x) = \frac{1}{12}x^3e^{10x}.$$

$$(C) \quad y_p(x) = \frac{1}{36}x^3e^{10x}.$$

$$(D) \quad y_p(x) = \frac{1}{60}x^3e^{10x}.$$

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44. Let t be a real variable. The matrix

$$\begin{pmatrix} 10e^{3t} & -5e^{6t} & -5e^{9t} \\ -5e^{3t} & 10e^{6t} & -5e^{9t} \\ -5e^{3t} & -5e^{6t} & 10e^{9t} \end{pmatrix}$$

is equal to which of the following products?

$$(A) \quad \begin{pmatrix} 10 & -5 & -5 \\ -5 & 10 & -5 \\ -5 & -5 & 10 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{6t} & 0 \\ 0 & 0 & e^{9t} \end{pmatrix}.$$

$$(B) \quad \begin{pmatrix} 10 & -5 & -5 \\ -5 & 10 & -5 \\ -5 & -5 & 10 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{-6t} & 0 \\ 0 & 0 & e^{-9t} \end{pmatrix}.$$

$$(C) \quad \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{6t} & 0 \\ 0 & 0 & e^{9t} \end{pmatrix} \begin{pmatrix} 10 & -5 & -5 \\ -5 & 10 & -5 \\ -5 & -5 & 10 \end{pmatrix}.$$

$$(D) \quad \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{-6t} & 0 \\ 0 & 0 & e^{-9t} \end{pmatrix} \begin{pmatrix} 10 & -5 & -5 \\ -5 & 10 & -5 \\ -5 & -5 & 10 \end{pmatrix}.$$

45. Let t be a real variable. The inverse matrix of the product

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 & 0 & 0 \\ 0 & e^{-4t} & 0 & 0 \\ 0 & 0 & e^{-6t} & 0 \\ 0 & 0 & 0 & e^{-8t} \end{pmatrix}$$

is equal to which of the following products?

$$(A) \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{4t} & 0 & 0 \\ 0 & 0 & e^{6t} & 0 \\ 0 & 0 & 0 & e^{8t} \end{pmatrix}.$$

$$(B) \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{4t} & 0 & 0 \\ 0 & 0 & e^{6t} & 0 \\ 0 & 0 & 0 & e^{8t} \end{pmatrix}.$$

$$(C) \quad \begin{pmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{4t} & 0 & 0 \\ 0 & 0 & e^{6t} & 0 \\ 0 & 0 & 0 & e^{8t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

$$(D) \quad \begin{pmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^{4t} & 0 & 0 \\ 0 & 0 & e^{6t} & 0 \\ 0 & 0 & 0 & e^{8t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}^{-1}.$$

46. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} \alpha^2 & \beta^2 \\ -\beta^2 & \alpha^2 \end{pmatrix} \mathbf{x},$$

where α and β are positive constants. Which of the following is the general solution of the system of differential equations?

(*A*) $\mathbf{x}(t) = C_1 e^{\alpha^2 t} \begin{pmatrix} \cos(\beta^2 t) \\ \sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t} \begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}.$

(*B*) $\mathbf{x}(t) = C_1 e^{\alpha^2 t} \begin{pmatrix} \cos(\beta^2 t) \\ -\sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t} \begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}.$

(*C*) $\mathbf{x}(t) = C_1 e^{\alpha^2 t} \begin{pmatrix} \cos(\beta^2 t) \\ \sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t} \begin{pmatrix} -\sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}.$

(*D*) $\mathbf{x}(t) = C_1 e^{\alpha^2 t} \begin{pmatrix} \cos(\beta^2 t) \\ \sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t} \begin{pmatrix} \sin(\beta^2 t) \\ -\cos(\beta^2 t) \end{pmatrix}.$

47. Consider the homogeneous system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix} \mathbf{x},$$

where α and β are positive constants. Which of the following is the general solution of the system?

$$(A) \quad \mathbf{x}(t) = C_1 e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3 e^{(\alpha^2-2\beta^2)t} \left(\begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \right.$$

$$(B) \quad \mathbf{x}(t) = C_1 e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 t e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3 e^{(\alpha^2-2\beta^2)t} \left(\begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \right.$$

$$(C) \quad \mathbf{x}(t) = C_1 e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 e^{(\alpha^2-2\beta^2)t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3 e^{(\alpha^2-2\beta^2)t} \left(\begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \right.$$

$$(D) \quad \mathbf{x}(t) = C_1 e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 t e^{(\alpha^2-2\beta^2)t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3 e^{(\alpha^2-2\beta^2)t} \left(\vphantom{\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}} \right.$$

48. Consider the nonhomogeneous linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{x} - \alpha^2 \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

Which of the following is a particular solution of the nonhomogeneous linear system of differential equations?

$$(A) \quad \mathbf{x}_p(t) = \alpha \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$(B) \quad \mathbf{x}_p(t) = \alpha \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

$$(C) \quad \mathbf{x}_p(t) = \alpha^2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$(D) \quad \mathbf{x}_p(t) = \alpha^2 \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

49. Consider the nonhomogeneous linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} \alpha & -\alpha & \alpha \\ -\alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \end{pmatrix} \mathbf{x} - \alpha^{124} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} e^{\alpha t},$$

where $\alpha > 0$ is a positive constant. Which of the following is a particular solution of the system?

$$(A) \quad \mathbf{x}_p(t) = \alpha^{123} e^{\alpha t} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$(B) \quad \mathbf{x}_p(t) = \alpha^{123} e^{\alpha t} \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

$$(C) \quad \mathbf{x}_p(t) = e^{\alpha t} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$(D) \quad \mathbf{x}_p(t) = e^{\alpha t} \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

50. Consider the nonhomogeneous linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \begin{pmatrix} \alpha^2 & \beta^2 \\ -\beta^2 & \alpha^2 \end{pmatrix}\mathbf{x}$$

$$+\ 8t^7 e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ -\sin(\beta^2 t) \end{pmatrix} + 10t^9 e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix},$$

where α and β are positive constants. Which of the following is the general solution of the nonhomogeneous linear system of differential equations?

(A) $$\mathbf{x}(t) = C_1 e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ -\sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}$$
$$+\ t^8 e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ -\sin(\beta^2 t) \end{pmatrix} + t^{10} e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}.$$

(B) $$\mathbf{x}(t) = C_1 e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ -\sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}$$
$$+\ 8t^7 e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ -\sin(\beta^2 t) \end{pmatrix} + 10t^9 e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}.$$

(C) $$\mathbf{x}(t) = C_1 e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ -\sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}$$
$$+\ t^{10} e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ -\sin(\beta^2 t) \end{pmatrix} + t^8 e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}.$$

(D) $$\mathbf{x}(t) = C_1 e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ -\sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}$$
$$+\ e^{8t} e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ -\sin(\beta^2 t) \end{pmatrix} + e^{10t} e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}.$$

Mathematics 43 Homework Assignments
Autumn Semester, 2023
Mathematics 205 - Linear Methods
Summer 2024 - Everyday Quizzes

Homework Assignment 1Chapter 2 - Everyday Quizzes

1. Let

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Compute (1) A^2. (2) B^2. (3) AB. (4) BA.

2. Let $\alpha \neq 0$ be a real nonzero constant. Define the matrices

$$A = \begin{pmatrix} \alpha & \alpha & \alpha & \alpha \\ \alpha & \alpha & -\alpha & -\alpha \\ \alpha & -\alpha & -\alpha & \alpha \\ \alpha & -\alpha & \alpha & -\alpha \end{pmatrix},$$

$$B = \frac{1}{4\alpha^2}\begin{pmatrix} \alpha & \alpha & \alpha & \alpha \\ \alpha & \alpha & -\alpha & -\alpha \\ \alpha & -\alpha & -\alpha & \alpha \\ \alpha & -\alpha & \alpha & -\alpha \end{pmatrix}.$$

(1) Compute A^2. (2) Compute B^2. (3) Compute AB. (4) Compute BA.

3. Let $\beta \neq 0$ be a real nonzero constant. Define the matrices

$$A = \begin{pmatrix} \beta & \beta & \beta & \beta \\ \beta & -\beta & \beta & -\beta \\ \beta & \beta & -\beta & -\beta \\ \beta & -\beta & -\beta & \beta \end{pmatrix},$$

$$B = \frac{1}{4\beta^2} \begin{pmatrix} \beta & \beta & \beta & \beta \\ \beta & -\beta & \beta & -\beta \\ \beta & \beta & -\beta & -\beta \\ \beta & -\beta & -\beta & \beta \end{pmatrix}.$$

Compute (1) A^2. (2) B^2. (3) AB. (4) BA.

4. Let

$$A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & -3 & 4 \\ 3 & 2 & -5 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 & 2 & 3 \\ 3 & -1 & 4 & 2 \\ 7 & -1 & 8 & 0 \end{pmatrix}.$$

Find the reduced row echelon form and the rank of each matrix.

5. Solve the following systems of equations

$$(1) \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$$(2) \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 0 \\ -2 \end{pmatrix}.$$

6. Let

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

(1) Find the inverse matrix A^{-1}.
(2) Solve the system of linear equations $A\mathbf{x} = \mathbf{b}$.

7. Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

(1) Find the inverse matrix A^{-1}.
(2) Solve the system of linear equations $A\mathbf{x} = \mathbf{b}$.

8. Let

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 5 \\ 0 \\ -1 \\ -2 \end{pmatrix}.$$

(1) Compute A^2.
(2) Find the inverse matrix A^{-1}.
(3) Solve the system of linear equations $A\mathbf{x} = \mathbf{b}$.

9. Let λ_1, λ_2, λ_3, λ_4 and λ_5 be real nonzero constants. Define

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix}.$$

(1) Find the inverse matrix A^{-1}.
(2) Solve the system of linear equations $A\mathbf{x} = \mathbf{b}$.

In the next two problems, let A be an $n \times n$ real constant matrix.

10. Which of the following statements is not equivalent to the others?

(A) The rank of the matrix A is equal to n.
(B) The inverse matrix A^{-1} exists.
(C) There exists a unique solution to the system of equations $A\mathbf{x} = \mathbf{b}$, for each fixed vector $\mathbf{b} \in \mathbb{R}^n$.
(D) There exist infinitely many solutions to the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$.

11. Which of the following statements is not equivalent to other statements?

(A) The rank of the matrix A is less than n.
(B) The inverse matrix A^{-1} does not exist.
(C) There exist infinitely many solutions to the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$.
(D) There exists a unique solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$, for each fixed $\mathbf{b} \in \mathbb{R}^n$.

The Final Answers to Homework Problems

Chapter 1

1.

$$(1) \qquad A^2 = I$$

$$(2) \qquad B^2 = I$$

$$(3) \qquad AB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$(4) \qquad BA = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

2.

$$(1) \qquad A^2 = 4\alpha^2 I = 4\alpha^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(2) \qquad B^2 = \frac{1}{4\alpha^2} I = \frac{1}{4\alpha^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(3) \qquad AB = I$$

$$(4) \qquad BA = I$$

3.

$$(1) \qquad A^2 = 4\beta^2 I = 4\beta^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(2) \qquad B^2 = \frac{1}{4\beta^2} I = \frac{1}{4\beta^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(3) \qquad AB = I$$

$$(4) \qquad BA = I$$

4. The reduced echelon forms are, respectively

$$A \to \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B \to \begin{pmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 1 & -1 & -7/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

5. There exists a unique solution to (1):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

There exists a unique solution to (2):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

6.

$$A^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}, \qquad \mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

7.

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

8.

$$A^2 = I, A^{-1} = A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix},$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

9.

$$A^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\lambda_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\lambda_4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\lambda_5} \end{pmatrix}, \mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

10. (D)

11. (D)

$$\lambda \quad \{\}\,\{\}\,\{\}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$1$$

$$ccccccccc$$

$$0\mathbf{v} =, , ., , \mathbf{v} =, \mathbf{v} =, , , \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \mathbf{v} =,$$

$$\left(\quad\right), \left(\quad\right)\left(\quad\right)\mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \{\}$$

$$\left(\qquad \right), \left(\qquad \right) \left(\qquad \right) \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \{\} \, \alpha\alpha\alpha\alpha\beta\beta\beta\beta$$

$$\left(\qquad \right), \left(\qquad \right) \left(\qquad \right) \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \{\} \, \alpha\alpha\alpha\alpha\beta\beta\beta\beta$$

$$\left(\qquad \right), \left(\qquad \right) \left(\qquad \right) \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \{\} \, \alpha\alpha\alpha\alpha\beta\beta\beta\beta$$

$$\left(\qquad \right), \left(\qquad \right) \left(\qquad \right) \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \{\} \, \alpha\alpha\alpha\alpha\beta\beta\beta\beta$$

$$\left(\qquad \right), \left(\qquad \right) \left(\qquad \right) \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \{\} \, \alpha\alpha\alpha\alpha\beta\beta\beta\beta$$

$$\left(\qquad \right), \left(\qquad \right) \left(\qquad \right) \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \{\} \, \alpha\alpha\alpha\alpha\beta\beta\beta\beta$$

Chapter 2 - Determinants - Everyday Quizzes

Let $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$ be real nonzero constants.

1. Let

$$A = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Use properties to compute the determinant of A.

2. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{pmatrix}.$$

Use properties to compute the determinant of A.

3. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

Use properties to compute the determinant of A.

4. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}.$$

Use properties to compute the determinant of A.

5. Let

$$A = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Use properties to compute the determinant of $\lambda I - A$.

6. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{pmatrix}.$$

Use properties to compute the determinant of $\lambda I - A$.

7. Let

$$A = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix}.$$

Use properties to compute the determinant of $\lambda I - A$.

8. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}.$$

Use properties to compute the determinant of $\lambda I - A$.

9. Let $\alpha + \beta \neq 0$ and $\alpha - 2\beta \neq 0$. Use properties of determinant and adjoint matrix to find the inverse matrix of $\lambda I - A$, where

$$A = \begin{pmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{pmatrix}.$$

10. Let $\alpha+\beta \neq 0$ and $\alpha-3\beta \neq 0$. Use properties of determinant and adjoint matrix to find the inverse matrix of $\lambda I - A$, where

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}.$$

11. Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants. Solve the following equations for solutions (for λ).

$$(1) \quad \det \begin{pmatrix} \lambda-2 & 2 & -2 \\ 2 & \lambda-2 & -2 \\ -2 & -2 & \lambda-2 \end{pmatrix} = 0.$$

$$(2) \quad \det \begin{pmatrix} \lambda-10 & 5 & 5 \\ 5 & \lambda-10 & 5 \\ 5 & 5 & \lambda-10 \end{pmatrix} = 0.$$

$$(3) \quad \det \begin{pmatrix} \lambda-\alpha & \beta & \beta \\ \beta & \lambda-\alpha & \beta \\ \beta & \beta & \lambda-\alpha \end{pmatrix} = 0.$$

$$(4) \quad \det \begin{pmatrix} \lambda-\alpha & \alpha & -\alpha \\ \alpha & \lambda-\alpha & -\alpha \\ -\alpha & -\alpha & \lambda-\alpha \end{pmatrix} = 0.$$

$$(5) \quad \det \begin{pmatrix} \lambda-\alpha & \beta & \beta & \beta \\ \beta & \lambda-\alpha & \beta & \beta \\ \beta & \beta & \lambda-\alpha & \beta \\ \beta & \beta & \beta & \lambda-\alpha \end{pmatrix} = 0.$$

$$(6) \qquad \det \begin{pmatrix} \lambda - \alpha & \beta & -\alpha & \beta \\ \beta & \lambda - \alpha & \beta & -\alpha \\ -\alpha & \beta & \lambda - \alpha & \beta \\ \beta & -\alpha & \beta & \lambda - \alpha \end{pmatrix} = 0.$$

Answers

(1) -32

(2) $(\alpha+\beta)^2(\alpha-2\beta)$

(3) 16

(4) $(\alpha+\beta)^3(\alpha-3\beta)$

(5) $(\lambda-4)^2(\lambda+2)$

(6) $(\lambda-\alpha-\beta)^2(\lambda-\alpha+2\beta)$

(7) $(\lambda-\alpha^2-\beta^2)(\lambda-\alpha^2+2\beta^2)$

(8) $(\lambda-\alpha-\beta)^3(\lambda-\alpha+3\beta)$

$\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha\beta\beta\beta\,\{\}\,\{\}\,\{\}\,\{\}$

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Compute the determinants

$$(1) \quad \det\begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}$$

$$(2) \quad \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

$$(3) \quad \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$$(4) \quad \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Let $\alpha \neq 0$, $\beta \neq 0$ and $\lambda \neq 0$ be real nonzero constants, such that $\lambda + \alpha - \beta \neq 0$ and $\lambda + \alpha + 3\beta \neq 0$. Show that the set

$$\mathcal{S} = \left\{ \begin{pmatrix} \lambda + \alpha \\ \beta \\ \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \lambda + \alpha \\ \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \\ \lambda + \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \\ \beta \\ \lambda + \alpha \end{pmatrix} \right\}$$

is a spanning set of the space $\mathbb{R}^4$.

Let $\alpha \neq 0$, $\beta \neq 0$ and $\lambda \neq 0$ be real nonzero constants. Given that the set

$$\mathcal{S} = \left\{ \begin{pmatrix} \lambda - \alpha \\ \beta \\ \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \lambda - \alpha \\ \beta \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \\ \lambda - \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \beta \\ \beta \\ \beta \\ \lambda - \alpha \end{pmatrix} \right\}$$

is linearly independent in $\mathbb{R}^4$. Find the conditions satisfied by the constants.

Let α, β and γ be sufficiently large positive integers, such that $\alpha > \beta > \gamma \gg 1$. Show that the set

$$\mathcal{S} = \{x^\alpha, x^\beta, x^\gamma\}$$

is linearly independent.

Let $\alpha \neq 0$, $\beta \neq 0$ and $\lambda \neq 0$ be real nonzero constants, such that $\lambda \neq \alpha + \beta$ and $\lambda \neq \alpha - \beta$.

Let $\alpha \neq 0$, $\beta \neq 0$ and $\lambda \neq 0$ be real nonzero constants, such that $\lambda \neq \alpha + \beta$ and $\lambda \neq \alpha - \beta$.

Let $\alpha \neq 0$, $\beta \neq 0$ and $\lambda \neq 0$ be real nonzero constants, such that $\lambda \neq \alpha + \beta$ and $\lambda \neq \alpha - \beta$.

real nonzero constants, such that $\lambda \neq \alpha + \beta$ and $\lambda \neq \alpha - \beta$.

Solve the following second order linear differential equations

$$\begin{aligned} &(1) \quad y'' + 9y' + 20y = 0 \\ &(2) \quad y'' + 12y' + 36y = 0 \\ &(3) \quad y'' + 48y' + 625y = 0 \\ &(4) \quad y'' + 900y = 0 \end{aligned}$$

1. Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}.$$

Find the eigenvalues, eigenvectors, eigenspaces of the matrix A. Find a matrix T, such that $T^{-1}AT$ is a diagonal matrix.

Solution: Let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\lambda I - A = \begin{pmatrix} \lambda - \alpha & \beta & \beta & \beta \\ \beta & \lambda - \alpha & \beta & \beta \\ \beta & \beta & \lambda - \alpha & \beta \\ \beta & \beta & \beta & \lambda - \alpha \end{pmatrix}$$

$$\to \begin{pmatrix} \lambda - \alpha & \beta & \beta & \beta \\ \alpha + \beta - \lambda & \lambda - \alpha - \beta & 0 & 0 \\ 0 & \alpha + \beta - \lambda & \lambda - \alpha - \beta & 0 \\ 0 & 0 & \alpha + \beta - \lambda & \lambda - \alpha - \beta \end{pmatrix}.$$

Now let us compute the determinant of the matrix $\lambda I - A$. We will use some properties of determinants and perform a few more elementary row operations. We have

$$\begin{aligned}
&\det(\lambda I - A)\\
=\ &(\lambda-\alpha-\beta)^3 \det \begin{pmatrix} \lambda-\alpha & \beta & \beta & \beta \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}\\
=\ &(\lambda-\alpha-\beta)^3 \det \begin{pmatrix} \lambda-\alpha+3\beta & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}\\
=\ &(\lambda-\alpha-\beta)^3(\lambda-\alpha+3\beta).
\end{aligned}$$

Therefore, there are two eigenvalues to the matrix A: $\lambda = \alpha+\beta$ and $\lambda = \alpha - 3\beta$. The eigenvectors corresponding to the eigenvalues are given below

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \lambda = \alpha - 3\beta;$$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \lambda = \alpha + \beta.$$

The eigenspaces are

$$\mathbf{E}_{\lambda=\alpha-3\beta} = \text{ span } \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\},$$

$$\mathbf{E}_{\lambda=\alpha+\beta} = \text{ span } \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Let

$$D = \begin{pmatrix} \alpha-3\beta & 0 & 0 & 0 \\ 0 & \alpha+\beta & 0 & 0 \\ 0 & 0 & \alpha+\beta & 0 \\ 0 & 0 & 0 & \alpha+\beta \end{pmatrix},$$

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Then

$$T^{-1}AT = D.$$

2. Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants. Let

$$A = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix}.$$

Find the eigenvalues, eigenvectors and eigenspaces of the matrix A. Find a matrix T, such that $T^{-1}AT$ is a diagonal matrix.
Solution: Let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\lambda I - A = \begin{pmatrix} \lambda - \alpha & \beta & -\alpha & \beta \\ \beta & \lambda - \alpha & \beta & -\alpha \\ -\alpha & \beta & \lambda - \alpha & \beta \\ \beta & -\alpha & \beta & \lambda - \alpha \end{pmatrix}$$

$$\to \begin{pmatrix} \lambda - 2\alpha + 2\beta & \lambda - 2\alpha + 2\beta & \lambda - 2\alpha + 2\beta & \lambda - 2\alpha + 2\beta \\ \beta & \lambda - \alpha & \beta & -\alpha \\ -\alpha & \beta & \lambda - \alpha & \beta \\ \beta & -\alpha & \beta & \lambda - \alpha \end{pmatrix}.$$

Now let us compute the determinant of the matrix $\lambda I - A$. We will use elementary properties of determinants and perform a few more elementary row operations. We have the following computations

$$
\begin{aligned}
& \det(\lambda I - A) \\
= \ & (\lambda - 2\alpha + 2\beta) \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ \beta & \lambda - \alpha & \beta & -\alpha \\ -\alpha & \beta & \lambda - \alpha & \beta \\ \beta & -\alpha & \beta & \lambda - \alpha \end{pmatrix} \\
= \ & (\lambda - 2\alpha + 2\beta) \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \lambda - \alpha - \beta & 0 & -\alpha - \beta \\ -\alpha - \beta & 0 & \lambda - \alpha - \beta & 0 \\ 0 & -\alpha - \beta & 0 & \lambda - \alpha - \beta \end{pmatrix} \\
= \ & (\lambda - 2\alpha + 2\beta) \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & \lambda - \alpha - \beta & 0 & -\alpha - \beta \\ -\alpha - \beta & 0 & \lambda - \alpha - \beta & 0 \\ 0 & \lambda - 2\alpha - 2\beta & 0 & \lambda - 2\alpha - 2\beta \end{pmatrix} \\
= \ & (\lambda - 2\alpha - 2\beta)(\lambda - 2\alpha + 2\beta) \det \begin{pmatrix} 1 & 1 & 1 & \\ 0 & \lambda - \alpha - \beta & 0 & - \\ -\alpha - \beta & 0 & \lambda - \alpha - \beta & \\ 0 & 1 & 0 & \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= (\lambda - 2\alpha - 2\beta)(\lambda - 2\alpha + 2\beta) \det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 \\ -\alpha - \beta & 0 & \lambda - \alpha - \beta & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \\
&= (\lambda - 2\alpha - 2\beta)(\lambda - 2\alpha + 2\beta) \det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \\
&= \lambda^2(\lambda - 2\alpha - 2\beta)(\lambda - 2\alpha + 2\beta) \det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \\
&= \lambda^2(\lambda - 2\alpha - 2\beta)(\lambda - 2\alpha + 2\beta) \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \lambda^2(\lambda - 2\alpha - 2\beta)(\lambda - 2\alpha + 2\beta).
\end{aligned}
$$

Therefore, there are three eigenvalues to the matrix A: $\lambda = 2\alpha + 2\beta$, $\lambda = 2\alpha - 2\beta$, $\lambda = 0$. The eigenvectors corresponding to the eigenvalues are given below:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \lambda = 2\alpha - 2\beta,$$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \qquad \lambda = 0,$$

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \lambda = 2\alpha + 2\beta.$$

The eigenspaces are

$$\mathbf{E}_{\lambda=2\alpha-2\beta} = \text{ span } \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\},$$

$$\mathbf{E}_{\lambda=0} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\},$$

$$\mathbf{E}_{\lambda=2\alpha+2\beta} = \text{ span } \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Let

$$D = \begin{pmatrix} 2\alpha - 2\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha + 2\beta \end{pmatrix}$$

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Then

$$T^{-1}AT = D.$$

$$\begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \alpha \end{pmatrix} \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \alpha \end{pmatrix} \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \alpha \end{pmatrix} \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \alpha \end{pmatrix}$$

$$\alpha\alpha\alpha\alpha$$

Chapter 14.

Chapter 15.

Chapter 16.2023

Chapter 3 - Vector Spaces - Everyday Quizzes

1. Let

$$\mathbf{W} = \left\{ \begin{pmatrix} 5x + 6y \\ 7x + 8y \\ 9x + 10y \end{pmatrix} : x \in \mathbb{R}, y \in \mathbb{R} \right\}.$$

Show that **W** is a vector subspace of the vector space $\mathbb{R}^3$.

2. Let

$$\mathbf{W} = \left\{ \begin{pmatrix} 5x + 6y + 1 \\ 7x + 8y + 2 \\ 9x + 10y + 3 \end{pmatrix} : x \in \mathbb{R}, y \in \mathbb{R} \right\}.$$

Show that **W** is NOT a vector subspace of the vector space $\mathbb{R}^3$.

3. Is the set of vectors

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

linearly independent? Justify your solution.

4. Is the set of vectors

$$\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \quad \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right\}$$

linearly independent? Justify your solution.

5. Let α and β be real constants, such that $\alpha + \beta \neq 0$ and $\alpha - 3\beta \neq 0$. Is the set of vectors

$$\left\{\begin{pmatrix}\alpha\\-\beta\\-\beta\\-\beta\end{pmatrix}, \quad \begin{pmatrix}-\beta\\\alpha\\-\beta\\-\beta\end{pmatrix}, \quad \begin{pmatrix}-\beta\\-\beta\\\alpha\\-\beta\end{pmatrix}, \quad \begin{pmatrix}-\beta\\-\beta\\-\beta\\\alpha\end{pmatrix}\right\}$$

linearly independent? Justify your solutions.

6. Let α and β be real constants. Is the set of vectors

$$\left\{\begin{pmatrix}\alpha\\-\beta\\\alpha\\-\beta\end{pmatrix}, \quad \begin{pmatrix}-\beta\\\alpha\\-\beta\\\alpha\end{pmatrix}, \quad \begin{pmatrix}\alpha\\-\beta\\\alpha\\-\beta\end{pmatrix}, \quad \begin{pmatrix}-\beta\\\alpha\\-\beta\\\alpha\end{pmatrix}\right\}$$

a spanning set of $\mathbb{R}^4$? Justify your solutions.

7. Let $\alpha \neq 0$, $\beta \neq 0$ and $\lambda \neq 0$ be real nonzero constants, such that $\lambda \neq \alpha + \beta$ and $\lambda \neq \alpha - 3\beta$. Is the set of vectors

$$\left\{ \begin{pmatrix} \lambda-\alpha \\ \beta \\ \beta \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \lambda-\alpha \\ \beta \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \beta \\ \lambda-\alpha \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \beta \\ \beta \\ \lambda-\alpha \end{pmatrix} \right\}$$

a spanning set of $\mathbb{R}^4$? Justify your solutions.

8. Find a basis and dimension of the vector space $\mathbb{R}^3$.

9. Find a basis and dimension of the vector space $M_2(\mathbb{R})$.

10. Find a basis and dimension of the vector space $P_5(\mathbb{R})$.

11. Let

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 4 & -6 & 8 \\ 3 & -6 & 9 & -12 \\ -4 & 8 & -12 & 16 \end{pmatrix}.$$

(1) Find the nullspace $NS(A)$, a basis and the dimension.
(2) Find the column space $CS(A)$, a basis and the dimension.
(3) Find the row space $RS(A)$, a basis and the dimension.

12. Let

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

(1) Find the nullspace $NS(A)$, a basis and the dimension.
(2) Find the column space $CS(A)$, a basis and the dimension.
(3) Find the row space $RS(A)$, a basis and the dimension.

13. Let α and β be real constants, such that $\alpha^2 > \beta^2$. Let

$$A = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix}.$$

Find the dimension of the nullspace $NS(A)$, the dimension of the column space $CS(A)$ and the dimension of the row space $RS(A)$.

14. Given a basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}. \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}$$

and a vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ in $\mathbb{R}^4$. Find the component vector $[\mathbf{v}]_{\mathcal{B}}$.

15. Given a basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}. \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\},$$

and the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find the component vector $[\mathbf{v}]_{\mathcal{B}}$.

16. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}. \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

be two bases of $\mathbb{R}^4$. Find the change-of-basis-matrices

$$P_{\mathcal{C}\leftarrow\mathcal{B}}, \qquad P_{\mathcal{B}\leftarrow\mathcal{C}}.$$

17. Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

be two bases of $\mathbb{R}^3$. Find the change-of-basis-matrices

$$P_{\mathcal{C} \leftarrow \mathcal{B}}, \qquad P_{\mathcal{B} \leftarrow \mathcal{C}}.$$

1. The addition is closed in $\mathbf{W}$ and the scalar multiplication is closed in $\mathbf{W}$. Therefore, $\mathbf{W}$ is a subspace of $\mathbb{R}^3$.

2. The addition is not closed $\mathbf{W}$. The scalar multiplication is not closed in $\mathbf{W}$ either. Therefore, $\mathbf{W}$ is not a subspace of $\mathbb{R}^3$.

3. The set of vectors is linearly independent, because the determinant

$$\det \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = -4 \neq 0.$$

4. The determinant

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = 16 > 0.$$

Therefore, the set of vectors is linearly independent in $\mathbb{R}^4$.

5. The determinant

$$\det\begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix} = (\alpha+\beta)^3(\alpha-3\beta) \neq 0.$$

Therefore, the set of vectors is linearly independent.

6. The determinant

$$\det\begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix} = 0.$$

Therefore, the set of vectors is not a spanning set.

7. The determinant

$$\det\begin{pmatrix} \lambda-\alpha & \beta & \beta & \beta \\ \beta & \lambda-\alpha & \beta & \beta \\ \beta & \beta & \lambda-\alpha & \beta \\ \beta & \beta & \beta & \lambda-\alpha \end{pmatrix} = (\lambda-\alpha-\beta)^3(\lambda-\alpha+3\beta) \neq 0.$$

Therefore, the set of vectors is a spanning set of $\mathbb{R}^4$.

8. Here is a basis for $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

The dimension $\dim(\mathbb{R}^3) = 3$.

9. Here is a basis for $M_2(\mathbb{R})$:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

The dimension $\dim[M_2(\mathbb{R})] = 4$.

10. Here is a basis for $P_5(\mathbb{R})$:

$$\mathcal{B} = \left\{1, x, x^2, x^3, x^4, x^5\right\}.$$

The dimension $\dim[P_5(\mathbb{R})] = 6$.

11. Let us perform elementary row operations to the matrix A. We have

$$\begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 4 & -6 & 8 \\ 3 & -6 & 9 & -12 \\ -4 & 8 & -12 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Here is a basis for the nullspace $NS(A)$:

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

The dimension $\dim[NS(A)] = 3$.

Here is the information for the row space $RS(A)$:

$$\begin{aligned} &RS(A) = \text{ span } \left\{\begin{pmatrix} 1 & -2 & 3 & -4 \end{pmatrix}\right\}, \\ &\text{Here is a basis } \mathcal{B} = \left\{\begin{pmatrix} 1 & -2 & 3 & -4 \end{pmatrix}\right\} \\ &\dim[RS(A)] = 1 \end{aligned}$$

Now let us perform elementary column operations to the matrix A. We have

$$\begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 4 & -6 & 8 \\ 3 & -6 & 9 & -12 \\ -4 & 8 & -12 & 16 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 \end{pmatrix}$$

Here is the information for the column space $CS(A)$:

$$CS(A) = \text{ span } \left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix} \right\},$$

$$\text{A basis is } \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \\ -4 \end{pmatrix} \right\}$$

$$\dim[CS(A)] = 1$$

12. Note that

$$A^2 = I, \qquad A^{-1} = A.$$

For the nullspace $NS(A)$:

$$NS(A) = \{\mathbf{0}\},$$
$$\dim[NS(A)] = 0.$$

For the column space $CS(A)$:

$$CS(A) = \mathbb{R}^4,$$
$$\dim[CS(A)] = 4.$$

Here is a basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

For the row space $RS(A)$:

$$RS(A) = \mathbb{R}^4$$
$$\dim[RS(A)] = 4$$

Here is a basis

$$\mathcal{B} = \{ \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix} \}$$

13. Note that A is a symmetric matrix. The first row is the same as the third row, and the second row is the same as the fourth row. Here is a basis for the nullspace $NS(A)$:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Here is a basis for the column space $CS(A)$:

$$\left\{ \begin{pmatrix} \alpha \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ \alpha \end{pmatrix} \right\}$$

Here is a basis for the row space $RS(A)$:

$$\{ \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \end{pmatrix}, \begin{pmatrix} -\beta & \alpha & -\beta & \alpha \end{pmatrix} \}$$

Therefore, the dimension of the nullspace $NS(A)$, the dimension of the column space $CS(A)$ and the dimension of the row space $RS(A)$ are all equal to 2.

14. The component vector is

$$\begin{aligned}
[\mathbf{v}]_{\mathcal{B}} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 10 \\ 0 \\ -2 \\ -4 \end{pmatrix}.
\end{aligned}$$

15. The component vector is

$$\begin{aligned}
[\mathbf{v}]_{\mathcal{B}} &= \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.
\end{aligned}$$

16. The change-of-basis-matrices are

$$
\begin{aligned}
P_{\mathcal{C}\leftarrow\mathcal{B}} &= C^{-1}B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
P_{\mathcal{B}\leftarrow\mathcal{C}} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\end{aligned}
$$

17. The change-of-basis-matrices are

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = C^{-1}B = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$P_{\mathcal{B}\leftarrow\mathcal{C}} = B^{-1}C = \frac{1}{2}\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Chapter 4 - Linear Transformations - Everyday Quizzes

Let $\alpha \neq 0$, $\beta \neq 0$ and $\lambda \neq 0$ be real nonzero constants, such that $\alpha^2 > \beta^2$.

For each of the following linear transformations:
(1) Find the kernel $\mathrm{Ker}(T)$, a basis and the dimension of $\mathrm{Ker}(T)$.
(2) Find the range $\mathrm{Rng}(T)$, a basis and the dimension of $\mathrm{Rng}(T)$.

1. $T : \mathbb{R}^4 \to \mathbb{R}^4$,

$$T(\mathbf{x}) = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix} \mathbf{x}.$$

2. $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$,

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

3. $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$,

$$T(a + bx + cx^2) = (a + b) + (b + c)x + (c + a)x^2.$$

4. $T : \mathbb{R}^4 \to M_2(\mathbb{R})$,

$$T \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Show work to determine if each of the following linear transformations is one-to-one, onto, both or neither.

5. $T : \mathbb{R}^4 \to \mathbb{R}^4$,

$$T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

6. $T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$,
$T(A) = A^T - A$.

7. $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$,
$T(a + bx + cx^2 + dx^3)$
$= a + (a + b)x + (a + b + c)x^2 + (a + b + c + d)x^3$.

8. $T : M_2(\mathbb{R}) \to \mathbb{R}^4$

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + b \\ b + c \\ c + d \\ d + a \end{pmatrix}.$$

9. Under what conditions on the parameters α, β and λ, is the

following linear transformation

$$\begin{aligned} &T : \mathbb{R}^4 \to \mathbb{R}^4, \\ &T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} \lambda - \alpha & \beta & \beta & \beta \\ \beta & \lambda - \alpha & \beta & \beta \\ \beta & \beta & \lambda - \alpha & \beta \\ \beta & \beta & \beta & \lambda - \alpha \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \end{aligned}$$

an isomorphism? Justify your answer.

11. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, given by

$$T(\mathbf{x}) = A\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where A is an $n \times n$ real matrix and $\det(A) \neq 0$. Which of the following statements is wrong?

(A) The linear transformation is one-to-one.

(B) The linear transformation is onto.

(C) The inverse linear transformation T^{-1} exists and it is given by

$$T^{-1}(x) = A^{-1}\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n.$$

(D) None of the above.

12. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, given by

$$T(\mathbf{x}) = A\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where A is an $n \times n$ real matrix and $\det(A) = 0$. Which of the following statements is true?

(A) The linear transformation is one-to-one.

(B) The linear transformation is onto.

(C) The inverse linear transformation T^{-1} exists.

(D) None of the above.

13. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, given by

$$T(\mathbf{x}) = A\mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^n,$$

where A is an $m \times n$ real matrix, $m > 2$ and $n > 2$ are positive integers. Let the rank of A be less than both $m - 1$ and $n - 1$. Which of the following statements is true?

(A) The linear transformation is one-to-one.

(B) The linear transformation is onto.

(C) The inverse linear transformation T^{-1} exists.

(D) None of the above.

14. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear transformation, given by

$$\begin{aligned} & T(a + bx + cx^2 + dx^3) \\ = \ & 9[(4a - 2b) + (9c - 3d)] + 16[(4a - 2b) - (9c - 3d)]x \\ + \ & [(4a - 2b) + (9c - 3d)]x^2 + [(4a - 2b) - (9c - 3d)]x^3. \end{aligned}$$

Which of the following sets of vectors represent the basis of the kernel and the basis of the range of the linear transformation T?

$(A) \quad \{1 + 2x, x^2 + x^3\}, \qquad \{x^2 + 3, 12x + x^3\}.$

$(B) \quad \{1 + 2x, x^2 + 2x^3\}, \qquad \{x^2 + 6, 14x + x^3\}.$

$(C) \quad \{1 + 2x, x^2 + 3x^3\}, \qquad \{x^2 + 9, 16x + x^3\}.$

$(D) \quad \{1 + 2x, x^2 + 4x^3\}, \qquad \{x^2 + 12, 18x + x^3\}.$

15. Which of the following linear transformations is NOT an

isomorphism?

$$(A) \quad T : \mathbb{R}^3 \to M_3(\mathbb{R})(\text{ Only Skew Symmetric Matrices }),$$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & -a & c \\ a & 0 & -b \\ -c & b & 0 \end{pmatrix}.$$

$$(B) \quad T : P_5(\mathbb{R}) \to M_3(\mathbb{R})(\text{ Only Symmetric Matrices }),$$

$$T(a + bx + cx^2 + \alpha x^3 + \beta x^4 + \gamma x^5) = \begin{pmatrix} \alpha & a & c \\ a & \beta & b \\ c & b & \gamma \end{pmatrix}.$$

$$(C) \quad T : M_2(\mathbb{R}) \to P_3(\mathbb{R}),$$

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + bx + cx^2 + dx^3.$$

$$(D) \quad T : \mathbb{R}^3 \to \mathbb{R}^3,$$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & -2 & 2 \\ 2 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

1. The kernel, a basis of the kernel and the dimension of the kernel are given by

$$\text{Ker }(T) = \text{ span }\left\{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}\right\},$$

$$\mathcal{B} = \left\{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}\right\},$$

$$\dim[\text{ Ker }(T)] = 2.$$

The range, a basis of the range and the dimension of the range are

$$\text{Rng }(T) = \text{ span }\left\{\begin{pmatrix} \alpha \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ \alpha \end{pmatrix}\right\},$$

$$\mathcal{B} = \left\{\begin{pmatrix} \alpha \\ -\beta \\ \alpha \\ -\beta \end{pmatrix}, \quad \begin{pmatrix} -\beta \\ \alpha \\ -\beta \\ \alpha \end{pmatrix}\right\},$$

$$\dim[\text{ Rng }(T)] = 2.$$

2. The kernel is the trivial space

$$\text{Ker }(T) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

The range, a basis of the range and the dimension of the range are given by

$$\begin{aligned}
&\text{Rng }(T) = \text{ span }\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}\\
&\mathcal{B} = \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \quad \begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad \begin{pmatrix}0&0\\1&0\end{pmatrix}, \quad \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}\\
&\dim[\text{ Rng }(T)] = 4
\end{aligned}$$

3. The kernel is the trivial space and it is given by

$$\text{Ker }(T) = \{0\}$$

The range, a basis of the range and the dimension of the range are given by

$$\begin{aligned}
&\text{Rng }(T) = \text{ span }\{1, x, x^2\}\\
&\mathcal{B} = \{1, x, x^2\}\\
&\dim[\text{ Rng }(T)] = 3
\end{aligned}$$

4. The kernel is the trivial space and it is given by

$$\text{Ker }(T) = \left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}\right\}.$$

The range, a basis of the range and the dimension of the range are given by

$$\begin{aligned}
&\text{Rng }(T) = \text{ span }\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\},\\
&\mathcal{B} = \left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \quad \begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad \begin{pmatrix}0&0\\1&0\end{pmatrix}, \quad \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\},\\
&\dim[\text{ Rng }(T)] = 4.
\end{aligned}$$

5. The linear transformation is one-to-one and onto.

6. The linear transformation is neither one-to-one nor onto.
7. The linear transformation is one-to-one and onto.
8. The linear transformation is neither one-to-one nor onto.
9. The conditions are

$$\lambda \neq \alpha + \beta, \qquad \lambda \neq \alpha - 3\beta.$$

11 (D). 12 (D). 13 (D). 14 (C). 15 (D).

Chapter 5 - Eigenvalues, Eigenvectors and Diagonalizations - E

Let $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$ be real nonzero constants. Find the eigenvalues and eigenvectors of the following matrices.

$$1. \quad A = \begin{pmatrix} 3 & -2 & -2 \\ -3 & -2 & -6 \\ 3 & 6 & 10 \end{pmatrix}.$$

$$2. \quad A = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

$$3. \quad A = \begin{pmatrix} 5 & 2 & 2 \\ -5 & 12 & 2 \\ 5 & -2 & 8 \end{pmatrix}$$

$$4. \ A = \begin{pmatrix} 5 & 12 & -6 \\ -3 & -10 & 6 \\ -3 & -12 & 8 \end{pmatrix}.$$

$$5. \quad A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

$$6. \quad A = \begin{pmatrix} 9 & 6 & -6 \\ 3 & 12 & -6 \\ -3 & -6 & 12 \end{pmatrix}.$$

$$7. \quad A = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

$$8. \quad A = \begin{pmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{pmatrix}.$$

$$9. \quad A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}.$$

$$10. \quad A = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix}.$$

11. For each of the following five matrices, find a diagonal matrix D and a matrix T, such that

$$T^{-1}AT = D.$$

$$(1) \quad A = \begin{pmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{pmatrix}$$

$$(2) \quad A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$(3) \quad A = \begin{pmatrix} \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \\ \alpha & \alpha & \alpha \end{pmatrix}$$

$$(4) \quad A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}$$

$$(5) \quad A = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix}$$

12. Let

$$A = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix}.$$

Define the following matrices

$$T_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix},$$

$$T_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Compute the following matrices

(1) T_1^2

(2) T_2^2

(3) T_3^2

(4) T_4^2

and

(5) $T_1^{-1}AT_1$
(6) $T_2^{-1}AT_2$
(7) $T_3^{-1}AT_3$
(8) $T_4^{-1}AT_4$

13. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{pmatrix}.$$

Which of the following statements about the algebraic multiplicty and the geometric multiplicity of the eigenvalues is true?

(A) Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = \alpha - 2\beta$ are 1. Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_2 = \alpha + \beta$ are 1.

(B) Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = \alpha - 2\beta$ are 2. Both the algebraic multiplicity and the geometric multiplicty of the eigenvalue $\lambda_2 = \alpha + \beta$ are 2.

(C) Both the algebraic multiplicity and the geometric multiplicty of the eigenvalue $\lambda_1 = \alpha - 2\beta$ are 1. Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_2 = \alpha + \beta$ are 2.

(D) Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_1 = \alpha - 2\beta$ are 2. Both the algebraic multiplicity and the geometric multiplicity of the eigenvalue $\lambda_2 = \alpha + \beta$ are 1.

14. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix},$$

where α and β are real nonzero constants. Which of the following statements about the eigenvalues and their algebraic multiplicities is true?

(A) The first eigenvalue is $\lambda_1 = \alpha - 3\beta$ and the algebraic multiplicity is 1. The second eigenvalue is $\lambda_2 = \alpha + \beta$ and the algebraic multiplicity is 1.

(B) The first eigenvalue is $\lambda_1 = \alpha - 3\beta$ and the algebraic multiplicity is 3. The second eigenvalue is $\lambda_2 = \alpha + \beta$ and the algebraic multiplicity is 3.

(C) The first eigenvalue is $\lambda_1 = \alpha - 3\beta$ and the algebraic multiplicity is 1. The second eigenvalue is $\lambda_2 = \alpha + \beta$ and the algebraic multiplicity is 3.

(D) The first eigenvalue is $\lambda = \alpha - 3\beta$ and the algebraic multiplicity is 3. The second eigenvalue is $\lambda = \alpha + \beta$ and the algebraic multiplicity is 1.

15. Let A be a 3×3 matrix. Let λ_i and ξ_i be the eigenvalue and eigenvector, respectively, that is,

$$\begin{aligned} A\xi_1 &= \lambda_1\xi_1, \\ A\xi_2 &= \lambda_2\xi_2, \\ A\xi_3 &= \lambda_3\xi_3. \end{aligned}$$

Define $T = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix}$. Which of the following is $T^{-1}AT$ equal to?

$$(A) \quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

$$(B) \quad \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

$$(C) \quad \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

$$(D) \quad \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

16. Let A be an $n \times n$ matrix. Let $A\xi = \lambda\xi$. Which of the following is NOT true?

$$(A) \quad A^2\xi = \lambda^2\xi.$$

$$(B) \quad A^3\xi = \lambda^3\xi.$$

$$(C) \quad A^4\xi = \lambda^4\xi.$$

$$(D) \quad A^m\xi = \lambda^m\xi^m.$$

17. Let $B = T^{-1}AT$, where A, B and T are real $n \times n$ matrices. Which of the following statements about the eigenvalues and eigenvectors of A and B is true?

(A) Both the eigenvalues and eigenvectors of A and B are the same.

(B) Both the eigenvalues and the eigenvectors of A and B are different.

(C) The eigenvalues of A and B are the same. The eigenvectors may be different.

(D) The eigenvectors of A and B are the same. The eigenvalues may be different.

18. Let A be a 4×4 real matrix. Two complex eigenvalues of A are given: $\lambda_1 = 10 - 5i$ and $\lambda_2 = -9i$. What are the other complex eigenvalues?

(A) $\lambda_3 = 10 + 5i,$ $\lambda_4 = 9i.$

(B) $\lambda_3 = 10 + 5i,$ $\lambda_4 = -9i.$

(C) $\lambda_3 = 10 - 5i,$ $\lambda_4 = 9i.$

(D) $\lambda_3 = 10 - 5i,$ $\lambda_4 = -9i.$

1. First of all, let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\begin{aligned}\lambda I - A &= \begin{pmatrix} \lambda - 3 & 2 & 2 \\ 3 & \lambda + 2 & 6 \\ -3 & -6 & \lambda - 10 \end{pmatrix} \\ &\to \begin{pmatrix} \lambda - 3 & 2 & 2 \\ 12 - 3\lambda & \lambda - 4 & 0 \\ 3\lambda - 12 & 0 & \lambda - 4 \end{pmatrix}.\end{aligned}$$

Secondly, the characteristic polynomial of A is

$$\det(\lambda I - A) = (\lambda - 4)^2(\lambda - 3).$$

There are two eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 3$. The linearly independent eigenvectors corresponding to the two eigenvalues are given by

$$\begin{aligned}\lambda_1 = 4, &\quad \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \\ \lambda_2 = 3, &\quad \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}.\end{aligned}$$

2. First of all, let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\begin{aligned} \lambda I - A &= \begin{pmatrix} \lambda-2 & -2 & -1 \\ -2 & \lambda-5 & -2 \\ -1 & -2 & \lambda-2 \end{pmatrix} \\ &\to \begin{pmatrix} \lambda-2 & -2 & -1 \\ 2-2\lambda & \lambda-1 & 0 \\ 1-\lambda & 0 & \lambda-1 \end{pmatrix}. \end{aligned}$$

Secondly, the characteristic polynomial of A is

$$\det(\lambda I - A) = (\lambda-1)^2(\lambda-7).$$

There are two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 7$. The linearly independent eigenvectors corresponding to the two eigenvalues are given by

$$\begin{aligned} \lambda_1 = 1, &\quad \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}, \\ \lambda_2 = 7, &\quad \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}. \end{aligned}$$

3. First of all, let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\begin{aligned}\lambda I - A &= \begin{pmatrix} \lambda - 5 & -2 & -2 \\ 5 & \lambda - 12 & -2 \\ -5 & 2 & \lambda - 8 \end{pmatrix} \\ &\to \begin{pmatrix} \lambda - 5 & -2 & -2 \\ 10 - \lambda & \lambda - 10 & 0 \\ \lambda - 10 & 0 & \lambda - 10 \end{pmatrix}.\end{aligned}$$

Secondly, the characteristic polynomial of A is

$$\det(\lambda I - A) = (\lambda - 10)^2(\lambda - 5).$$

There are two eigenvalues $\lambda_1 = 10$ and $\lambda_2 = 5$. The linearly independent eigenvectors corresponding to the two eigenvalues are given by

$$\begin{aligned}\lambda_1 = 10, &\quad \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}, \\ \lambda_2 = 5, &\quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.\end{aligned}$$

4. First of all, let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\begin{aligned}\lambda I - A &= \begin{pmatrix} \lambda-5 & -12 & 6 \\ 3 & \lambda+10 & -6 \\ 3 & 12 & \lambda-8 \end{pmatrix} \\ &\to \begin{pmatrix} \lambda-5 & -12 & 6 \\ \lambda-2 & \lambda-2 & 0 \\ \lambda-2 & 0 & \lambda-2 \end{pmatrix}.\end{aligned}$$

Secondly, the characteristic polynomial of A is

$$\det(\lambda I - A) = (\lambda-2)^2(\lambda+1).$$

There are two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. The linearly independent eigenvectors corresponding to the two eigenvalues are given by

$$\begin{aligned}\lambda_1 = 2, &\qquad \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \\ \lambda_2 = -1, &\qquad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.\end{aligned}$$

5. First of all, let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\begin{aligned} \lambda I - A &= \begin{pmatrix} \lambda - 1 & -1 & 1 \\ 1 & \lambda - 3 & 1 \\ 1 & -1 & \lambda - 1 \end{pmatrix} \\ &\to \begin{pmatrix} \lambda - 1 & -1 & 1 \\ 2 - \lambda & \lambda - 2 & 0 \\ 2 - \lambda & 0 & \lambda - 2 \end{pmatrix}. \end{aligned}$$

Secondly, the characteristic polynomial of A is

$$\det(\lambda I - A) = (\lambda - 2)^2(\lambda - 1).$$

There are two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$. The linearly independent eigenvectors corresponding to the two eigenvalues are given by

$$\begin{aligned} \lambda_1 = 2, &\quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\ \lambda_2 = 1, &\quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

6. First of all, let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$
\begin{aligned}
\lambda I - A &= \begin{pmatrix} \lambda - 9 & -6 & 6 \\ -3 & \lambda - 12 & 6 \\ 3 & 6 & \lambda - 12 \end{pmatrix} \\
&\to \begin{pmatrix} \lambda - 9 & -6 & 6 \\ 6 - \lambda & \lambda - 6 & 0 \\ \lambda - 6 & 0 & \lambda - 6 \end{pmatrix}.
\end{aligned}
$$

Secondly, the characteristic polynomial of A is

$$\det(\lambda I - A) = (\lambda - 6)^2(\lambda - 21).$$

There are two eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 21$. The linearly independent eigenvectors corresponding to the two eigenvalues are given by

$$
\begin{aligned}
\lambda_1 = 6, &\quad \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\
\lambda_2 = 21, &\quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.
\end{aligned}
$$

7. First of all, let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\begin{aligned}\lambda I - A &= \begin{pmatrix} \lambda-2 & 2 & -2 \\ 2 & \lambda-2 & -2 \\ -2 & -2 & \lambda-2 \end{pmatrix} \\ &\to \begin{pmatrix} \lambda-2 & 2 & -2 \\ 4-\lambda & \lambda-4 & 0 \\ \lambda-4 & 0 & \lambda-4 \end{pmatrix}.\end{aligned}$$

Secondly, the characteristic polynomial of A is

$$\det(\lambda I - A) = (\lambda-4)^2(\lambda+2).$$

There are two eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -2$. The linearly independent eigenvectors corresponding to the two eigenvalues are given by

$$\begin{aligned}\lambda_1 = 4, &\quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\ \lambda_2 = -2, &\quad \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.\end{aligned}$$

8. First of all, let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\begin{aligned} \lambda I - A &= \begin{pmatrix} \lambda-\alpha & \beta & \beta \\ \beta & \lambda-\alpha & \beta \\ \beta & \beta & \lambda-\alpha \end{pmatrix} \\ &\to \begin{pmatrix} \lambda-\alpha & \beta & \beta \\ \alpha+\beta-\lambda & \lambda-\alpha-\beta & 0 \\ 0 & \alpha+\beta-\lambda & \lambda-\alpha-\beta \end{pmatrix}. \end{aligned}$$

Secondly, the characteristic polynomial of A is

$$\det(\lambda I - A) = (\lambda-\alpha-\beta)^2(\lambda-\alpha+2\beta).$$

There are two eigenvalues $\lambda_1 = \alpha+\beta$ and $\lambda_2 = \alpha-2\beta$. The linearly independent eigenvectors corresponding to the two eigenvalues are given by

$$\begin{aligned} \lambda_1 = \alpha+\beta, &\quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \\ \lambda_2 = \alpha-2\beta, &\quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

9. First of all, let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\begin{aligned} \lambda I - A \;&=\; \begin{pmatrix} \lambda-\alpha & \beta & \beta & \beta \\ \beta & \lambda-\alpha & \beta & \beta \\ \beta & \beta & \lambda-\alpha & \beta \\ \beta & \beta & \beta & \lambda-\alpha \end{pmatrix} \\ &\to \begin{pmatrix} \lambda-\alpha & \beta & \beta & \beta \\ \alpha+\beta-\lambda & \lambda-\alpha-\beta & 0 & 0 \\ 0 & \alpha+\beta-\lambda & \lambda-\alpha-\beta & 0 \\ 0 & 0 & \alpha+\beta-\lambda & \lambda-\alpha-\beta \end{pmatrix}. \end{aligned}$$

Secondly, the characteristic polynomial of A is

$$\det(\lambda I - A) = (\lambda-\alpha-\beta)^3(\lambda-\alpha+3\beta).$$

There are two eigenvalues $\lambda_1 = \alpha+\beta$ and $\lambda_2 = \alpha-3\beta$. The linearly independent eigenvectors corresponding to the two eigenvalues are given by

$$\lambda_1 = \alpha+\beta, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix},$$

$$\lambda_2 = \alpha-3\beta, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

10. First of all, let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\lambda I - A = \begin{pmatrix} \lambda-\alpha & \beta & -\alpha & \beta \\ \beta & \lambda-\alpha & \beta & -\alpha \\ -\alpha & \beta & \lambda-\alpha & \beta \\ \beta & -\alpha & \beta & \lambda-\alpha \end{pmatrix}$$

$$\to \begin{pmatrix} \lambda-2\alpha+2\beta & \lambda-2\alpha+2\beta & \lambda-2\alpha+2\beta & \lambda-2\alpha+2\beta \\ \beta & \lambda-\alpha & \beta & -\alpha \\ -\alpha & \beta & \lambda-\alpha & \beta \\ \beta & -\alpha & \beta & \lambda-\alpha \end{pmatrix}.$$

Secondly, the characteristic polynomial of A is

$$\det(\lambda I - A) = (\lambda - 2\alpha - 2\beta)\lambda^2(\lambda - 2\alpha + 2\beta).$$

There are three eigenvalues $\lambda_1 = 2\alpha - 2\beta$, $\lambda_2 = 0$ and $\lambda_3 = 2\alpha + 2\beta$. The linearly independent eigenvectors corresponding to the two eigenvalues are given by

$$\lambda_1 = 2\alpha - 2\beta, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\lambda_2 = 0, \quad \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix},$$

$$\lambda_3 = 2\alpha + 2\beta, \quad \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

11 (1). First of all, let us define the matrices D and T. Define

$$D = \begin{pmatrix} \alpha - 2\beta & 0 & 0 \\ 0 & \alpha + \beta & 0 \\ 0 & 0 & \alpha + \beta \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then

$$T^{-1}AT = D.$$

11 (2). First of all, let us define the matrices D and T. Define

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Then

$$T^{-1}AT = D.$$

11 (3). First of all, let us perform elementary row operations to the matrix $\lambda I - A$. We have

$$\begin{aligned}\lambda I - A &= \begin{pmatrix} \lambda-\alpha & -\alpha & -\alpha \\ -\alpha & \lambda-\alpha & -\alpha \\ -\alpha & -\alpha & \lambda-\alpha \end{pmatrix} \\ &\to \begin{pmatrix} \lambda-\alpha & -\alpha & -\alpha \\ -\lambda & \lambda & 0 \\ 0 & -\lambda & \lambda \end{pmatrix}.\end{aligned}$$

Secondly, the characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^2(\lambda - 3\alpha).$$

There are two eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 3\alpha$. The linearly independent eigenvectors corresponding to the two eigenvalues are given by

$$\begin{aligned}\lambda_1 = 0, &\quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \\ \lambda_2 = 3\alpha, &\quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.\end{aligned}$$

Define

$$D = \begin{pmatrix} 3\alpha & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then

$$T^{-1}AT = D.$$

11 (4). First of all, let us define the matrices D and T. Define

$$D = \begin{pmatrix} \alpha - 3\beta & 0 & 0 & 0 \\ 0 & \alpha + \beta & 0 & 0 \\ 0 & 0 & \alpha + \beta & 0 \\ 0 & 0 & 0 & \alpha + \beta \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Then

$$T^{-1}AT = D.$$

11 (5). First of all, let us define the matrices D and T. Define

$$D = \begin{pmatrix} 2\alpha - 2\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\alpha + 2\beta \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Then

$$T^{-1}AT = D.$$

Let

$$A = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix},$$

where $\alpha \neq 0$ and $\beta \neq 0$ are real nonzero constants, such that $\alpha^2 > \beta^2$.

Let us perform some very simple elementary row operations to the matrix $\lambda I - A$ and then compute the determinant. We have

$$\begin{aligned} \lambda I - A \;&=\; \begin{pmatrix} \lambda-\alpha & \beta & -\alpha & \beta \\ \beta & \lambda-\alpha & \beta & -\alpha \\ -\alpha & \beta & \lambda-\alpha & \beta \\ \beta & -\alpha & \beta & \lambda-\alpha \end{pmatrix} \\ &\to \begin{pmatrix} \lambda-\alpha & \beta & -\alpha & \beta \\ \beta & \lambda-\alpha & \beta & -\alpha \\ -\lambda & 0 & \lambda & 0 \\ 0 & -\lambda & 0 & \lambda \end{pmatrix}. \end{aligned}$$

Now let us use some simple properties of determinants to compute

$\det(\lambda I - A)$. We have

$$\begin{aligned} & \det(\lambda I - A) \\ = \ & \det \begin{pmatrix} \lambda-\alpha & \beta & -\alpha & \beta \\ \beta & \lambda-\alpha & \beta & -\alpha \\ -\lambda & 0 & \lambda & 0 \\ 0 & -\lambda & 0 & \lambda \end{pmatrix} \\ = \ & \lambda^2 \det \begin{pmatrix} \lambda-\alpha & \beta & -\alpha & \beta \\ \beta & \lambda-\alpha & \beta & -\alpha \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \\ = \ & \lambda^2 \det \begin{pmatrix} \lambda-2\alpha & 2\beta & 0 & 0 \\ 2\beta & \lambda-2\alpha & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \\ = \ & \lambda^2 \det \begin{pmatrix} \lambda-2\alpha & 2\beta \\ 2\beta & \lambda-2\alpha \end{pmatrix} \\ = \ & \lambda^2(\lambda-2\alpha+2\beta)(\lambda-2\alpha-2\beta). \end{aligned}$$

Therefore, there are four eigenvalues to the matrix A:

$$\lambda_1 = \lambda_2 = 0, \qquad \lambda_3 = 2\alpha + 2\beta, \qquad \lambda_4 = 2\alpha - 2\beta.$$

The eigenvectors corresponding to these eigenvalues are

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \qquad \text{for the eigenvalue } \lambda = 0,$$

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \text{for the eigenvalue } \lambda = 2\alpha + 2\beta,$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \text{for the eigenvalue } \lambda = 2\alpha - 2\beta.$$

Define the following matrices

$$T = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\alpha + 2\beta & 0 \\ 0 & 0 & 0 & 2\alpha - 2\beta \end{pmatrix}.$$

Then we have

$$T^{-1}AT = D.$$

Chapter 8 - Everyday Quizzes

1. The auxiliary equation of the differential equation

$$y'''' - 20y''' + 150y'' - 500y' + 625y = 0$$

is $(\lambda - 5)^4 = 0$. What is the solution of the differential equation?

$(A)\quad y(x) = C_1e^{3x} + C_2e^{4x} + C_3e^{5x} + C_4e^{6x}.$

$(B)\quad y(x) = C_1e^{7x} + C_2e^{8x} + C_3e^{9x} + C_4e^{10x}.$

$(C)\quad y(x) = C_1e^{5x} + C_2xe^{5x} + C_3x^2e^{5x} + C_4x^3e^{5x}.$

$(D)\quad y(x) = C_1e^{4x} + C_2xe^{4x} + C_3x^2e^{4x} + C_4x^3e^{4x}.$

2. Consider the differential equation

$$y'' + 4y = 16x + 16.$$

Which of the following is the general solution of the differential equation?

$(A)\quad y(x) = C_1\cos(2x) + C_2\sin(2x) + 2x + 2.$

$(B)\quad y(x) = C_1\cos(2x) + C_2\sin(2x) + 4x + 4.$

$(C)\quad y(x) = C_1e^{2x} + C_2e^{-2x} + 4x + 4.$

$(D)\quad y(x) = C_1e^{2x} + C_2e^{-2x} + 2x + 2.$

3. Consider the differential equation

$$y'' - 11y' + 30y = 14e^{7x}.$$

Which of the following is the general solution of the differential equation?

$$(A) \quad y(x) = C_1e^{5x} + C_2e^{6x} + 7e^{7x}.$$

$$(B) \quad y(x) = C_1e^{5x} + C_2e^{6x} + 7xe^{7x}.$$

$$(C) \quad y(x) = C_1e^{5x} + C_2e^{6x} + 7x^2e^{7x}.$$

$$(D) \quad y(x) = C_1e^{5x} + C_2e^{6x} + 7x^3e^{7x}.$$

4. Consider the differential equation

$$y'' + \alpha^2 y = (\alpha^2 - \beta^2)\cos(\beta x) + (\alpha^2 - \beta^2)\sin(\beta x),$$

where $\alpha > 0$ and $\beta > 0$ are positive constants, and $\alpha > \beta$. Which of the following is the general solution of the differential equation?

$$(A) \quad y(x) = C_1\cos(\alpha x) + C_2\sin(\alpha x) + \alpha\cos(\beta x) + \alpha\sin(\beta x).$$

$$(B) \quad y(x) = C_1\cos(\alpha x) + C_2\sin(\alpha x) + \beta\cos(\beta x) + \beta\sin(\beta x).$$

$$(C) \quad y(x) = C_1\cos(\alpha x) + C_2\sin(\alpha x) + x\cos(\beta x) + x\sin(\beta x).$$

$$(D) \quad y(x) = C_1\cos(\alpha x) + C_2\sin(\alpha x) + \cos(\beta x) + \sin(\beta x).$$

5. Consider the differential equation

$$y'' + 100y = 20\cos(10x) - 20\sin(10x).$$

Which of the following is the general solution of the differential equation?

(A) $y(x) = C_1e^{2x} + C_2e^{-2x} + \cos(10x) + \sin(10x).$

(B) $y(x) = C_1e^{2x} + C_2e^{-2x} + x\cos(10x) + x\sin(10x).$

(C) $y(x) = C_1\cos(10x) + C_2\sin(10x) + \cos(10x) + \sin(10x).$

(D) $y(x) = C_1\cos(10x) + C_2\sin(10x) + x\cos(10x) + x\sin(10x).$

6. Consider the differential equation

$$y'' - 7y' + 12y = 8(1+x)e^{4x}.$$

Which of the following is the general solution of the differential equation?

(A) $y(x) = C_1e^{3x} + C_2e^{4x} + 4e^{4x}.$

(B) $y(x) = C_1e^{3x} + C_2e^{4x} + 4xe^{4x}.$

(C) $y(x) = C_1e^{3x} + C_2e^{4x} + 4x^2e^{4x}.$

(D) $y(x) = C_1e^{3x} + C_2e^{4x} + 4x^3e^{4x}.$

7. Consider the differential equation

$$y'' - 2\alpha y' + \alpha^2 y = \beta(m+2)(m+1)x^m e^{\alpha x},$$

where $\alpha > 0$, $\beta > 0$ and $m > 0$ are positive constants. Which of the following is the general solution of the differential equation?

$(A) \quad y(x) = C_1 e^{\alpha x} + C_2 x e^{\alpha x} + \beta x^m e^{\alpha x}.$

$(B) \quad y(x) = C_1 e^{\alpha x} + C_2 x e^{\alpha x} + \beta x^{m+1} e^{\alpha x}.$

$(C) \quad y(x) = C_1 e^{\alpha x} + C_2 x e^{\alpha x} + \beta x^{m+2} e^{\alpha x}.$

$(D) \quad y(x) = C_1 e^{\alpha x} + C_2 x e^{\alpha x} + \beta x^{m+3} e^{\alpha x}.$

8. Consider the differential equation

$$y'' - 2\alpha y' + \alpha^2 y = \beta(m+2)(m+1)x^m e^{\alpha x} + \gamma(n+4)(n+3)x^{n+2} e^{\alpha x},$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $m > 0$ and $n > 0$ are positive constants. Which of the following is the general solution of the differential equation?

$(A) \quad y(x) = C_1 e^{\alpha x} + C_2 x e^{\alpha x} + \beta x^{m-1} e^{\alpha x} + \gamma x^{n+1} e^{\alpha x}.$

$(B) \quad y(x) = C_1 e^{\alpha x} + C_2 x e^{\alpha x} + \beta e^m e^{\alpha x} + \gamma x^{n+2} e^{\alpha x}.$

$(C) \quad y(x) = C_1 e^{\alpha x} + C_2 x e^{\alpha x} + \beta x^{m+1} e^{\alpha x} + \gamma x^{n+3} e^{\alpha x}.$

$(D) \quad y(x) = C_1 e^{\alpha x} + C_2 x e^{\alpha x} + \beta x^{m+2} e^{\alpha x} + \gamma x^{n+4} e^{\alpha x}.$

9.

10.

11. Consider the differential equation

$$y'' - 2y' + y = 2e^x.$$

Which of the following is the general solution?

$$(A) \quad y(x) = C_1 e^x + C_2 x e^x + x e^x.$$

$$(B) \quad y(x) = C_1 e^x + C_2 x e^x + x^2 e^x.$$

$$(C) \quad y(x) = C_1 e^x + C_2 x e^x + x^3 e^x.$$

$$(D) \quad y(x) = C_1 e^x + C_2 x e^x + x^4 e^x.$$

12. Consider the differential equation

$$y'' - 10y' + 25y = 12xe^{5x}.$$

Which of the following is the general solution of the differential equation?

$$(A) \quad y(x) = C_1 e^{5x} + C_2 x e^{5x} + x^2 e^{5x}.$$

$$(B) \quad y(x) = C_1 e^{5x} + C_2 x e^{5x} + x^3 e^{5x}.$$

$$(C) \quad y(x) = C_1 e^{5x} + C_2 x e^{5x} + 2x^2 e^{5x}.$$

$$(D) \quad y(x) = C_1 e^{5x} + C_2 x e^{5x} + 2x^3 e^{5x}.$$

13. Consider the differential equation

$$y'' + 2y' + y = 2e^{-x}.$$

Which of the following is the general solution of the differential equation?

$$(A) \quad y(x) = C_1 e^{-x} + C_2 x e^{-x} + e^{-x}.$$

$$(B) \quad y(x) = C_1 e^{-x} + C_2 x e^{-x} + x e^{-x}.$$

$$(C) \quad y(x) = C_1 e^{-x} + C_2 x e^{-x} + x^2 e^{-x}.$$

$$(D) \quad y(x) = C_1 e^{-x} + C_2 x e^{-x} + x^3 e^{-x}.$$

14. Consider the differential equation

$$y'' + 4y' + 4y = 14\cos x - 2\sin x.$$

Which of the following is the general solution of the differential equation?

$$(A) \quad y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + x\cos x + x\sin x.$$

$$(B) \quad y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + 2\cos x + 2\sin x.$$

$$(C) \quad y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + 3\cos x + 3\sin x.$$

$$(D) \quad y(x) = C_1 e^{-2x} + C_2 x e^{-2x} + 4\cos x + 4\sin x.$$

15. Consider the differential equation

$$y'' + 4y = 8\cos(2x) - 8\sin(2x).$$

Which of the following is the general solution of the differential equation?

(A) $y(x) = C_1\cos(2x) + C_2\sin(2x) + 2x\cos(2x) + 2x\sin(2x)$.

(B) $y(x) = C_1\cos(2x) + C_2\sin(2x) - 2x\cos(2x) - 2x\sin(2x)$.

(C) $y(x) = C_1\cos(2x) + C_2\sin(2x) + x^2\cos(2x) + x^2\sin(2x)$.

(D) $y(x) = C_1\cos(2x) + C_2\sin(2x) - x^2\cos(2x) - x^2\sin(2x)$.

Chapter 9 - Everyday Quizzes

Let $\alpha \neq 0$, $\beta \neq 0$ and $\gamma \neq 0$ be real nonzero constants, such that $\alpha^2 > \beta^2$.

1. Solve the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{pmatrix}\mathbf{u}.$$

2. Solve the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}\mathbf{u}.$$

3. Solve the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix}\mathbf{u}$$

4. Let α and β be real nonzero constants, such that $\alpha^2 > \beta^2$. Solve the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix}\mathbf{u}.$$

5. Consider the homogeneous linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha^2 & -\beta^2 & -\beta^2 \\ -\beta^2 & \alpha^2 & -\beta^2 \\ -\beta^2 & -\beta^2 & \alpha^2 \end{pmatrix} \mathbf{u},$$

where α and β are real nonzero constant. Which of the following is the solution of the system of equations?

$$(A) \quad \mathbf{u}(t) = C_1 e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3 e^{(\alpha^2-2\beta^2)t} \left(\begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \right.$$

$$(B) \quad \mathbf{u}(t) = C_1 e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 t e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3 e^{(\alpha^2-2\beta^2)t} \left(\begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \right.$$

$$(C) \quad \mathbf{u}(t) = C_1 e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 e^{(\alpha^2-2\beta^2)t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3 e^{(\alpha^2-2\beta^2)t} \left(\begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \right.$$

$$(D) \quad \mathbf{u}(t) = C_1 e^{(\alpha^2+\beta^2)t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + C_2 t e^{(\alpha^2-2\beta^2)t} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + C_3 e^{(\alpha^2-2\beta^2)t} \Bigg($$

6. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha^2 & \beta^2 \\ -\beta^2 & \alpha^2 \end{pmatrix}\mathbf{u},$$

where α and β are real nonzero constants. Which of the following is the solution of the differential equation?

(A) $\quad \mathbf{u}(t) = C_1 e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ \sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}.$

(B) $\quad \mathbf{u}(t) = C_1 e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ -\sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}.$

(C) $\quad \mathbf{u}(t) = C_1 e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ \sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t}\begin{pmatrix} -\sin(\beta^2 t) \\ \cos(\beta^2 t) \end{pmatrix}.$

(D) $\quad \mathbf{u}(t) = C_1 e^{\alpha^2 t}\begin{pmatrix} \cos(\beta^2 t) \\ \sin(\beta^2 t) \end{pmatrix} + C_2 e^{\alpha^2 t}\begin{pmatrix} \sin(\beta^2 t) \\ -\cos(\beta^2 t) \end{pmatrix}.$

7. Consider the nonhomogeneous system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \mathbf{u} - \begin{pmatrix} 4 \\ 8 \\ 12 \end{pmatrix}.$$

Which of the following is the general solution of the system of differential equations?

$$(A) \quad \mathbf{u}(t) = C_1 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-2t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$(B) \quad \mathbf{u}(t) = C_1 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-2t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

$$(C) \quad \mathbf{u}(t) = C_1 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-2t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$(D) \quad \mathbf{u}(t) = C_1 e^{4t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-2t} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$$

8. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mathbf{u}$$

$$+ \ \beta\cos(\beta t)e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} - \beta\sin(\beta t)e^{\beta t}\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix},$$

where α and β are real nonzero constants. Which of the following is the general solution of the system of differential equations?

(A) $$\mathbf{u}(t) = C_1 e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + C_2 e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

$$+ \ \sin(\beta t)e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + \beta\cos(\beta t)e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}.$$

(B) $$\mathbf{u}(t) = C_1 e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + C_2 e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

$$+ \ \beta\sin(\beta t)e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + \cos(\beta t)e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}.$$

(C) $$\mathbf{u}(t) = C_1 e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + C_2 e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

$$+ \ \beta\sin(\beta t)e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + \beta\cos(\beta t)e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}.$$

(D) $$\mathbf{u}(t) = C_1 e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + C_2 e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

$$+ \ \sin(\beta t)e^{\alpha t}\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + \cos(\beta t)e^{\alpha t}\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}.$$

9. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \begin{pmatrix} \beta & -\beta & \beta \\ -\beta & \beta & \beta \\ \beta & \beta & \beta \end{pmatrix} \mathbf{u}$$

$$- \quad \beta \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} - \beta e^{\beta t} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

where β is a real nonzero constant. Which of the following is the general solution of the system of equations?

$$(A) \qquad \mathbf{u}(t) = C_1 e^{2\beta t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{2\beta t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-\beta t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$+ \ \beta \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + e^{\beta t}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$(B) \qquad \mathbf{u}(t) = C_1 e^{2\beta t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{2\beta t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-\beta t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$+ \ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \beta e^{\beta t}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$(C) \qquad \mathbf{u}(t) = C_1 e^{2\beta t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{2\beta t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-\beta t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$+ \ \beta \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \beta e^{\beta t}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

$$(D) \qquad \mathbf{u}(t) = C_1 e^{2\beta t}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + C_2 e^{2\beta t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-\beta t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$+ \ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + e^{\beta t}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Chapter 10 - Everyday Quizzes

1. Let

$$f(t) = e^{\alpha t}\cos(\beta t).$$

Which of the following is the Laplace transform of f?

$$(A) \quad \frac{\alpha}{(\lambda-\alpha)^2+\beta^2}.$$

$$(B) \quad \frac{\beta}{(\lambda-\alpha)^2+\beta^2}.$$

$$(C) \quad \frac{\lambda-\alpha}{(\lambda-\alpha)^2+\beta^2}.$$

$$(D) \quad \frac{\lambda-\beta}{(\lambda-\alpha)^2+\beta^2}.$$

2. Let

$$f(t) = e^{\alpha t}\sin(\beta t).$$

Which of the following is the Laplace transform of f?

$$(A) \quad \frac{\alpha}{(\lambda-\alpha)^2+\beta^2}.$$

$$(B) \quad \frac{\beta}{(\lambda-\alpha)^2+\beta^2}.$$

$$(C) \quad \frac{\lambda-\alpha}{(\lambda-\alpha)^2+\beta^2}.$$

$$(D) \quad \frac{\lambda-\beta}{(\lambda-\alpha)^2+\beta^2}.$$

3. Let

$$f(t) = t^m e^{\alpha t}.$$

Which of the following is the Laplace transform of f?

$$(A) \quad \frac{m!}{(\lambda - \alpha)^m}.$$

$$(B) \quad \frac{m!}{(\lambda - \alpha)^{m+1}}.$$

$$(C) \quad \frac{(m+1)!}{(\lambda - \alpha)^m}.$$

$$(B) \quad \frac{(m+1)!}{(\lambda - \alpha)^{m+1}}.$$

4. Let

$$f(t) = \frac{1}{t^{1/2}}.$$

Which of the following is the Lapalce transform of f?

$$(A) \quad \left(\frac{\pi}{\lambda}\right)^{1/2}.$$

$$(B) \quad \left(\frac{\pi}{\lambda}\right)^{1/4}.$$

$$(C) \quad \left(\frac{\pi}{\lambda}\right)^{2}.$$

$$(D) \quad \left(\frac{\pi}{\lambda}\right)^{3}.$$

Chapter 6 - Everyday Quizzes

1. Let $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$ be vectors. Show that

$$\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2,$$

if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

2. Let $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$ be any real vectors. Show that

$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2.$$

3. Let $\mathbf{v}_1 \in \mathbb{R}^n$ and $\mathbf{v}_2 \in \mathbb{R}^n$ be linearly independent vectors. Define

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{v}_1, \\ \mathbf{a}_2 &= \mathbf{v}_2 - \frac{\mathbf{a}_1 \cdot \mathbf{v}_2}{\mathbf{a}_1 \cdot \mathbf{a_1}}\mathbf{a}_1. \end{aligned}$$

Show that

$$\{\mathbf{a}_1, \mathbf{a}_2\}$$

is an orthogonal set in $\mathbb{R}^n$.

4. Let

$$\mathbf{a} = \begin{pmatrix} 25 \\ 24 \\ -7 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 25 \\ -24 \\ 7 \end{pmatrix}.$$

Show that $\mathbf{a} \perp \mathbf{b}$.

5. Let

$$\mathbf{a} = \begin{pmatrix} 13 \\ 12 \\ -5 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 13 \\ -12 \\ 5 \end{pmatrix}.$$

What is the angle θ between $\mathbf{a}$ and $\mathbf{b}$?

6. Let

$$S = \{\mathbf{a}_1, \qquad \mathbf{a}_2, \qquad \mathbf{a}_3, \qquad \mathbf{a}_4\},$$

where

$$\mathbf{a}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{a}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \qquad \mathbf{a}_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \qquad \mathbf{a}_4 = \frac{1}{2}\left(\begin{matrix} 1 \\ - \\ 1 \\ - \end{matrix}\right.$$

Show that S is an orthonormal set in $\mathbb{R}^4$. Is S linearly independent?

7. Let

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{a}_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Let

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}.$$

Find the projection of the vector $\mathbf{v}$ onto the subspace

$$\text{span}\ \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}.$$

8. Let $\mathbf{v} \in \mathbb{R}^n$ be a real nonzero column vector. Let $\mathbf{v}^T$ be the row vector by taking the transpose of the vector $\mathbf{v}$. Define the

matrix

$$P = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v} \cdot \mathbf{v}}.$$

Prove that P is a projection matrix. Namely, prove that $P^2 = P$.

9. Let

$$P = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Prove that P is a projection matrix.

10. Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

Use the set of vectors

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

and the Gram-Schmidt process to find an orthogonal set

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}.$$

11. Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants. Solve the system of equations for a least square solution

$$\begin{pmatrix} \alpha & \alpha & -\beta \\ -\beta & -\beta & \alpha \\ \alpha & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \\ \beta \\ \alpha \end{pmatrix}.$$

Chapter 7 - Erevyday Quizzes

Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants, such that $\alpha^2 > \beta^2$. For each of the following matrices (1) - (6), find an orthonormal matrix U, such that

$$U^T U = I, \qquad U^T A U = \text{ a diagonal matrix}$$

where the eigenvalues on the main diagonal are increasing as the index increases.

$$(1) \quad A = \begin{pmatrix} \alpha & -\beta \\ -\beta & \alpha \end{pmatrix}$$

$$(2) \quad A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$(3) \quad A = \begin{pmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{pmatrix}$$

$$(4) \quad A = \begin{pmatrix} \alpha & \beta & \beta & \beta \\ \beta & \alpha & \beta & \beta \\ \beta & \beta & \alpha & \beta \\ \beta & \beta & \beta & \alpha \end{pmatrix}$$

$$(5) \quad A = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix}$$

$$(6) \quad A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

Let $\alpha \neq 0$ and $\beta \neq 0$ be real nonzero constants. For each of the following matrices, find conditions on α and β, so that the matrix

A is positively definite.

$$(7)\qquad A=\begin{pmatrix}\alpha & -\beta & -\beta & -\beta\\ -\beta & \alpha & -\beta & -\beta\\ -\beta & -\beta & \alpha & -\beta\\ -\beta & -\beta & -\beta & \alpha\end{pmatrix}$$

$$(8)\qquad A=\begin{pmatrix}\alpha & -\beta & \alpha & -\beta\\ -\beta & \alpha & -\beta & \alpha\\ \alpha & -\beta & \alpha & -\beta\\ -\beta & \alpha & -\beta & \alpha\end{pmatrix}$$

$$(9)\qquad A=\begin{pmatrix}\alpha & -\beta & -\beta\\ -\beta & \alpha & -\beta\\ -\beta & -\beta & \alpha\end{pmatrix}$$

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$(+)t\alpha\alpha\alpha\alpha\alpha\alpha(2\alpha^2+2\beta^2)t\alpha\gamma\gamma\gamma\lambda_1\cdots\lambda_1\cdots$

$\lambda_n\lambda_n$

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$$\left(\quad \right), \left(\quad \right) \left(\quad \right) \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \{\} \alpha\alpha\alpha\alpha\beta\beta\beta\beta$$

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$$\mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \mathbf{v} =, \{\} \alpha\alpha\alpha\alpha\beta\beta\beta\beta,$$

$$(A) \quad \mathbf{u}(t) = C_1 e^{\beta t} + C_2 e^{\beta t} + C_3 e^{\beta t} + e^{\beta t}.$$

Mathematics 23
Answers of Mathematics 205:

(A) (B) (C) (A) (B) (C) (D) (A) (B) (C) 01 - 10

(B) (C) (D) (A) (B) (C) (D) (B) (C) (D) 11 - 20

(A) (B) (C) (A) (B) (C) (D) (A) (B) (C) 21 - 30

(B) (C) (D) (A) (B) (C) (D) (B) (C) (D) 31 - 40

(A) (A) (A) (A) (A) (D) (D) (D) (D) (D) 41 - 50

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Let

$$(1)\qquad \mathbf{a} = \begin{pmatrix} 7 \\ 24 \\ 25 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 7 \\ 24 \\ -25 \end{pmatrix},$$

$$(2)\qquad \mathbf{a} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}.$$

Compute the dot product $\mathbf{a} \cdot \mathbf{b}$ and the cross product $\mathbf{a} \times \mathbf{b}$ in each case.

Which of the following statements is true?

(A) $|\mathbf{a} \cdot \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$

(B) $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$

(C) $|\mathbf{a} + \mathbf{b}|^2 \le |\mathbf{a}|^2 + |\mathbf{b}|^2, \qquad |\mathbf{a} - \mathbf{b}|^2 \le |\mathbf{a}|^2 + |\mathbf{b}|^2$

(D) $|\mathbf{a} + \mathbf{b}|^2 \le 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2, \qquad |\mathbf{a} - \mathbf{b}|^2 \le 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$

Define the function

$$f(x, y) = -(x^2 + y^2 - 4)(x^2 + y^2 - 100),$$

on

$$1 \le x^2 + y^2 \le 900.$$

Find the absolute maximum and the absolute minimum of the function f.

Let $m > 2$ be a positive constant. Define the function

$$f(x, y) = (x^2 + y^2)^{2m} e^{-(x^2+y^2)},$$

on

$$m \le x^2 + y^2 \le 10m.$$

Find the absolute maximum and the absolute minimum of the function f.

Find the maximum and minimum of the function

$$f(x, y, z) = x + y + z,$$

subject to the condition

$$x^2 + y^2 + z^2 = 300.$$

1. Let $R > 0$ be a positive constant. Solve the following triple integrals by using cylindrical coordinates

$$(1) \quad \int_0^R \int_{-\sqrt{R^2-x^2}}^0 \int_0^{\sqrt{x^2+y^2}} 2ze^{-(x^2+y^2)} \mathrm{d}z \mathrm{d}y \mathrm{d}x$$

$$(2) \quad \int_{-R}^0 \int_{-\sqrt{R^2-x^2}}^0 \int_0^{10} 2ze^{-(x^2+y^2)} \mathrm{d}z \mathrm{d}y \mathrm{d}x$$

2. Let $R > 0$ be a positive constant. Solve the following triple integrals by using spherical coordinates

$$(1) \quad \int_0^R \int_0^{\sqrt{R^2-x^2}} \int_0^{\sqrt{R^2-x^2-y^2}} 2(x^2 + y^2 + z^2)^{1/2} e^{-(x^2+y^2+z^2)} \mathrm{d}z \mathrm{d}y \mathrm{d}x$$

$$(2) \quad \int_{-R}^0 \int_{-\sqrt{R^2-x^2}}^0 \int_{-\sqrt{R^2-x^2-y^2}}^0 2(x^2 + y^2 + z^2)^{3/2} e^{-(x^2+y^2+z^2)} \mathrm{d}z \mathrm{d}y \mathrm{d}x$$

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Example 1. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix},$$

where $f(u,v)$ and $g(u,v)$ are continuous functions of u and v. Suppose that

$$\begin{aligned} &f(0,0)=0, \qquad g(0,0)=0, \\ &2uf(u,v)+2vg(u,v) \leq -C(|u|^2+|v|^2)^{m+1}, \end{aligned}$$

for all $(u,v) \in \mathbb{R}^2$, where $C>0$ and $m>0$ are fixed positive constants. Show that the fixed point $(u,v)=(0,0)$ is asymptotically stable. Multiplying the first equation by $2u$ and multiplying the second equation by $2v$, adding the results together, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\{[u(t)]^2+[v(t)]^2\} = 2u(t)f(u(t),v(t))+2v(t)g(u(t),v(t)) \\ \leq\ & -C\{[u(t)]^2+[v(t)]^2\}^{m+1}, \end{aligned}$$

for all t. Define

$$E(t)=|u(t)|^2+|v(t)|^2,$$

on $(0,\infty)$. Then

$$E'(t) \leq -C[E(t)]^{m+1}.$$

It is easy to show that if $E(t_0)=0$, then $E(t)=0$, for all $t>t_0$, where $t_0 \geq 0$ is a constant. Without loss of generality, let

$$E(t)=|u(t)|^2+|v(t)|^2>0.$$

The above inequality is equivalent to

$$-\frac{mE'(t)}{[E(t)]^{m+1}} \geq mC.$$

That is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{[E(t)]^m}\right\} \geq mC.$$

Integrating this inequality with respect to t gives

$$\frac{1}{[E(t)]^m} \geq \frac{1}{[E(0)]^m} + mCt.$$

In another word, we have

$$E(t) \leq \left\{\frac{[E(0)]^m}{1+[E(0)]^m mCt}\right\}^{1/m},$$

for all $t > 0$. Therefore, we have

$$\lim_{t\to\infty} E(t) = 0.$$

This completes the proof that the fixed point $(u, v) = (0, 0)$ is asymptotically stable. □

Example 2. Consider the real system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix},$$

where $f(u, v)$ and $g(u, v)$ are continuous functions of (u, v), such that

$$\begin{aligned} &f(0,0) = 0, \qquad g(0,0) = 0, \\ &2uf(u,v) + 2vg(u,v) \leq -C(|u|^{2m+2} + |v|^{2m+2}), \end{aligned}$$

for all $(u, v) \in \mathbb{R}^2$, where $C > 0$ and $m > 0$ are fixed positive constants. Prove that the fixed point $(u, v) = (0, 0)$ is asymptotically stable.

Proof. Multiplying the first equation by $2u$ and multiplying the second equation by $2v$, adding the results together, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\{[u(t)]^2+[v(t)]^2\} = 2u(t)f(u(t),v(t))+2v(t)g(u(t),v(t))\\
\leq\ & -C\{[u(t)]^{2m+2}+[v(t)]^{2m+2}\}\\
\leq\ & -C_0\{[u(t)]^2+[v(t)]^2\}^{m+1},
\end{aligned}$$

for all $t>0$, where $C_0>0$ is a positive constant.

Let

$$E(t)=|u(t)|^2+|v(t)|^2.$$

Then

$$E'(t)\leq -C_0[E(t)]^{m+1}.$$

It is very easy to show that if $E(t_0)=0$, then $E(t)=0$, for all $t>t_0$, for some constant $t_0\geq 0$. Without loss of generality, let $E(t)>0$, for all $t>0$. Now we have

$$-\frac{mE'(t)}{[E(t)]^{m+1}}\geq mC_0.$$

Equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{[E(t)]^m}\right\}\geq mC_0.$$

Integrating this inequality with respect to t, we have

$$\frac{1}{[E(t)]^m}\geq\frac{1}{[E(0)]^m}+mC_0t.$$

That is

$$E(t)\leq\left\{\frac{[E(0)]^m}{1+[E(0)]^m mC_0t}\right\}^{1/m},$$

for all $t > 0$. Therefore, we have the limit

$$\lim_{t\to\infty} E(t) = 0.$$

The stability of the fixed point $(u, v) = (0, 0)$ is completed. $\square$

Due November 4

1. Find two examples of continuous functions $f(u, v)$ and $g(u, v)$, such that

$$2uf(u, v) + 2vg(u, v) \leq -C(|u|^2 + |v|^2)^{m+1},$$

for all $(u, v) \in \mathbb{R}^2$, where $m > 0$ and $C > 0$ fixed are positive constants.

2. Find two examples of continuous functions $f(u, v)$ and $g(u, v)$, such that

$$2uf(u, v) + 2vg(u, v) \leq -C(|u|^{2m+2} + |v|^{2m+2}),$$

for all $(u, v) \in \mathbb{R}^2$, where $m > 0$ and $C > 0$ are fixed positive constants.

3. Let $m > 0$ be a fixed positive constant. Find a positive constant $C = C(m) > 0$, such that

$$|u|^{2m+2} + |v|^{2m+2} \geq C(|u|^2 + |v|^2)^{m+1},$$

for all $(u, v) \in \mathbb{R}^2$.

4. Show that the fixed point $u = 0$ of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u + u = u^2 + \sin(u^4) + \ln(1 + u^8)$$

is exponentially stable.

5. Consider the real system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + B[R^2 - (u^2 + v^2)]\begin{pmatrix} u \\ v \end{pmatrix},$$

where $A \neq 0$, $B \neq 0$ and $R \neq 0$ are real constants. Find the fixed point of the system and determine its stability or instability.

6. Let

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & 0 & \alpha \end{pmatrix}, \qquad B = \begin{pmatrix} \beta & 1 & 0 & 0 & 0 \\ 0 & \beta & 1 & 0 & 0 \\ 0 & 0 & \beta & 1 & 0 \\ 0 & 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix},$$

where α and β are real constants.

(1) Let $\alpha = -2 - \varepsilon + 5\mathrm{i}$ and $\beta = -1 - \varepsilon + 10\mathrm{i}$, where $0 < \varepsilon \ll 1$ is a sufficiently small positive constant. Show that

$$\|\exp(At)\| \leq C\exp(-2t),$$
$$\|\exp(Bt)\| \leq C\exp(-t),$$

for all $t > 0$, where $C > 0$ is a positive constant.

(2) Let $\alpha = 1 + \varepsilon - 4\mathrm{i}$ and $\beta = 2 + \varepsilon - 8\mathrm{i}$, where $0 < \varepsilon \ll 1$ is a sufficiently small positive constant. Show that

$$\|\exp(At)\| \leq C\exp(t),$$
$$\|\exp(Bt)\| \leq C\exp(2t),$$

for all $t < 0$, where $C > 0$ is a positive constant.

Can you find the minimum value of the constant C in each of the above four estimates?

Let A, B and R be real nonzero constants. Consider the initial value problem for the nonlinear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + B[R^2 - u^2 - v^2]\begin{pmatrix} u \\ v \end{pmatrix},$$
$$\begin{pmatrix} u(t_0) \\ v(t_0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

Show that if $\begin{pmatrix} u(t_1) \\ v(t_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for some t_1, then

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

for all t.

Show that if A, B, then there exists a unique stable periodic solution to the system of differential equations.

The Final Exam of Mathematics 22 - 2022

The Final Exam is Three Hours long. The Final Exam is worth 200 points. There are fifty multiple choice problems. Each problem is worth four points. In each problem, there are four choices marked with (A), (B), (C) and (D). Only one choice is correct. Pick up the one you think is correct and put the answer inside ( ). If you do not know the answer to a problem, you may make a reasonable best possible choice. No supporting work is necessary for any problem.

In the Final Exam, $\alpha > 0$, $\beta > 0$, $R > 0$ are positive constants. They may satisfy some conditions in certain problems.

No calculator, cell phone, computer, i-pad or i-touch is allowed in this exam. You are not allowed to use the assistance of anybody, any website, any resources. Anybody who cheats in the Final Exam will receive an **F** as the Final Grade! Firm!

1. Given the smooth function $y = f(x)$, where $a \le x \le b$ and the region R below the graph of the function. Which of the following formulas represents the volume V of the solid generated by rotating about the x-axis the region R?

$$(A) \quad V = \int_a^b \sqrt{1 + [f'(x)]^2}\mathrm{d}x.$$

$$(B) \quad V = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\mathrm{d}x.$$

$$(C) \quad V = \int_a^b \pi[f(x)]^2\mathrm{d}x.$$

$$(D) \quad V = \int_a^b 2\pi x f(x)\mathrm{d}x.$$

2. Given the smooth function $f(x) = \arctan x$, where $0 \leq x \leq 1$, and the region R below the graph of the function. Which of the following quantities represents the volume V of the solid generated by rotating about the y-axis the region R?

$$(A) \quad V = \int_0^1 \pi(\arctan x)^2 \mathrm{d}x.$$

$$(B) \quad V = \int_0^1 \pi \arctan x \mathrm{d}x.$$

$$(C) \quad V = \int_0^1 2\pi \arctan x \mathrm{d}x.$$

$$(D) \quad V = \int_0^1 2\pi x \arctan x \mathrm{d}x.$$

3. Given the smooth curve $y = f(x)$, where $a \leq x \leq b$. Which of the following formulas represents the arclength L of the curve?

$$(A) \quad L = \int_a^b \pi[f(x)]^2 \mathrm{d}x.$$

$$(B) \quad L = \int_a^b 2\pi x f(x) \mathrm{d}x.$$

$$(C) \quad L = \int_a^b \sqrt{1 + [f'(x)]^2} \mathrm{d}x.$$

$$(D) \quad L = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2} \mathrm{d}x.$$

4. Which of the following formulas represents the area S of the surface generated by rotating the smooth curve $y = f(x)$, where $a \leq x \leq b$, about the x-axis?

$$(A) \quad S = \int_a^b \pi[f(x)]^2 \mathrm{d}x.$$

$$(B) \quad S = \int_a^b 2\pi x f(x) \mathrm{d}x.$$

$$(C) \quad S = \int_a^b \sqrt{1 + [f'(x)]^2} \mathrm{d}x$$

$$(D) \quad S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \mathrm{d}x.$$

5. Consider the separable differential equation

$$y \cos y \mathrm{d}y = \ln(x + \sqrt{x^2 + \alpha^2})\mathrm{d}x.$$

Which of the following is the general implicit solution?

(A) $\quad y \sin y + \cos y = +x \ln(x + \sqrt{x^2 + \alpha^2}) + \sqrt{x^2 + \alpha^2} + C.$

(B) $\quad y \sin y + \cos y = +x \ln(x + \sqrt{x^2 + \alpha^2}) - \sqrt{x^2 + \alpha^2} + C.$

(C) $\quad y \sin y + \cos y = -x \ln(x + \sqrt{x^2 + \alpha^2}) + \sqrt{x^2 + \alpha^2} + C.$

(D) $\quad y \sin y + \cos y = -x \ln(x + \sqrt{x^2 + \alpha^2}) - \sqrt{x^2 + \alpha^2} + C.$

6. Which of the following differential equations has the general solution $y = e^{-x^2}(x^2 + C)$?

$$\begin{aligned} &(A) \quad y' + y = 4x^3 e^{-x}. \\ &(B) \quad y' + 2xy = 2xe^{-x^2}. \\ &(C) \quad y' - 2xy = 2xe^{x^2}. \\ &(D) \quad xy' + y = 2x. \end{aligned}$$

1. The integral

$$\int x \cos x \mathrm{d}x$$

is equal to which of the following functions?

$$\begin{array}{ll} (A) & x\sin(x) + \cos x + C. \\ (B) & x\sin(x) - \cos x + C. \\ (C) & x\cos(x) + \sin x + C. \\ (D) & x\cos(x) - \sin x + C. \end{array}$$

2. The integral

$$\int \arcsin x \mathrm{d}x$$

is equal to which of the following functions?

$$(A) \quad x \arcsin x + \sqrt{1-x^2} + C.$$

$$(B) \quad x \arcsin x - \sqrt{1-x^2} + C.$$

$$(C) \quad x \arcsin x + \arcsin x + C.$$

$$(D) \quad x \arcsin x - \arcsin x + C.$$

3. The integral

$$\int xe^x \mathrm{d}x$$

is equal to which of the following functions?

$$\begin{array}{ll}(A) & xe^x + e^x + C. \\ (B) & xe^x - e^x + C. \\ (C) & xe^x + x + C. \\ (D) & xe^x - x + C.\end{array}$$

4. The integral

$$\int \ln x \mathrm{d}x$$

is equal to which of the following functions?

$$\begin{array}{ll} (A) & x \ln x + \ln x + C. \\ (B) & x \ln x - \ln x + C. \\ (C) & x \ln x + x + C. \\ (D) & x \ln x - x + C. \end{array}$$

5. The integral

$$\int \arctan x \mathrm{d}x$$

is equal to which of the following functions?

$$\begin{aligned} &(A) \quad x \arctan x + 2\ln(1+x^2) + C. \\ &(B) \quad x \arctan x - 2\ln(1+x^2) + C. \\ &(C) \quad x \arctan x + \frac{1}{2}\ln(1+x^2) + C. \\ &(D) \quad x \arctan x - \frac{1}{2}\ln(1+x^2) + C. \end{aligned}$$

6. The integral

$$\int -\frac{1}{x^2}\ln x \mathrm{d}x$$

is equal to which of the following functions?

$$\begin{aligned}
&(A) \quad \frac{1}{x}\ln x + \frac{1}{x} + C.\\
&(B) \quad \frac{1}{x}\ln x - \frac{1}{x} + C.\\
&(C) \quad \frac{1}{x^2}\ln x + \frac{1}{x^2} + C.\\
&(D) \quad \frac{1}{x^2}\ln x - \frac{1}{x^2} + C.
\end{aligned}$$

7. The integral

$$\int (2+2x)\arctan x \mathrm{d}x$$

is equal to which of the following functions?

$$\begin{aligned}
&(A) \quad (1+2x+x^2)\arctan x + \ln(1+x^2) + x + C.\\
&(B) \quad (1+2x+x^2)\arctan x + \ln(1+x^2) - x + C.\\
&(C) \quad (1+2x+x^2)\arctan x - \ln(1+x^2) + x + C.\\
&(D) \quad (1+2x+x^2)\arctan x - \ln(1+x^2) - x + C.
\end{aligned}$$

8. The integral

$$\int -3x(\cos x)^2 \sin x \mathrm{d}x$$

is equal to which of the following functions?

$$(A) \quad x(\cos x)^3 + \frac{1}{3}(\sin x)^3 + \sin x + C.$$
$$(B) \quad x(\cos x)^3 + \frac{1}{3}(\sin x)^3 - \sin x + C.$$
$$(C) \quad x(\cos x)^3 - \frac{1}{3}(\sin x)^3 + \sin x + C.$$
$$(D) \quad x(\cos x)^3 - \frac{1}{3}(\sin x)^3 - \sin x + C.$$

9. Let $m \geq 1$ be any positive integer. The integral

$$\int (\sin x)^{m+2} \mathrm{d}x$$

is equal to which of the following functions?

(A) $\quad \dfrac{m+1}{m+2} \displaystyle\int (\sin x)^m \mathrm{d}x + \dfrac{m+1}{m+2} (\sin x)^{m+1} \cos x.$

(B) $\quad \dfrac{m+1}{m+2} \displaystyle\int (\sin x)^m \mathrm{d}x - \dfrac{m+1}{m+2} (\sin x)^{m+1} \cos x.$

(C) $\quad \dfrac{m+1}{m+2} \displaystyle\int (\sin x)^m \mathrm{d}x + \dfrac{1}{m+2} (\sin x)^{m+1} \cos x.$

(D) $\quad \dfrac{m+1}{m+2} \displaystyle\int (\sin x)^m \mathrm{d}x - \dfrac{1}{m+2} (\sin x)^{m+1} \cos x.$

Let $m \geq 1$ be any positive integer. Which of the following is true?

$$\int (\cos x)^{m+2} \mathrm{d}x = \frac{m+1}{m+2} \int (\cos x)^{m} \mathrm{d}x + \frac{1}{m+2} (\cos x)^{m+1} \sin x.$$

$$\int (\sec x)^{m+2} \mathrm{d}x = \frac{m}{m+1} \int (\sec x)^{m} \mathrm{d}x + \frac{1}{m+1} (\sec x)^{m} \tan x.$$

$$\int (\tan x)^{m+2} \mathrm{d}x = - \int (\tan x)^{m} \mathrm{d}x + \frac{1}{m+1} (\tan x)^{m+1}.$$

$$\int x^{m+1} e^{-x^2} \mathrm{d}x = -\frac{1}{2} x^{m} e^{-x^2} + \frac{m}{2} \int x^{m-1} e^{-x^2} \mathrm{d}x.$$

10. The integral

$$\int -\frac{2}{x^3}\arctan x \mathrm{d}x$$

is equal to which of the following functions?

$$(A) \quad \frac{1}{x^2}\arctan x + \arctan x + \frac{1}{x} + C.$$
$$(B) \quad \frac{1}{x^2}\arctan x + \arctan x - \frac{1}{x} + C.$$
$$(C) \quad \frac{1}{x^2}\arctan x - \arctan x + \frac{1}{x} + C.$$
$$(D) \quad \frac{1}{x^2}\arctan x - \arctan x - \frac{1}{x} + C.$$

11. The integral

$$\int \frac{2}{(1+x^2)^2}\mathrm{d}x$$

is equal to which of the following functions?

$$(A) \quad +\arctan x + \frac{x}{1+x^2} + C.$$

$$(B) \quad +\arctan x - \frac{x}{1+x^2} + C.$$

$$(C) \quad -\arctan x + \frac{x}{1+x^2} + C.$$

$$(D) \quad -\arctan x - \frac{x}{1+x^2} + C.$$

12. The integral

$$\int \sqrt{x^2 + \alpha^2} \mathrm{d}x$$

is equal to which of the following functions?

$$\begin{aligned}
&(A) \quad 2x\sqrt{x^2+\alpha^2} + 2\ln(x + \sqrt{x^2+\alpha^2}) + C.\\
&(B) \quad 2x\sqrt{x^2+\alpha^2} - 2\ln(x + \sqrt{x^2+\alpha^2}) + C.\\
&(C) \quad \frac{1}{2}x\sqrt{x^2+\alpha^2} + \frac{1}{2}\alpha^2\ln(x + \sqrt{x^2+\alpha^2}) + C.\\
&(D) \quad \frac{1}{2}x\sqrt{x^2+\alpha^2} - \frac{1}{2}\alpha^2\ln(x + \sqrt{x^2+\alpha^2}) + C.
\end{aligned}$$

13. Let $\alpha > 0$ be any positive constant. The integral

$$\int (x^2+\alpha^2)\sqrt{x^2+\alpha^2}\mathrm{d}x$$

is equal to which of the following functions?

(A) $\quad \frac{1}{4}x(x^2+\alpha^2)\sqrt{x^2+\alpha^2}+\frac{3}{8}\alpha^2 x\sqrt{x^2+\alpha^2}+\frac{3}{8}\alpha^4\ln(x+\sqrt{x^2+\alpha^2})+C.$

(B) $\quad \frac{1}{4}x(x^2+\alpha^2)\sqrt{x^2+\alpha^2}+\frac{3}{8}\alpha^2 x\sqrt{x^2+\alpha^2}-\frac{3}{8}\alpha^4\ln(x+\sqrt{x^2+\alpha^2})+C.$

(C) $\quad \frac{1}{4}x(x^2+\alpha^2)\sqrt{x^2+\alpha^2}-\frac{3}{8}\alpha^2 x\sqrt{x^2+\alpha^2}+\frac{3}{8}\alpha^4\ln(x+\sqrt{x^2+\alpha^2})+C.$

(D) $\quad \frac{1}{4}x(x^2+\alpha^2)\sqrt{x^2+\alpha^2}-\frac{3}{8}\alpha^2 x\sqrt{x^2+\alpha^2}-\frac{3}{8}\alpha^4\ln(x+\sqrt{x^2+\alpha^2})+C$

14. The integral

$$\int \sqrt{\alpha^2 - x^2}\mathrm{d}x$$

is equal to which of the following functions?

$$\begin{aligned}
&(A) \quad 2x\sqrt{\alpha^2 - x^2} + 2\alpha^2 \arcsin\left(\frac{x}{\alpha}\right) + C.\\
&(B) \quad 2x\sqrt{\alpha^2 - x^2} - 2\alpha^2 \arcsin\left(\frac{x}{\alpha}\right) + C.\\
&(C) \quad \frac{1}{2}x\sqrt{\alpha^2 - x^2} + \frac{1}{2}\alpha^2 \arcsin\left(\frac{x}{\alpha}\right) + C.\\
&(D) \quad \frac{1}{2}x\sqrt{\alpha^2 - x^2} - \frac{1}{2}\alpha^2 \arcsin\left(\frac{x}{\alpha}\right) + C.
\end{aligned}$$

15. The integral

$$\int \frac{6}{(x^2+15x+50)(x^2+15x+56)}\mathrm{d}x$$

is equal to which of the following functions?

$$(A) \quad \ln\left|\frac{x+5}{x+10}\right| + \frac{1}{5}\ln\left|\frac{x+7}{x+8}\right| + C.$$

$$(B) \quad \ln\left|\frac{x+5}{x+10}\right| - \frac{1}{5}\ln\left|\frac{x+7}{x+8}\right| + C.$$

$$(C) \quad \frac{1}{5}\ln\left|\frac{x+5}{x+10}\right| + \ln\left|\frac{x+7}{x+8}\right| + C.$$

$$(D) \quad \frac{1}{5}\ln\left|\frac{x+5}{x+10}\right| - \ln\left|\frac{x+7}{x+8}\right| + C.$$

16. The integral

$$\int \left[\frac{2}{x(1+x^2)} - \frac{1}{x^2(1+x^2)}\right] \mathrm{d}x$$

is equal to which of the following functions?

$$(A) \quad \ln\frac{x^2}{1+x^2} + \arctan x + \frac{1}{x} + C.$$

$$(B) \quad \ln\frac{x^2}{1+x^2} + \arctan x - \frac{1}{x} + C.$$

$$(C) \quad \ln\frac{x^2}{1+x^2} - \arctan x + \frac{1}{x} + C.$$

$$(D) \quad \ln\frac{x^2}{1+x^2} - \arctan x - \frac{1}{x} + C.$$

17. The integral

$$\int e^{\alpha x}\cos(\beta x)\mathrm{d}x$$

is equal to which of the following functions?

$(A)\quad (\alpha^2+\beta^2)e^{\alpha x}[\alpha\cos(\beta x)+\beta\sin(\beta x)]+C.$

$(B)\quad (\alpha^2+\beta^2)e^{\alpha x}[\alpha\cos(\beta x)-\beta\sin(\beta x)]+C.$

$(C)\quad \dfrac{1}{\alpha^2+\beta^2}e^{\alpha x}[\alpha\cos(\beta x)+\beta\sin(\beta x)]+C.$

$(D)\quad \dfrac{1}{\alpha^2+\beta^2}e^{\alpha x}[\alpha\cos(\beta x)-\beta\sin(\beta x)]+C.$

18. The integral

$$\int e^{\alpha x} \sin(\beta x) \mathrm{d}x$$

is equal to which of the following functions?

$$(A) \quad (\alpha^2 + \beta^2) e^{\alpha x}[\alpha \sin(\beta x) + \beta \cos(\beta x)] + C.$$

$$(B) \quad (\alpha^2 + \beta^2) e^{\alpha x}[\alpha \sin(\beta x) - \beta \cos(\beta x)] + C.$$

$$(C) \quad \frac{1}{\alpha^2 + \beta^2} e^{\alpha x}[\alpha \sin(\beta x) + \beta \cos(\beta x)] + C.$$

$$(D) \quad \frac{1}{\alpha^2 + \beta^2} e^{\alpha x}[\alpha \sin(\beta x) - \beta \cos(\beta x)] + C.$$

19. The integral

$$\int (\arcsin x)^2 \mathrm{d}x$$

is equal to which of the following functions?

(A) $x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x + 2x + C.$

(B) $x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C.$

(C) $x(\arcsin x)^2 - 2\sqrt{1-x^2}\arcsin x + 2x + C.$

(D) $x(\arcsin x)^2 - 2\sqrt{1-x^2}\arcsin x - 2x + C.$

20. The integral

$$\int (\arccos x)^2 \mathrm{d}x$$

is equal to which of the following functions?

(A) $\quad x(\arccos x)^2 + 2\sqrt{1-x^2}\arccos x + 2x + C.$

(B) $\quad x(\arccos x)^2 + 2\sqrt{1-x^2}\arccos x - 2x + C.$

(C) $\quad x(\arccos x)^2 - 2\sqrt{1-x^2}\arccos x + 2x + C.$

(D) $\quad x(\arccos x)^2 - 2\sqrt{1-x^2}\arccos x - 2x + C.$

21. The integral

$$\int (\arcsin x)(\arccos x)\mathrm{d}x$$

is equal to which of the following functions?

(A) $x(\arcsin x)(\arccos x) + \sqrt{1-x^2}(\arcsin x - \arccos x) + 2x + C.$

(B) $x(\arcsin x)(\arccos x) + \sqrt{1-x^2}(\arcsin x - \arccos x) - 2x + C.$

(C) $x(\arcsin x)(\arccos x) - \sqrt{1-x^2}(\arcsin x - \arccos x) + 2x + C.$

(D) $x(\arcsin x)(\arccos x) - \sqrt{1-x^2}(\arcsin x - \arccos x) - 2x + C.$

22. The integral

$$\int -\frac{1}{x^2}(\ln x)^2 \mathrm{d}x$$

is equal to which of the following functions?

$$(A)\quad \frac{1}{x}(\ln x)^2 + \frac{2}{x}\ln x + \frac{2}{x} + C.$$

$$(B)\quad \frac{1}{x}(\ln x)^2 + \frac{2}{x}\ln x - \frac{2}{x} + C.$$

$$(C)\quad \frac{1}{x}(\ln x)^2 - \frac{2}{x}\ln x + \frac{2}{x} + C.$$

$$(D)\quad \frac{1}{x}(\ln x)^2 - \frac{2}{x}\ln x - \frac{2}{x} + C.$$

23. Consider the parametric equations

$$x = R(\theta - \sin\theta), \qquad y = R(1 - \cos\theta), \qquad 0 \le \theta \le 2\pi.$$

What is the arclength L of the parametric curve equal to?
(A) $L = 2R$. (B) $L = 4R$. (C) $L = 6R$. (D) $L = 8R$.

24. Given the smooth parametric equations

$$x = f(\theta), \qquad y = g(\theta), \qquad \alpha \leq \theta \leq \beta.$$

Which of the following formulas represents the area of the surface S generated by rotating the parametric curve about the x-axis?

$$(A) \quad S = \int_{\alpha}^{\beta} \sqrt{[f'(\theta)]^2 + [g'(\theta)]^2}\mathrm{d}\theta.$$

$$(B) \quad S = \int_{\alpha}^{\beta} 2\pi g(\theta)\sqrt{[f'(\theta)]^2 + [g'(\theta)]^2}\mathrm{d}\theta.$$

$$(C) \quad S = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\mathrm{d}\theta.$$

$$(D) \quad S = \int_{\alpha}^{\beta} \frac{1}{2}\pi[f(\theta)]^2\mathrm{d}\theta.$$

25. Given the polar curve

$$r = 10(1 + \cos\theta), \qquad 0 \le \theta \le 2\pi.$$

Which of the following is equal to the arclength L of the curve?

$$(A) \quad L = 10\int_0^{2\pi} \sqrt{2 + 2\sin\theta}\mathrm{d}\theta.$$

$$(B) \quad L = 10\int_0^{2\pi} \sqrt{2 - 2\sin\theta}\mathrm{d}\theta.$$

$$(C) \quad L = 10\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\mathrm{d}\theta.$$

$$(D) \quad L = 10\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\mathrm{d}\theta.$$

26. Given the polar equations of two circles

$$r = 20\cos\theta, \qquad r = 20\sin\theta.$$

Which of the following is equal to the area A of the region inside both circles?

$$(A) \quad A = \frac{1}{2}\int_0^{\pi/4} (20\cos\theta)^2 \mathrm{d}\theta + \frac{1}{2}\int_0^{\pi/4} (20\sin\theta)^2 \mathrm{d}\theta.$$

$$(B) \quad A = \frac{1}{2}\int_0^{\pi/2} (20\cos\theta)^2 \mathrm{d}\theta + \frac{1}{2}\int_0^{\pi/2} (20\sin\theta)^2 \mathrm{d}\theta.$$

$$(C) \quad A = \frac{1}{2}\int_{\pi/4}^{\pi/2} (20\cos\theta)^2 \mathrm{d}\theta + \frac{1}{2}\int_0^{\pi/4} (20\sin\theta)^2 \mathrm{d}\theta.$$

$$(D) \quad A = \frac{1}{2}\int_0^{\pi/4} (20\cos\theta)^2 \mathrm{d}\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} (20\sin\theta)^2 \mathrm{d}\theta.$$

27. Consider the positive sequence

$$\left\{ \left(1+\frac{1}{n^2}\right)^{n^2} : n = 1, 2, 3, \cdots\cdots \right\}.$$

Which of the following statements is NOT true?

(A) The sequence is bounded and increasing.

(B) The sequence is convergent.

(C) The limit of the sequence is e.

(D) The sequence is divergent.

28. Consider the positive sequence

$$\left\{ \left(1 + \frac{1}{n^4} \right)^{n^2} : n = 1, 2, 3, \cdots\cdots \right\}$$

Which of the following is the limit L of the sequence, as $n \to \infty$?
(A) $L = 0$. (B) $L = 1$. (C) $L = 2$. (D) $L = e$.

29. Consider the geometric series

$$\sum_{n=0}^{\infty} a^2(r^2)^n,$$

where a and r are real constants. Which of the following statements is NOT true?

(A) The series is convergent, if $|r| < 1$.

(B) The series is divergent, if $|r| \geq 1$.

(C) The sum is $a^2/(1-r^2)$, if $|r| < 1$.

(D) The series is convergent for all r.

30. Consider the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where p is a real parameter. Which of the following statements is NOT true?

(A) The p-series is convergent, if $p > 1$.
(B) The p-series is divergent, if $p \leq 1$.
(C) The p-series is convergent for all real p.
(D) The convergence of the p-series may be determined by the integral test.

31. Consider the positive series

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n^4}\right)^{n^4}.$$

Which of the following statements is true?

(A) The series is a geometric series and it is convergent.

(B) The series is a p-series and it is convergent.

(C) The series is divergent by the divergence test.

(D) The series is convergent by using the comparison test.

32. Consider the positive series

$$\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n^2}.$$

Which of the following tests is the best idea to determine the convergence of the series?

(A) The divergence test.

(B) The integral test.

(C) The comparison test.

(D) The ratio test.

33. Consider the positive series

$$\sum_{n=1}^{\infty} \frac{3^n + 5^n}{7^n + 9^n}$$

Which of the following tests is the best idea to determine the convergence of the series?

(A) The divergence test.

(B) The integral test.

(C) The comparison test.

(D) The ratio test.

34. Consider the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^2}.$$

Which of the following tests is the best idea to determine the convergence of the series?

(A) The integral test.
(B) The comparison test.
(C) The alternating series test.
(D) The ratio test.

35. Consider the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

Which of the following tests is the best idea to determine the convergence of the series?
(A) The divergence test.
(B) The integral test.
(C) The comparison test.
(D) The ratio test.

36. Consider the series

$$\sum_{n=1}^{\infty} \left(\frac{5^n + 6^n}{7^n + 8^n} \right)^n .$$

Which of the following tests is the best idea to determine the convergence of the series?

(A) The integral test.

(B) The comparison test.

(C) The ratio test.

(D) The root test.

37. Which of the following series is absolutely convergent?

$$(A) \quad \sum_{n=1}^{\infty}(-1)^n \frac{n!}{n^n}.$$

$$(B) \quad \sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n}.$$

$$(C) \quad \sum_{n=1}^{\infty} \frac{1+(\sin n)^2}{n}.$$

$$(D) \quad \sum_{n=1}^{\infty} 1.$$

38. Consider the series

$$\sum_{n=1}^{\infty}(-1)^n \arcsin\left(\frac{7^n+8^n}{9^n+10^n}\right).$$

Which of the following statements is true?

(A) The series is absolutely convergent.

(B) The series is conditionally convergent.

(C) The series is divergent.

(D) The convergence of the series cannot be determined by any known tests in calculus.

39. Which of the following series is conditionally convergent?

$$(A) \quad \sum_{n=2}^{\infty}(-1)^n\frac{1}{n\ln n}.$$

$$(B) \quad \sum_{n=2}^{\infty}(-1)^n\frac{1}{n(\ln n)^2}.$$

$$(C) \quad \sum_{n=1}^{\infty}\frac{3^n+4^n}{5^n+6^n}.$$

$$(D) \quad \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n.$$

40. Which of the following series is divergent?

$$(A)\quad \sum_{n=1}^{\infty} \frac{1}{n^2+2^n}.$$

$$(B)\quad \sum_{n=1}^{\infty} \frac{3^n+7^n}{5^n+9^n}.$$

$$(C)\quad \sum_{n=1}^{\infty} \frac{1}{(1+\frac{1}{n})^{n^2}}.$$

$$(D)\quad \sum_{n=1}^{\infty} (-1)^n \arccos\left(\frac{1}{n^2}\right).$$

41. Consider the series

$$\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1+n^2\arcsin(\frac{1}{n^2})}{1+2\arccos(\frac{1}{n^2})}\right\}^n.$$

Which of the following statements is true?

(A) The series is absolutely convergent.

(B) The series is conditionally convergent.

(C) The series is divergent.

(D) The convergence or divergence of the series cannot be determined by any known test in calculus.

42. Which of the following power series is absolutely convergent on the interval $I = (-\infty, \infty)$?

$$(A) \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

$$(B) \quad \sum_{n=0}^{\infty} (-1)^n \frac{(x-8)^n}{n^2 + 2^n}.$$

$$(C) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

$$(D) \quad \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{n+1} x^{2n+2}.$$

43. Consider the power series

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

Which of the following is the radius R of convergence of the power series?
(A) $R = 1$. (B) $R = 2$. (C) $R = e$. (D) $R = \infty$.

44. Consider the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}.$$

Which of the following intervals is the interval of convergence of this series?

(A) $I = (-5, 5)$.
(B) $I = (-8, 8)$.
(C) $I = (-10, 10)$.
(D) $I = (-\infty, \infty)$.

45. Consider the power series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}(x-10)^n.$$

Which of the following statements is true?

(A) The radius of convergence is $R = 1$ and the interval of convergence is $I = (-1, 1)$.

(B) The radius of convergence is $R = 2$ and the interval of convergence is $I = (-2, 2)$.

(C) The radius of convergence is $R = e$ and the interval of convergence is $I = (10 - e, 10 + e)$.

(D) The radius of convergence is $R = \pi$ and the interval of convergence is $I = (-\pi, \pi)$.

46. Consider the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n+2}.$$

Which of the following is equal to the sum S of the power series?

$$\begin{aligned} &(A) \quad S = \cos(x). \\ &(B) \quad S = \cos(x^2). \\ &(C) \quad S = x^2 \cos(x). \\ &(D) \quad S = x^2 \cos(x^2). \end{aligned}$$

47. Which of the following is the power series representation of the function $f(x) = x^2 \arctan(x^2)$?

$$(A) \quad x^2 \arctan(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

$$(B) \quad x^2 \arctan(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+2}.$$

$$(C) \quad x^2 \arctan(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+2}.$$

$$(D) \quad x^2 \arctan(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{4n+4}.$$

48. Which of the following is the MacLaurin series of the function $f(x) = x^2e^{-x^2}$?

$$(A) \quad x^2e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1}.$$

$$(B) \quad x^2e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+2}.$$

$$(C) \quad x^2e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+3}.$$

$$(D) \quad x^2e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+4}.$$

49. Consider the following power series representation

$$\ln\left(\frac{1+x}{1-x}\right) = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{n+1} x^{n+1}.$$

On which of the following intervals is the representation of the power series valid?

$(A)\quad I = (-\infty, \infty).$

$(B)\quad I = (-\infty, 0).$

$(C)\quad I = (0, \infty).$

$(D)\quad I = (-1, 1).$

50. Consider the power series

$$\sum_{n=0}^{\infty}(-1)^n x^n \left[\sin\left(\frac{5^n+6^n}{7^n+8^n}\right)\right]^n.$$

Which of the following statements about the series is true?

(A) The series is absolutely convergent on the interval $I=(-1,1)$.

(B) The series is absolutely convergent on the interval $I=(-2,2)$.

(C) The series is absolutely convergent on the interval $I=(-10,10)$.

(D) The series is absolutely convergent on the interval $I=(-\infty,\infty)$.

51. Consider the power series

$$\sum_{n=0}^{\infty}(-1)^n x^n \left\{\arccos\left(\frac{1}{n}\right)+\arccos\left(\frac{1}{n^2}\right)\right\}^n.$$

Which of the following statements is true?

(A) The series is absolutely convergent on the interval $I=(-1,1)$.

(B) The series is absolutely convergent on the interval $I=(0,2)$.

(C) The series is absolutely convergent on the interval $I=(0,\pi)$.

(D) The series is absolutely convergent on the interval $I=(-1/\pi,1/\pi)$.

Example 1. Consider the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix},$$

where $f(u,v)$ and $g(u,v)$ are continuous functions of u and v. Suppose that

$$\begin{aligned} &f(0,0)=0, \qquad g(0,0)=0, \\ &2uf(u,v)+2vg(u,v) \leq -C(|u|^2+|v|^2)^{m+1}, \end{aligned}$$

for all $(u,v) \in \mathbb{R}^2$, where $C>0$ and $m>0$ are fixed positive constants. Show that the fixed point $(u,v)=(0,0)$ is asymptotically stable. Multiplying the first equation by $2u$ and multiplying the second equation by $2v$, adding the results together, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\{[u(t)]^2+[v(t)]^2\} &= 2u(t)f(u(t),v(t))+2v(t)g(u(t),v(t)) \\ &\leq -C\{[u(t)]^2+[v(t)]^2\}^{m+1}, \end{aligned}$$

for all t. Define

$$E(t)=|u(t)|^2+|v(t)|^2,$$

on $(0,\infty)$. Then

$$E'(t) \leq -C[E(t)]^{m+1}.$$

It is easy to show that if $E(t_0)=0$, then $E(t)=0$, for all $t>t_0$, where $t_0 \geq 0$ is a constant. Without loss of generality, let

$$E(t)=|u(t)|^2+|v(t)|^2>0.$$

The above inequality is equivalent to

$$-\frac{mE'(t)}{[E(t)]^{m+1}} \geq mC.$$

That is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{[E(t)]^m}\right\} \geq mC.$$

Integrating this inequality with respect to t gives

$$\frac{1}{[E(t)]^m} \geq \frac{1}{[E(0)]^m} + mCt.$$

In another word, we have

$$E(t) \leq \left\{\frac{[E(0)]^m}{1+[E(0)]^m mCt}\right\}^{1/m},$$

for all $t > 0$. Therefore, we have

$$\lim_{t\to\infty} E(t) = 0.$$

This completes the proof that the fixed point $(u, v) = (0, 0)$ is asymptotically stable. $\square$

Example 2. Consider the real system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix},$$

where $f(u, v)$ and $g(u, v)$ are continuous functions of (u, v), such that

$$\begin{aligned} &f(0,0) = 0, \qquad g(0,0) = 0, \\ &2uf(u,v) + 2vg(u,v) \leq -C(|u|^{2m+2} + |v|^{2m+2}), \end{aligned}$$

for all $(u, v) \in \mathbb{R}^2$, where $C > 0$ and $m > 0$ are fixed positive constants. Prove that the fixed point $(u, v) = (0, 0)$ is asymptotically stable.

Proof. Multiplying the first equation by $2u$ and multiplying the second equation by $2v$, adding the results together, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\{[u(t)]^2+[v(t)]^2\} = 2u(t)f(u(t),v(t))+2v(t)g(u(t),v(t))\\ \le\ & -C\{[u(t)]^{2m+2}+[v(t)]^{2m+2}\}\\ \le\ & -C_0\{[u(t)]^2+[v(t)]^2\}^{m+1},\end{aligned}$$

for all $t>0$, where $C_0>0$ is a positive constant.

Let

$$E(t)=|u(t)|^2+|v(t)|^2.$$

Then

$$E'(t)\le -C_0[E(t)]^{m+1}.$$

It is very easy to show that if $E(t_0)=0$, then $E(t)=0$, for all $t>t_0$, for some constant $t_0\ge 0$. Without loss of generality, let $E(t)>0$, for all $t>0$. Now we have

$$-\frac{mE'(t)}{[E(t)]^{m+1}}\ge mC_0.$$

Equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{[E(t)]^m}\right\}\ge mC_0.$$

Integrating this inequality with respect to t, we have

$$\frac{1}{[E(t)]^m}\ge\frac{1}{[E(0)]^m}+mC_0t.$$

That is

$$E(t)\le\left\{\frac{[E(0)]^m}{1+[E(0)]^m mC_0t}\right\}^{1/m},$$

for all $t > 0$. Therefore, we have the limit

$$\lim_{t\to\infty} E(t) = 0.$$

The stability of the fixed point $(u, v) = (0, 0)$ is completed. $\square$

Due November 4

1. Find two examples of continuous functions $f(u, v)$ and $g(u, v)$, such that

$$2uf(u, v) + 2vg(u, v) \leq -C(|u|^2 + |v|^2)^{m+1},$$

for all $(u, v) \in \mathbb{R}^2$, where $m > 0$ and $C > 0$ fixed are positive constants.

2. Find two examples of continuous functions $f(u, v)$ and $g(u, v)$, such that

$$2uf(u, v) + 2vg(u, v) \leq -C(|u|^{2m+2} + |v|^{2m+2}),$$

for all $(u, v) \in \mathbb{R}^2$, where $m > 0$ and $C > 0$ are fixed positive constants.

3. Let $m > 0$ be a fixed positive constant. Find a positive constant $C = C(m) > 0$, such that

$$|u|^{2m+2} + |v|^{2m+2} \geq C(|u|^2 + |v|^2)^{m+1},$$

for all $(u, v) \in \mathbb{R}^2$.

4. Show that the fixed point $u = 0$ of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u + u = u^2 + \sin(u^4) + \ln(1 + u^8)$$

is exponentially stable.

5. Consider the real system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + B[R^2 - (u^2 + v^2)]\begin{pmatrix} u \\ v \end{pmatrix},$$

where $A \neq 0$, $B \neq 0$ and $R \neq 0$ are real constants. Find the fixed point of the system and determine its stability or instability.

6. Let

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & 0 & \alpha \end{pmatrix}, \qquad B = \begin{pmatrix} \beta & 1 & 0 & 0 & 0 \\ 0 & \beta & 1 & 0 & 0 \\ 0 & 0 & \beta & 1 & 0 \\ 0 & 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & \beta \end{pmatrix},$$

where α and β are real constants.

(1) Let $\alpha = -2 - \varepsilon + 5\mathrm{i}$ and $\beta = -1 - \varepsilon + 10\mathrm{i}$, where $0 < \varepsilon \ll 1$ is a sufficiently small positive constant. Show that

$$\begin{aligned} \|\exp(At)\| &\leq C\exp(-2t), \\ \|\exp(Bt)\| &\leq C\exp(-t), \end{aligned}$$

for all $t > 0$, where $C > 0$ is a positive constant.

(2) Let $\alpha = 1 + \varepsilon - 4\mathrm{i}$ and $\beta = 2 + \varepsilon - 8\mathrm{i}$, where $0 < \varepsilon \ll 1$ is a sufficiently small positive constant. Show that

$$\begin{aligned} \|\exp(At)\| &\leq C\exp(t), \\ \|\exp(Bt)\| &\leq C\exp(2t), \end{aligned}$$

for all $t < 0$, where $C > 0$ is a positive constant.

Can you find the minimum value of the constant C in each of the above four estimates?

Let A, B and R be real nonzero constants. Consider the initial value problem for the nonlinear system of differential equations

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + B[R^2 - u^2 - v^2]\begin{pmatrix} u \\ v \end{pmatrix}, \\ &\begin{pmatrix} u(t_0) \\ v(t_0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \end{aligned}$$

Show that if $\begin{pmatrix} u(t_1) \\ v(t_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for some t_1, then

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

for all t.

Show that if A, B, then there exists a unique stable periodic solution to the system of differential equations.

Homework 1 - Due September 2

Let α and β be real nonzero constants. Solve the following scalar differential equations

$$(1) \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = y(\alpha - \beta y)$$

$$(2) \qquad \frac{\mathrm{d}y}{\mathrm{d}x} + xy = e^{\frac{1}{2}x^2} y^2$$

$$(3) \qquad 4x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 8\sqrt{x} + 8\sqrt{x} \ln x, x > 0$$

$$(4) \qquad x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3x \frac{\mathrm{d}y}{\mathrm{d}x} + 4y = x^2 \ln x, x > 0$$

$$(5) \qquad 2xe^y \mathrm{d}x + (x^2 e^y + \cos y)\mathrm{d}y = 0$$

$$(6) \qquad [\sin(xy) + xy\cos(xy) + 2x]\mathrm{d}x + [x^2\cos(xy) + 2y]\mathrm{d}y = 0$$

$$(7) \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - 2y}{2x - y}$$

$$(8) \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + 2y}{2x + y}$$

$$(9) \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}^2 x} = \frac{1}{x}\left(\frac{\mathrm{d}y}{\mathrm{d}x} + x^2 \cos x\right), x > 0$$

$$(10) \qquad y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y^2$$

Let α, β, γ, δ, ε be real nonzero constants.

1. Let P_1, P_2, P_3, $\cdots$, P_m be $n \times n$ real constant matrices, such that

$$P_1 + P_2 + P_3 + \cdots\cdots + P_m = I,$$
$$P_iP_j = P_jP_i = O, \qquad i = 1, 2, 3, \cdots, m; \qquad j = 1, 2, 3, \cdots, m; \qquad i \neq$$

Compute

$$e^{P_1t}e^{P_2t}e^{P_3t}\cdots\cdots e^{P_mt}.$$

Solutions:

$$e^{P_1t}e^{P_2t}e^{P_3t}\cdots e^{P_mt} = e^{(P_1+P_2+P_3+\cdots+P_m)t} = e^{It} = e^tI.$$

2. Let

$$A = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

(1) Compute e^{At}.
(2) Solve the system of differential equations $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = A\mathbf{u}$, where you may let $n = 5$.

Solutions:

$$A = \lambda I + C.$$

$$e^{At} = e^{(\lambda t)I}e^{Ct} = e^{\lambda t}\left\{I + Ct + \frac{1}{2!}C^2t^2 + \frac{1}{3!}C^3t^3 + \frac{1}{4!}C^4t^4 + \frac{1}{5!}C^5t^5\right\},$$

where $C^m = O$, for all positive integers $m > 5$.

The solution of the system of differential equations is given by

$$\mathbf{u}(t) = e^{At}\mathbf{c} = e^{\lambda t}\left\{I + Ct + \frac{1}{2!}C^2t^2 + \frac{1}{3!}C^3t^3 + \frac{1}{4!}C^4t^4 + \frac{1}{5!}C^5t^5\right\}\mathbf{c},$$

where $\mathbf{c}$ is any real constant vector.

3. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Compute e^{At}.

Solutions: Note that

$$A^2 = 2I, \qquad A^3 = 2A, \qquad A^4 = 4I, \cdots\cdots, A^{2m} = 2^m I, \qquad A^{2m+1} = 2^m A$$

By definition, we have

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \frac{1}{5!}A^5t^5 + \cdots\cdots \\ &+ \frac{1}{(2m)!}A^{2m}t^{2m} + \frac{1}{(2m+1)!}A^{2m+1}t^{2m+1} + \cdots\cdots \\ &= \frac{1}{2}\left\{e^{\sqrt{2}t} + e^{-\sqrt{2}t}\right\} I + \frac{1}{2\sqrt{2}}\left\{e^{\sqrt{2}t} - e^{-\sqrt{2}t}\right\} A. \end{aligned}$$

4. Let

$$f(x) = 1 + x + x^2 + x^3 + \cdots\cdots + x^{10},$$

and

$$A = \begin{pmatrix} \alpha & -\beta & -\beta \\ -\beta & \alpha & -\beta \\ -\beta & -\beta & \alpha \end{pmatrix}.$$

Find all eigenvalues and the corresponding eigenvectors of $f(A)$.

Solutions: The eigenvalues and the corresponding eigenvectors of A are

$$\lambda_1 = \alpha + \beta, \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = \alpha - 2\beta, \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore, the eigenvalues and the corresponding eigenvectors are

$$f(\lambda_1) = f(\alpha + \beta), \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$f(\lambda_2) = f(\alpha - 2\beta), \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

5. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}.$$

Compute e^{At}.

Solutions: Let

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Then $T^{-1} = T$ and

$$T^{-1}AT = \begin{pmatrix} \alpha - 3\beta & 0 & 0 & 0 \\ 0 & \alpha + \beta & 0 & 0 \\ 0 & 0 & \alpha + \beta & 0 \\ 0 & 0 & 0 & \alpha + \beta \end{pmatrix}.$$

Now

$$e^{(T^{-1}AT)t} = \begin{pmatrix} e^{(\alpha-3\beta)t} & 0 & 0 & 0 \\ 0 & e^{(\alpha+\beta}t & 0 & 0 \\ 0 & 0 & e^{(\alpha+\beta)t} & 0 \\ 0 & 0 & 0 & e^{(\alpha+\beta)t} \end{pmatrix}.$$

Therefore we have

$$e^{At} = \frac{1}{4}e^{(\alpha-3\beta)t} \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} + \frac{1}{4}e^{(\alpha+\beta)t} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

6. Let

$$A = \begin{pmatrix} \alpha & \beta & \gamma & \delta & \varepsilon \\ 0 & \alpha & \beta & \gamma & \delta \\ 0 & 0 & \alpha & \beta & \gamma \\ 0 & 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & 0 & \alpha \end{pmatrix}.$$

Compute e^{At}.

$$A = \alpha I + \beta C + \gamma C^2 + \delta C^3 + \varepsilon C^4,$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{aligned} e^{At} &= e^{(\alpha t)I} e^{(\beta t)C} e^{(\gamma t)C^2} e^{(\delta t)C^3} e^{(\varepsilon t)C^4} \\ &= e^{\alpha t}[I + (\beta t)C + \frac{1}{2!}(\beta t)^2 C^2 + \frac{1}{3!}(\beta t)^3 C^3 + \frac{1}{4!}(\beta t)^4 C^4] \\ &\cdot [I + (\gamma t)C^2 + \frac{1}{2}(\gamma t)^2 C^4][I + (\delta t)C^3][I + (\varepsilon t)C^4]. \end{aligned}$$

1. Let

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}.$$

Compute e^{At}. Express the final result as the linear combination of mutually perpendicular projection matrices with exponential functions as the coefficients. Hint $A^2 = A$.

2. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}.$$

Compute e^{At}. Express your final result as the linear combination of two mutually orthogonal projection matrices with exponential functions as the coefficients.

A general hint for the following eigenvalue problems: You may multiply each given eigenvalue problem by some appropriate function to transform it back to the standard eigenvalue problem $f'' + \lambda f = 0$ on $(0, L)$, with some simple boundary conditions.

3. Solve the following eigenvalue problems

$$\begin{aligned} &f'' + 2f' + (\lambda + 1)f = 0, \text{ on } (0, L), \\ &f(0) = 0, \qquad f(L) = 0. \end{aligned}$$

4. Solve the following eigenvalue problems

$$\begin{aligned} &f'' + 2f' + (\lambda + 1)f = 0, \text{ on } (0, L), \\ &f(0) + f'(0) = 0, \qquad f(L) + f'(L) = 0. \end{aligned}$$

5. Solve the following eigenvalue problems

$$\begin{aligned} &f'' + 4xf' + (\lambda + 2 + 4x^2)f = 0, \text{ on } (0, L), \\ &f(0) = 0, \qquad f(L) = 0. \end{aligned}$$

6. Solve the following eigenvalue problems

$$\begin{aligned} &f'' + 4xf' + (\lambda + 2 + 4x^2)f = 0, \text{ on } (0, L), \\ &f'(0) = 0, \qquad 2Lf(L) + f'(L) = 0. \end{aligned}$$

7. Solve the following eigenvalue problems

$$\begin{aligned} &(1 + x^2)f'' + 4xf' + [2 + \lambda(1 + x^2)]f = 0, \text{ on } (0, L), \\ &f(0) = 0, \qquad f(L) = 0. \end{aligned}$$

8. Solve the following eigenvalue problems

$$\begin{aligned} &(1 + x^2)f'' + 4xf' + [2 + \lambda(1 + x^2)]f = 0, \text{ on } (0, L), \\ &f'(0) = 0, \qquad 2Lf(L) + (1 + L^2)f'(L) = 0. \end{aligned}$$

9. Solve the following eigenvalue problems

$$f'' + 2f' + (\lambda + 1)f = 0, \text{ on } (0, L),$$
$$f(0) + f'(0) = 0, \qquad f(L) = 0.$$

10. Solve the following eigenvalue problems

$$f'' + 2f' + (\lambda + 1)f = 0, \text{ on } (0, L),$$
$$f(0) = 0, \qquad f(L) + f'(L) = 0.$$

Due Thursday September 23 Let α and β be real nonzero constants.

Use the method of Laplace transform to solve the following integral equations

$$1 \quad y(t) = 4t + 4\int_0^t (t - \tau)y(\tau)\mathrm{d}\tau.$$
$$[\mathcal{L}y](\lambda) = \frac{4}{\lambda^2 - 4}, \qquad y(t) = e^{2t} - e^{-2t}.$$

$$2 \quad y(t) = \exp(\alpha t) + \beta\int_0^t \exp(\alpha t - \alpha\tau)y(\tau)\mathrm{d}\tau.$$
$$[\mathcal{L}y](\lambda) = \frac{1}{\lambda - \alpha - \beta}, \qquad y(t) = e^{(\alpha+\beta)t}.$$

$$3 \quad y(t) = 2\beta t\exp(\alpha t) + \beta^2\int_0^t (t - \tau)\exp(\alpha t - \alpha\tau)y(\tau)\mathrm{d}\tau.$$
$$[\mathcal{L}y](\lambda) = \frac{1}{\lambda - \alpha - \beta} - \frac{1}{\lambda - \alpha + \beta}, \qquad y(t) = \exp[(\alpha + \beta)t] - \exp[(\alpha -$$

$$4 \quad y(t) = 2\cos(\beta t) + 2\beta\int_0^t \sin(\beta t - \beta\tau)y(\tau)\mathrm{d}\tau.$$
$$[\mathcal{L}y](\lambda) = \frac{1}{\lambda - \beta} + \frac{1}{\lambda + \beta}, y(t) = \exp(\beta t) + \exp(-\beta t).$$

5 $\quad y(t) = \cos(\beta t) - 3\beta \int_0^t \sin(\beta t - \beta\tau) y(\tau) \mathrm{d}\tau.$

$$[\mathcal{L}y](\lambda) = \frac{\lambda}{\lambda^2 + 4\beta^2}, \qquad y(t) = \cos(2\beta t).$$

6 $\quad y(t) = \sin(\beta t) + 2\beta \int_0^t \cos(\beta t - \beta\tau) y(\tau) \mathrm{d}\tau.$

$$[\mathcal{L}y](\lambda) = \frac{\beta}{(\lambda - \beta)^2}, y(t) = \beta t \exp(\beta t).$$

7 $\quad y(t) = \cos(\beta t) + 2\beta \int_0^t \cos(\beta t - \beta\tau) y(\tau) \mathrm{d}\tau.$

$$[\mathcal{L}y](\lambda) = \frac{\lambda}{(\lambda - \beta)^2}, \qquad y(t) = \exp(\beta t) + \beta t \exp(\beta t).$$

8 $\quad y(t) = 2\sin(\beta t) + 2\beta \int_0^t \sin(\beta t - \beta\tau) y(\tau) \mathrm{d}\tau.$

$$[\mathcal{L}y](\lambda) = \frac{1}{\lambda - \beta} - \frac{1}{\lambda + \beta}, \qquad y(t) = \exp(\beta t) - \exp(-\beta t).$$

9. Solve the eigenvalue problem

$$[\mathcal{L}f](\lambda) = \mu f(\lambda).$$

$$\mu = \sqrt{\pi}, \qquad f(t) = \frac{1}{t^{1/2}}.$$

That is, find a real constant μ and a nontrivial function $f = f(t)$, such that $[\mathcal{L}f](\lambda) = \mu f(\lambda)$, for all $\lambda > 0$.

10. Find the Laplace transform of the following functions

$$(1) \quad f(t) = \exp(\alpha t)[a\cos(\beta t) + b\sin(\beta t)],$$

$$(2) \quad g(t) = t\exp(\alpha t)[a\cos(\beta t) + b\sin(\beta t)].$$

Let $m \geq 0$ be an integer. Define the function

$$\psi_m(x) = \exp\left(\frac{1}{2}x^2\right)\frac{\mathrm{d}^m}{\mathrm{d}x^m}\exp(-x^2).$$

(1) Show that if the integer m is even, then the function $\psi_m(x)$ is an even function of x.
(2) Show that if the integer m is odd, then the function $\psi_m(x)$ is an odd function of x.
(3) Show that it is a solution of the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}y = (x^2 - 2m - 1)y.$$

1. Let $m_1 \geq 0$, $m_2 \geq 0$, $m_3 \geq 0$, $\cdots$, $m_n \geq 0$ be nonnegative integers, let $\mathbf{m} = (m_1, m_2, m_3, \cdots, m_n)$ and $|\mathbf{m}| = m_1 + m_2 + m_3 + \cdots + m_n$. For example, if $\mathbf{m} = (1, 2, 3, \cdots, n)$, then $|\mathbf{m}| = 1 + 2 + 3 + \cdots + n$. Let $\mathbf{x} = (x_1, x_2, x_3, \cdots, x_n) \in \mathbb{R}^n$ and

$|\mathbf{x}|^2 = |x_1|^2 + |x_2|^2 + |x_3|^2 + \cdots + |x_n|^2$. Define the function

$$f(\mathbf{m}, \mathbf{x}) = \exp\left(\frac{1}{2}|\mathbf{x}|^2\right) \frac{\partial^{m_1}}{\partial x_1^{m_1}} \frac{\partial^{m_2}}{\partial x_2^{m_2}} \frac{\partial^{m_3}}{\partial x_3^{m_3}} \cdots \frac{\partial^{m_n}}{\partial x_n^{m_n}} \exp(-|\mathbf{x}|^2)$$

$$\begin{aligned} = & \exp\left(\frac{1}{2}|x_1|^2\right) \frac{\partial^{m_1}}{\partial x_1^{m_1}} \exp(-|x_1|^2) \\ \cdot & \exp\left(\frac{1}{2}|x_2|^2\right) \frac{\partial^{m_2}}{\partial x_2^{m_2}} \exp(-|x_2|^2) \\ \cdot & \exp\left(\frac{1}{2}|x_3|^2\right) \frac{\partial^{m_3}}{\partial x_3^{m_3}} \exp(-|x_3|^2) \cdots\cdots \\ \cdot & \exp\left(\frac{1}{2}|x_n|^2\right) \frac{\partial^{m_n}}{\partial x_n^{m_4}} \exp(-|x_n|^2) \end{aligned}$$

Show that

$$\triangle f(\mathbf{m}, \mathbf{x}) = (|\mathbf{x}|^2 - 2|\mathbf{m}| - n) f(\mathbf{m}, \mathbf{x}).$$

Suppose that $\mathbf{m} = 10$ and $n = 10$. Find as many as possible solutions of the partial differential equation given above.

2. Let $\alpha > 0$ be a positive constant. Use the method of Fourier transform to solve the Cauchy problem for the differential equation

$$\frac{\partial u}{\partial t} - \alpha \triangle u = 0, \qquad u(\mathbf{x}, 0) = \frac{\beta}{(4\pi\alpha)^{n/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right).$$

3. Let $\alpha > 0$ be a positive constant. Use the method of Fourier transform to solve the Cauchy problem for the differential equation

$$\begin{aligned} & \frac{\partial u}{\partial t} - \alpha \triangle u = f(\mathbf{x}, t), \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}), \\ & u(\mathbf{x}, 0) = \frac{\beta}{(4\pi\alpha)^{n/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right), \\ & f(\mathbf{x}, t) = \gamma \ln(1+t) \frac{1}{[4\pi\alpha(1+t)]^{n/2}} \exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right]. \end{aligned}$$

4. Let $\alpha > 0$ be a positive constant. Use the method of Fourier transform to solve the Cauchy problem for the differential equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \alpha \triangle u = f(\mathbf{x}, t), \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}),\\
&u(\mathbf{x}, 0) = \frac{1}{(4\pi\alpha)^{n/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right),\\
&f(\mathbf{x}, t) = \frac{\gamma}{[4\pi\alpha(1+t)]^{n/2}} \exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right].
\end{aligned}$$

5. Let $\alpha > 0$ be a positive constant. Use the method of Fourier transform to solve the Cauchy problem for the differential equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \alpha \triangle u = f(\mathbf{x}, t), \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}),\\
&u(\mathbf{x}, 0) = \frac{1}{(4\pi\alpha)^{n/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right),\\
&f(\mathbf{x}, t) = \frac{\gamma}{1+t} \frac{1}{[4\pi\alpha(1+t)]^{n/2}} \exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right].
\end{aligned}$$

6. Let $\alpha > 0$ be a positive constant. Use the method of Fourier transform to solve the Cauchy problem for the differential equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \alpha \triangle u = f(\mathbf{x}, t), \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}),\\
&u(\mathbf{x}, 0) = \frac{1}{(4\pi\alpha)^{n/2}} \exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right),\\
&f(\mathbf{x}, t) = \frac{\gamma}{(1+t)^2} \frac{1}{[4\pi\alpha(1+t)]^{n/2}} \exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right].
\end{aligned}$$

7. Let $\alpha > 0$ be a positive constant. Use the method of Fourier

transform to solve the Cauchy problem for the differential equation

$$\frac{\partial u}{\partial t}-\alpha\triangle u=f(\mathbf{x},t),\qquad u(\mathbf{x},0)=u_0(\mathbf{x}),$$

$$u(\mathbf{x},0)=\frac{1}{(4\pi\alpha)^{n/2}}\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right),$$

$$f(\mathbf{x},t)=\frac{\gamma}{(e+t)[\ln(e+t)]^2}\frac{1}{[4\pi\alpha(1+t)]^{n/2}}\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right].$$

8. Let $\alpha>0$ be a positive constant. Use the method of Fourier transform to solve the Cauchy problem for the differential equation

$$\frac{\partial u}{\partial t}-\alpha\triangle u=f(\mathbf{x},t),\qquad u(\mathbf{x},0)=u_0(\mathbf{x}),$$

$$u(\mathbf{x},0)=\frac{1}{(4\pi\alpha)^{n/2}}\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right),$$

$$f(\mathbf{x},t)=\frac{\ln(1+t)}{1+t}\frac{1}{[4\pi\alpha(1+t)]^{n/2}}\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right].$$

Exam. Let $\alpha>0$ be a positive constant. Use the method of Fourier transform to solve the Cauchy problem for the differential equation

$$\frac{\partial u}{\partial t}-\alpha\triangle u=f(\mathbf{x},t),\qquad u(\mathbf{x},0)=u_0(\mathbf{x}),$$

$$u(\mathbf{x},0)=\frac{1}{(4\pi\alpha)^{n/2}}\exp\left(-\frac{|\mathbf{x}|^2}{4\alpha}\right),$$

$$f(\mathbf{x},t)=\frac{1}{[4\pi\alpha(1+t)]^{n/2}}\exp\left[-\frac{|\mathbf{x}|^2}{4\alpha(1+t)}\right].$$

Consider the initial value problem $y(t_0)=y_0$ to each of the following differential equations

$$
\begin{aligned}
&(1) \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \sin(y+t). \\
&(2) \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \cos(y^2+t^2). \\
&(3) \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \ln(1+t^2+y^2). \\
&(4) \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{y^2}. \\
&(5) \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{(y-5)^4}. \\
&(6) \quad \frac{\mathrm{d}y}{\mathrm{d}t} = y^3.
\end{aligned}
$$

Which differential equations satisfy the Lipschitz continuity condition?
Which differential equations always have a global solution on $(-\infty, \infty)$?
Which differential equations may only have a local solution?
Which differential equations may have more than one solutions?
Justify your solutions.

Let

$$
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_n \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \cdots \\ y_n \end{pmatrix},
$$

be vectors with n components, let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix},$$

be $n \times n$ matrices. Define the norm of $\mathbf{v}$ and the norm of A by

$$\|\mathbf{x}\| = \{|x_1|^2 + |x_2|^2 + |x_3|^2 + \cdots + |x_n|^2\}^{1/2},$$

$$\|A\| = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right\}^{1/2}.$$

Prove that there hold the following estimates

$$\begin{aligned} &(1) \quad \|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\| \\ &(2) \quad \|A+B\| \le \|A\| + \|B\| \\ &(3) \quad \|A\mathbf{x}\| \le \|A\|\|\mathbf{x}\| \\ &(4) \quad \|A(\mathbf{x}+\mathbf{y})\| \le \|A\|\|\mathbf{x}\| + \|A\|\|\mathbf{y}\| \\ &(5) \quad \|(A+B)\mathbf{x}\| \le \|A\|\|\mathbf{x}\| + \|B\|\|\mathbf{x}\| \\ &(6) \quad \|AB\mathbf{x}\| \le \|A\|\|B\|\|\mathbf{x}\| \\ &(7) \quad \|AB\| \le \|A\|\|B\| \end{aligned}$$

for all vectors $\mathbf{x}$ and $\mathbf{y}$, for all matrices A and B.

A Summary of the Properties of Harmonic Functions

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of the Laplace's equation

$$\triangle u = 0 \text{ in } \Omega.$$

The Mean Value Formula.
The Strong Maximum Principle.
The Smoothness $u \in C^\infty(\Omega)$.
The Estimates of All Order Derivatives.
The Analyticity.
The Harnack Inequality.
If $\Omega = \mathbb{R}^n$ and u is bounded, then u is a constant function.

Dirichlet's Principle

Define the admissible set

$$\mathcal{A} = \{v \in C^2(\overline{\Omega}) : v = \phi \text{ on } \partial\Omega\}.$$

Define the energy functional

$$\mathrm{E}(v) = \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})v(\mathbf{x}),$$

for all $v \in \mathcal{A}$.

Dirichlet's Principle: Let

$$u \in C^2(\overline{\Omega}), \qquad u = \phi \text{ on } \partial\Omega.$$

Then the smooth function u is a solution of the boundary value problem

$$\begin{aligned} -\triangle u &= f, \text{ in } \Omega, \\ u &= \phi, \text{ on } \partial\Omega, \end{aligned}$$

if and only if

$$\mathrm{E}[u] = \min_{v\in\mathcal{A}} \mathrm{E}[v].$$

Proof. Let $u \in C^2(\overline{\Omega})$ be the solution of the boundary value problem

$$\begin{aligned} -\triangle u = f(\mathbf{x}), &\quad \text{in} \quad \Omega, \\ u = \phi, &\quad \text{on} \quad \partial\Omega. \end{aligned}$$

The purpose of this step is to show that

$$\mathrm{E}[u] \le \mathrm{E}[v],$$

for all $v \in \mathcal{A}$.

Let $v \in \mathcal{A}$. Then $u - v \in H_0^1(\Omega)$. Now

$$\int_\Omega [-\triangle u(\mathbf{x}) - f(\mathbf{x})][u(\mathbf{x}) - v(\mathbf{x})]\mathrm{d}\mathbf{x} = 0.$$

By using integration by parts, we get

$$\int_\Omega \nabla u(\mathbf{x}) \cdot \nabla[u(\mathbf{x}) - v(\mathbf{x})]\mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})[u(\mathbf{x}) - v(\mathbf{x})]\mathrm{d}\mathbf{x} = 0.$$

Rearranging terms and using Cauchy-Schwartz's inequality, we have

$$\begin{aligned} &\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x} \\ \le \frac{1}{2}\int_\Omega |\nabla u(\mathbf{x})|^2\mathrm{d}\mathbf{x} &+ \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2\mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}. \end{aligned}$$

That is

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x}\\ \leq\, &\frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}.\end{aligned}$$

Therefore, we see that

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x}\\ =\ &\min_{v\in H_0^1(\Omega)}\left\{\int_\Omega |\nabla v(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}\right\}.\end{aligned}$$

On the other hand, let $u \in C^2(\overline{\Omega})$, such that

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x}\\ =\ &\min_{v\in\mathcal{A}}\left\{\frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}\right\}.\end{aligned}$$

Let $\lambda \in \mathbb{R}$ and let $a \in C_0^\infty(\Omega)$. Then $v = u + \lambda a \in \mathcal{A}$. Note that

$$\mathrm{E}(u+\lambda a) = \frac{1}{2}\int_\Omega |\nabla u(\mathbf{x}) + \lambda a(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})[u(\mathbf{x}) + \lambda a(\mathbf{x})]\mathrm{d}\mathbf{x}$$

$$\begin{aligned}=\ &\frac{1}{2}\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x}\\ +\ &\lambda\left[\int_\Omega \nabla u(\mathbf{x})\cdot\nabla a(\mathbf{x})\mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})a(\mathbf{x})\mathrm{d}\mathbf{x}\right]\\ +\ &\frac{1}{2}\lambda^2\int_\Omega |\nabla a(\mathbf{x})|^2 \mathrm{d}\mathbf{x}.\end{aligned}$$

We may view this as a function of λ and we know that it attains a local minimum at $\lambda = 0$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathrm{E}(u+\lambda a) &= \lambda \int_\Omega |\nabla a(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \\ &\quad + \int_\Omega \nabla u(\mathbf{x}) \cdot \nabla a(\mathbf{x}) \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x}) a(\mathbf{x}) \mathrm{d}\mathbf{x}.\end{aligned}$$

The derivative at $\lambda = 0$ must be zero. That is

$$\int_\Omega \nabla u(\mathbf{x}) \cdot \nabla a(\mathbf{x}) \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x}) a(\mathbf{x}) \mathrm{d}\mathbf{x} = 0.$$

Using integration by parts leads to

$$\int_\Omega [-\triangle u - f(\mathbf{x})] a(\mathbf{x}) \mathrm{d}\mathbf{x} = 0.$$

Note that this is true for all $a \in C_0^\infty(\Omega)$, we must have

$$-\triangle u = f, \qquad \text{in } \Omega.$$

Note: Here u is called a minimizer of the energy functional. Dirichlet's principle is an example of the calculus of variations applied to Laplace's equation.

Theorem. There exists a unique function

$$u \in C^2(\overline{\Omega}),$$

such that

$$\mathrm{E}[u] = \min_{v \in \mathcal{A}} \mathrm{E}[v].$$

Therefore, $u \in C^2(\overline{\Omega})$ is the unique solution of the boundary value problem.

Proof. First of all, let us show that the infimum of the energy functional exists. By using the Poincare's inequality, we have the following estimates

$$\begin{aligned}
& \frac{1}{2}\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x} \\
\geq\ & \frac{1}{2}\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \\
-\ & \frac{\varepsilon}{2}\int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \frac{2}{\varepsilon}\int_\Omega |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \\
\geq\ & \frac{1}{2}\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \\
-\ & C\frac{\varepsilon}{2}\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \frac{2}{\varepsilon}\int_\Omega |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \\
\geq\ & -\frac{2}{\varepsilon}\int_\Omega |f(\mathbf{x})|^2 \mathrm{d}\mathbf{x},
\end{aligned}$$

for all functions $u \in H_0^1(\Omega)$, where $C > 0$ is a positive constant, independent of u, $\varepsilon > 0$ is a small positive constant. We let $C\varepsilon = 1$. Therefore, the infimum

$$\mathrm{I} \overset{\text{def}}{=} \inf_{u\in H_0^1(\Omega)} \left\{ \frac{1}{2}\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x} \right\} > -\infty,$$

exists.

Now by definition, for any positive integer $k \geq 1$, there exists a function $u_k \in H_0^1(\Omega)$, such that

$$\mathrm{I} \leq \frac{1}{2}\int_\Omega |\nabla u_k(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u_k(\mathbf{x})\mathrm{d}\mathbf{x} < \mathrm{I} + \frac{1}{2^{2k+3}},$$

for all $k = 1, 2, 3, \cdots\cdots$.

Below, we will show that the sequence of functions $\{u_k : k = 1, 2, 3, \cdots\cdots\}$ is a Cauchy sequence in $H_0^1(\Omega)$.

There holds the following identity

$$\begin{aligned}
&\int_\Omega \left|\nabla \frac{u(\mathbf{x})-v(\mathbf{x})}{2}\right|^2 \mathrm{d}\mathbf{x} + \int_\Omega \left|\nabla \frac{u(\mathbf{x})+v(\mathbf{x})}{2}\right|^2 \mathrm{d}\mathbf{x} \\
- \;& 2\int_\Omega f(\mathbf{x})\frac{u(\mathbf{x})+v(\mathbf{x})}{2}\mathrm{d}\mathbf{x} \\
= \;& \frac{1}{2}\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x} \\
+ \;& \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x},
\end{aligned}$$

for all functions $u \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$.

If we let $u = u_k$ and $v = u_l$, then we have

$$\begin{aligned}
&\int_\Omega \left|\nabla \frac{u_k(\mathbf{x})-u_l(\mathbf{x})}{2}\right|^2 \mathrm{d}\mathbf{x} + \int_\Omega \left|\nabla \frac{u_k(\mathbf{x})+u_l(\mathbf{x})}{2}\right|^2 \mathrm{d}\mathbf{x} \\
- \;& 2\int_\Omega f(\mathbf{x})\frac{u_k(\mathbf{x})+u_l(\mathbf{x})}{2}\mathrm{d}\mathbf{x} \\
= \;& \frac{1}{2}\int_\Omega |\nabla u_k(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u_k(\mathbf{x})\mathrm{d}\mathbf{x} \\
+ \;& \frac{1}{2}\int_\Omega |\nabla u_l(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u_l(\mathbf{x})\mathrm{d}\mathbf{x},
\end{aligned}$$

for all positive integers $k \geq 1$ and $l \geq 1$.

Recall that there hold the following estimates

$$\mathrm{I} \leq \frac{1}{2}\int_\Omega \left|\nabla \frac{u_k(\mathbf{x})+u_l(\mathbf{x})}{2}\right|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})\frac{u(\mathbf{x})+u_l(\mathbf{x})}{2}\mathrm{d}\mathbf{x},$$

$$\frac{1}{2}\int_\Omega |\nabla u_k(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u_k(\mathbf{x})\mathrm{d}\mathbf{x} \leq \mathrm{I} + \frac{1}{2^{2k+3}},$$

$$\frac{1}{2}\int_\Omega |\nabla u_l(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x})u_l(\mathbf{x})\mathrm{d}\mathbf{x} \leq \mathrm{I} + \frac{1}{2^{2l+3}},$$

for all positive integers $k \geq 1$ and $l \geq 1$.

Therefore, we obtain the important estimate

$$\int_\Omega \left| \nabla \frac{u_k(\mathbf{x}) - u_l(\mathbf{x})}{2} \right|^2 \mathrm{d}\mathbf{x} \leq \frac{1}{2^{2k+3}} + \frac{1}{2^{2l+3}},$$

for all $k \geq 1$ and $l \geq 1$.

Furthermore, by using the Poincare's inequality, we have

$$\int_\Omega |u_k(\mathbf{x}) - u_l(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \leq C \int_\Omega |\nabla[u_k(\mathbf{x}) - u_l(\mathbf{x})]|^2 \mathrm{d}\mathbf{x} \leq C \left(\frac{1}{2^{2k+3}} + \frac{1}{2^{2l+3}} \right),$$

for all $k \geq 1$ and $l \geq 1$.

Now it is easy to see that the sequence $\{u_k : k = 1, 2, 3, \cdots\cdots\}$ is a Cauchy sequence in $H_0^1(\Omega)$. Note that $H_0^1(\Omega)$ is a Hilbert space. Therefore, there exists a unique function $u_0 \in H_0^1(\Omega)$, such that

$$\lim_{k\to\infty} \left\{ \int_\Omega |u_k(\mathbf{x}) - u_0(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \int_\Omega |\nabla[u_k(\mathbf{x}) - u_0(\mathbf{x})]|^2 \mathrm{d}\mathbf{x} \right\} = 0.$$

Letting $k \to \infty$ in the estimates

$$\mathrm{I} \leq \frac{1}{2} \int_\Omega |\nabla u_k(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x}) u_k(\mathbf{x}) \mathrm{d}\mathbf{x} < \mathrm{I} + \frac{1}{2^{2k+3}},$$

we get

$$\begin{aligned} & \lim_{k\to\infty} \left\{ \frac{1}{2} \int_\Omega |\nabla u_k(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x}) u_k(\mathbf{x}) \mathrm{d}\mathbf{x} \right\} \\ = & \frac{1}{2} \int_\Omega |\nabla u_0(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x}) u_0(\mathbf{x}) \mathrm{d}\mathbf{x} \\ = & \inf_{u \in H_0^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x}) u(\mathbf{x}) \mathrm{d}\mathbf{x} \right\} \\ = & \min_{u \in H_0^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} - \int_\Omega f(\mathbf{x}) u(\mathbf{x}) \mathrm{d}\mathbf{x} \right\}. \end{aligned}$$

The Existence and Uniqueness of a Weak Solution

The Lax-Milgram Theorem. Let $\mathbf{H}$ be a real Hilbert space, with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. Let

$$B : \mathbf{H} \times \mathbf{H} \to \mathbb{R},$$

be a bilinear mapping. Suppose that there exist positive constants $C_1 > 0$ and $C_2 > 0$, such that

$$B[\mathbf{u}, \mathbf{u}] \geq C_1 \|\mathbf{u}\|^2,$$

for all vectors $\mathbf{u} \in \mathbf{H}$;

$$|B[\mathbf{u}, \mathbf{v}]| \leq C_2 \|\mathbf{u}\| \|\mathbf{v}\|,$$

for all $\mathbf{u} \in \mathbf{H}$ and $\mathbf{v} \in \mathbf{H}$.

For any continuous linear functional

$$f \in \mathbf{H}^*,$$

there exists a unique vector $\mathbf{v}_0 \in \mathbf{H}$, such that

$$B[\mathbf{u}, \mathbf{v}_0] = f(\mathbf{u}),$$

for all $\mathbf{u} \in \mathbf{H}$.

The key point in the proof of the Lax-Milgram theorem is that by using the Riesz representation theorem, there exists an operator $A : \mathbf{H} \to \mathbf{H}$, such that

$$B[\mathbf{u}, \mathbf{v}] = (\mathbf{u}, A\mathbf{v}),$$

for all $\mathbf{u} \in \mathbf{H}$ and $\mathbf{v} \in \mathbf{H}$.

We can prove that the operator $A : \mathbf{H} \to \mathbf{H}$ enjoys the following properties:

(1) A is linear.

(2) A is bounded.

(3) $C_1 \leq \|A\| \leq C_2$, where $C_1 > 0$ and $C_2 > 0$ are positive constants.

A is one-to-one.

(4) A is onto.

(5) $A\mathbf{H}$ is closed in $\mathbf{H}$.

(6) $A\mathbf{H} = \mathbf{H}$.

The proof of the Lax-Milgram theorem is completed by using the Riesz representation theorem again.

Let Ω be a bounded open set in $\mathbb{R}^n$, and the boundary $\partial\Omega$ satisfies interior ball condition. Let α_{ij}, β_i and γ be bounded continuous functions defined in Ω, such that $\alpha_{ij}(\mathbf{x}) = \alpha_{ji}(\mathbf{x})$, for all $i = 1, 2, 3, \cdots, n$, $j = 1, 2, 3, \cdots, n$ and $\mathbf{x} \in \Omega$. Suppose that there exists a positive constant $\Lambda > 0$, such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}(\mathbf{x}) \xi_i \xi_j \geq \Lambda |\xi|^2,$$

for all vectors $\xi \in \mathbb{R}^n$ and for all $\mathbf{x} \in \Omega$.

This condition is called the uniform ellipticity.

Define the following linear differential operators

$$\begin{aligned}
\mathcal{L}_0 u &= \gamma(\mathbf{x})u, \\
\mathcal{L}_1 u &= \sum_{i=1}^{n} \beta_i(\mathbf{x})\frac{\partial}{\partial x_i}u, \\
\mathcal{L}_2 u &= -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u\right].
\end{aligned}$$

Define the second order elliptic linear differential operator

$$\mathcal{L}u = -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u\right] + \sum_{i=1}^{n} \beta_i(\mathbf{x})\frac{\partial}{\partial x_i}u + \gamma(\mathbf{x})u.$$

Obviously

$$\mathcal{L}u = \mathcal{L}_0 u + \mathcal{L}_1 u + \mathcal{L}_2 u,$$

for all functions $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Define the following bilinear functionals

$$\begin{aligned}
B_0[u,v] &= \int_\Omega \gamma(\mathbf{x})u(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}, \\
B_1[u,v] &= \sum_{i=1}^{n}\int_\Omega \beta_i(\mathbf{x})\frac{\partial}{\partial x_i}u(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x}, \\
B_2[u,v] &= \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega \alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u(\mathbf{x})\frac{\partial}{\partial x_j}v(\mathbf{x})\mathrm{d}\mathbf{x},
\end{aligned}$$

and

$$\begin{aligned}
B[u,v] \;=\;& \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega \alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u(\mathbf{x})\frac{\partial}{\partial x_j}v(\mathbf{x}) \\
&+ \sum_{i=1}^{n}\int_\Omega \beta_i(\mathbf{x})\frac{\partial}{\partial x_i}u(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x} + \int_\Omega \gamma(\mathbf{x})u(\mathbf{x})v(\mathbf{x})\mathrm{d}\mathbf{x},
\end{aligned}$$

for all functions $u \in H_0^1(\Omega)$.

Obviously, there holds

$$B[u, v] = B_0[u, v] + B_1[u, v] + B_2[u, v],$$

for all functions $u \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$.

The elementary estimates for the bilinear functionals: There hold the following estimates

(1)

$$\begin{aligned} B_0[u,u] &\geq -\|\gamma\|_{L^\infty(\Omega)} \int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x}, \\ |B_0[u,v]| &\leq \|\gamma\|_{L^\infty(\Omega)} \left\{ \int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\}^{1/2} \left\{ \int_\Omega |v(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\}^{1/2}, \end{aligned}$$

(2)

$$\begin{aligned} & B_1[u,u] \\ \geq & -\left\{ \sum_{i=1}^n \|\beta_i\|^2_{L^\infty(\Omega)} \right\}^{1/2} \left\{ \int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\}^{1/2} \left\{ \int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\}^{1/2}, \\ & |B_1[u,v]| \\ \leq & \left\{ \sum_{i=1}^n \|\beta_i\|^2_{L^\infty(\Omega)} \right\}^{1/2} \left\{ \int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\}^{1/2} \left\{ \int_\Omega |v(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\}^{1/2}, \end{aligned}$$

(3)

$$\begin{aligned} & B_2[u,u] \geq \Lambda \int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x}, \\ & |B_2[u,v]| \\ \leq & \left\{ \sum_{i=1}^n \sum_{j=1}^n \|\alpha_{ij}\|^2_{L^\infty(\Omega)} \right\}^{1/2} \left\{ \int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\}^{1/2} \left\{ \int_\Omega |\nabla v(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\}^{1/2}, \end{aligned}$$

(4)

$$
\begin{aligned}
B[u,u] \;&\geq\; \left\{\Lambda - \left[C\sum_{i=1}^{n}\|\beta_i\|^2_{L^\infty(\Omega)}\right]^{1/2} - C\|\gamma\|_{L^\infty(\Omega)}\right\} \\
&\quad\cdot\; \left\{\int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right\}^{1/2}, \\
B[u,v] \;&\leq\; \left\{2\sum_{i=1}^{n}\sum_{j=1}^{n}\|\alpha_{ij}\|^2_{L^\infty(\Omega)} + 2\sum_{i=1}^{n}\|\beta_i\|^2_{L^\infty(\Omega)} + \|\gamma\|_{L^\infty(\Omega)}\right\} \\
&\quad\cdot\; \left\{\int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right\}^{1/2} \\
&\quad\cdot\; \left\{\int_\Omega |v(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \int_\Omega |\nabla v(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\right\}^{1/2},
\end{aligned}
$$

for all functions $u \in H^1_0(\Omega)$ and $v \in H^1_0(\Omega)$, where we have used the Poincare's inequality

$$\int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \leq C \int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x},$$

for all functions $u \in H^1_0(\Omega)$, where $C > 0$ is a positive constant, independent of u.

There exists a positive constant $\lambda_0 > 0$, such that

$$B[u,u] + \lambda_0 \int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \geq \frac{1}{2}\Lambda \int_\Omega |\nabla u(\mathbf{x})|^2 \mathrm{d}\mathbf{x},$$

for all functions $u \in H^1_0(\Omega)$.

In other words, the inverse operator $(\mathcal{L} + \lambda_0 I)^{-1}$ of the second order linear differential operator $\mathcal{L} + \lambda_0 I$ exists. Moreover, the inverse $(\mathcal{L} + \lambda_0 I)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is a compact operator, by using the above estimate.

Consider the boundary value problem for the second order linear elliptic equation

$$-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u\right]+\sum_{i=1}^{n}\beta_i(\mathbf{x})\frac{\partial}{\partial x_i}u+\gamma(\mathbf{x})u=f(\mathbf{x}),\quad \text{in}\quad \Omega,$$

$$u(\mathbf{x})=0,\quad \text{on}\quad \partial\Omega.$$

Either the inverse operator $\mathcal{L}^{-1}$ of $\mathcal{L}$ exists and is a compact operator, or the inverse of $\mathcal{L}$ does not exist. Note that for the second choice, $\lambda = 0$ is an eigenvalue of the operator $\mathcal{L}$.

Either there exists a unique weak solution $u \in H_0^1(\Omega)$, for each $f \in L^2(\Omega)$.

Or, there exist nontrivial solutions to the homogeneous equation

$$-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial u}{\partial x_i}\right]+\sum_{i=1}^{n}\beta_i(\mathbf{x})\frac{\partial}{\partial x_i}+\gamma(\mathbf{x})u=0,\quad \text{in}\quad \Omega,$$

$$u(\mathbf{x})=0,\quad \text{on}\quad \partial\Omega.$$

The Maximum Principle

Let Ω be a bounded connected open set in $\mathbb{R}^n$. Let α_{ij}, β_i and γ be bounded continuous functions defined in Ω, such that $\alpha_{ij}(\mathbf{x}) = \alpha_{ji}(\mathbf{x})$, for all $i = 1, 2, 3, \cdots, n$, $j = 1, 2, 3, \cdots, n$ and for all $\mathbf{x} \in \Omega$.

Suppose that there exists a positive constant $\Lambda > 0$, such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(\mathbf{x})\xi_i\xi_j \geq \Lambda|\xi|^2,$$

for all vectors $\xi \in \mathbb{R}^n$ and for all $\mathbf{x} \in \Omega$.

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$, or simply let $u \in C^2(\overline{\Omega})$. Define the following differential operators

$$\mathcal{L}_0 u = \gamma(\mathbf{x}) u,$$

$$\mathcal{L}_1 u = \sum_{i=1}^{n} \beta_i(\mathbf{x}) \frac{\partial}{\partial x_i} u,$$

$$\mathcal{L}_2 u = -\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} u,$$

$$\mathcal{L} u = -\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{i=1}^{n} \beta_i(\mathbf{x}) \frac{\partial}{\partial x_i} u + \gamma(\mathbf{x}) u,$$

These operators are linear transformations from $C^2(\Omega)$ to $C(\Omega)$. Clearly

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2.$$

The Weak Maximum Principle: If the function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies the conditions

$$-\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{i=1}^{n} \beta_i(\mathbf{x}) \frac{\partial}{\partial x_i} u + \gamma(\mathbf{x}) u \leq 0, \quad \text{in} \quad \Omega,$$

$$\gamma(\mathbf{x}) u(\mathbf{x}) \geq 0, \quad \text{in} \quad \Omega,$$

then

$$\max_{\overline{\Omega}} u(\mathbf{x}) = \max_{\partial\Omega} u(\mathbf{x}).$$

One of the key points in the proof is to construct a few auxiliary functions a and w:

$$a(\lambda, \mathbf{a}, \mathbf{x}) = \exp[\lambda(a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots\cdots + a_n x_n)],$$

and

$$w(\mathbf{x}) = u(\mathbf{x}) + \varepsilon a(\lambda, \mathbf{a}, \mathbf{x}),$$

for all $\mathbf{x} \in \Omega$, where a_1, a_2, a_3, $\cdots$, a_n and λ are real nonzero constants, $0 < \varepsilon \ll 1$ is a positive, but sufficiently small constant.

Let us compute the partial derivatives of a. We have

$$\begin{aligned} \frac{\partial}{\partial x_i} a(\lambda, \mathbf{a}, \mathbf{x}) &= a_i \lambda \exp[\lambda(a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots\cdots + a_n x_n)], \\ \frac{\partial^2}{\partial x_i \partial x_j} a(\lambda, \mathbf{a}, \mathbf{x}) &= a_i a_j \lambda^2 \exp[\lambda(a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots\cdots + a_n x_n)]. \end{aligned}$$

Now

$$\begin{aligned} & -\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} a(\mathbf{x}) + \sum_{i=1}^{n} \beta_i(\mathbf{x}) \frac{\partial}{\partial x_i} a(\mathbf{x}) + \gamma(\mathbf{x}) a(\mathbf{x}) \\ = & \left\{ -\lambda^2 \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}(\mathbf{x}) a_i a_j + \lambda \sum_{i=1}^{n} \beta_i(\mathbf{x}) a_i + \gamma(\mathbf{x}) \right\} \\ \cdot & \exp[\lambda(a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots\cdots + a_n x_n)] \\ \leq & \left\{ -\Lambda(|\lambda \mathbf{a}|)^2 + (|\lambda \mathbf{a}|) \left[\sum_{i=1}^{n} \|\beta_i\|^2_{L^\infty(\Omega)} \right]^{1/2} + \|\gamma\|_{L^\infty(\Omega)} \right\} \\ \cdot & \exp[\lambda(a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots\cdots + a_n x_n)] < 0, \end{aligned}$$

for all $\mathbf{x} \in \Omega$, where $|\lambda \mathbf{a}| \gg 1$.

The Hopf Lemma: Let the function $u \in C^2(\overline{\Omega})$ satisfy the following conditions

$$-\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} u(\mathbf{x}) + \sum_{i=1}^{n} \beta_i(\mathbf{x}) \frac{\partial}{\partial x_i} u(\mathbf{x}) + \gamma(\mathbf{x}) u(\mathbf{x}) \leq 0, \quad \text{in} \quad \Omega,$$

$$\gamma(\mathbf{x}) u(\mathbf{x}) \geq 0, \quad \text{in} \quad \Omega.$$

Suppose that there exists a ball $B(\mathbf{x}_0, R) \subset \Omega$ and there is a point $\mathbf{x}_1 \in \partial\Omega$, such that

$$\begin{aligned} &|\mathbf{x}_1 - \mathbf{x}_0| = R, \\ &u(\mathbf{x}) < u(\mathbf{x}_1), \end{aligned}$$

for all $\mathbf{x} \in B(\mathbf{x}_0, R)$. Then

$$\frac{\partial}{\partial \mathbf{n}} u(\mathbf{x}_1) = \frac{\mathbf{x}_1 - \mathbf{x}_0}{R} \cdot \nabla u(\mathbf{x}_1) > 0,$$

where the outward unit normal vector $\mathbf{n} = \frac{1}{R}(\mathbf{x}_1 - \mathbf{x}_0)$.

The key points in the proof are to construct the auxiliary functions $a(\lambda, \mathbf{x})$ and $w(\lambda, \mathbf{x})$:

$$a(\lambda, \mathbf{x}) = \exp(-\lambda|\mathbf{x} - \mathbf{x}_0|^2) - \exp(-\lambda R^2),$$

and

$$w(\lambda, \mathbf{x}) = u(\mathbf{x}) - u(\mathbf{x}_0) + \varepsilon a(\lambda, \mathbf{x}),$$

in the annular open region

$$\Omega_A = \left\{ \mathbf{x} \in \mathbb{R}^n : \frac{1}{2}R < |\mathbf{x} - \mathbf{x}_0| < R \right\}.$$

We can easily show that

$$\begin{aligned} &-\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} a(\mathbf{x}) \\ &+\sum_{i=1}^{n} \beta_i(\mathbf{x}) a(\mathbf{x}) + \gamma(\mathbf{x}) a(\mathbf{x}) < 0, \quad \text{in} \quad \Omega_A, \\ &\qquad a(\mathbf{x}) > 0, \quad \text{on} \quad |\mathbf{x} - \mathbf{x}_0| = \frac{1}{2}R, \\ &\qquad a(\mathbf{x}) = 0, \quad \text{on} \quad |\mathbf{x} - \mathbf{x}_0| = R, \end{aligned}$$

for all sufficiently large $\lambda \gg 1$. Moreover

$$\begin{aligned} & -\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(\mathbf{x})\frac{\partial^2}{\partial x_i \partial x_j}w(\mathbf{x}) \\ & +\sum_{i=1}^{n}\beta_i(\mathbf{x})\frac{\partial}{\partial x_i}w(\mathbf{x}) + \gamma(\mathbf{x})w(\mathbf{x}) \leq 0, \quad \text{in} \quad \Omega_A, \\ & \qquad w(\mathbf{x}) \leq 0, \quad \text{on} \quad |\mathbf{x}-\mathbf{x}_0| = \frac{1}{2}R, \\ & \qquad w(\mathbf{x}) \leq 0, \quad \text{on} \quad |\mathbf{x}-\mathbf{x}_0| = R, \end{aligned}$$

for all sufficiently large $\lambda \gg 1$ and for all sufficiently small $0 < \varepsilon \ll 1$.

By using the weak maximum principle, we see that

$$u(\mathbf{x}) + \varepsilon a(\lambda, \mathbf{x}) \leq 0,$$

in Ω_A. Note that $u(\mathbf{x}_1) + \varepsilon a(\lambda, \mathbf{x}_1) = 0$. Note that the outward unit normal vector is $\mathbf{n} = \frac{1}{R}(\mathbf{x}_1 - \mathbf{x}_0)$. Therefore

$$\frac{\partial}{\partial \mathbf{n}}w(\mathbf{x}_1) = \frac{1}{R}(\mathbf{x}_1 - \mathbf{x}_0) \cdot \nabla w(\mathbf{x}_1) \geq 0.$$

Let us compute the gradient of the function a:

$$\nabla a(\lambda, \mathbf{x}) = -2\lambda(\mathbf{x} - \mathbf{x}_0)\exp(-\lambda|\mathbf{x}-\mathbf{x}_0|^2).$$

Finally, we obtain

$$\begin{aligned} \frac{\partial}{\partial \mathbf{n}}u(\mathbf{x}_1) &= \frac{1}{R}(\mathbf{x}_1 - \mathbf{x}_0) \cdot \nabla u(\mathbf{x}_1) \\ &> \frac{2\lambda\varepsilon}{R}|\mathbf{x}_1 - \mathbf{x}_0|^2 \exp(-\lambda|\mathbf{x}_1 - \mathbf{x}_0|^2) > 0. \end{aligned}$$

The Strong Maximum Principle: Let the function $u \in$

$C^2(\Omega) \cap C(\overline{\Omega})$ satisfy the following conditions

$$-\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{i=1}^{n} \beta_i(\mathbf{x}) \frac{\partial}{\partial x_i} u + \gamma(\mathbf{x}) u \leq 0, \quad \text{in} \quad \Omega,$$

$$\gamma(\mathbf{x}) u(\mathbf{x}) \geq 0, \quad \text{in} \quad \Omega.$$

Let $\Omega \subset \mathbb{R}^n$ be connected. If there exists a point $\mathbf{x}_0 \in \Omega$, such that

$$u(\mathbf{x}_0) = \max_{\overline{\Omega}} u(\mathbf{x}),$$

then u is a constant function.

The Eigenvalue Problems of a Differential Operator

Let Ω be a bounded connected open set in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$. Let α_{ij} and γ be bounded smooth functions defined in Ω, such that $\alpha_{ij}(\mathbf{x}) = \alpha_{ji}(\mathbf{x})$, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$ and for all $\mathbf{x} \in \Omega$. Suppose that there exists a positive constant $\Lambda > 0$, such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}(\mathbf{x}) \xi_i \xi_j \geq \Lambda |\xi|^2,$$

for all $\xi \in \mathbb{R}^n$ and for all $\mathbf{x} \in \Omega$. Suppose that $\gamma(\mathbf{x}) \geq 0$, for all $\mathbf{x} \in \Omega$.

Define the linear differential operator

$$\mathcal{L}u = -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left[\alpha_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} u \right] + \gamma(\mathbf{x}) u,$$

for all $u \in C^2(\Omega)$.

Define the bilinear functional

$$B[u, v] = \sum_{i=1}^{n}\sum_{j=1}^{n} \int_{\Omega} \alpha_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} u(\mathbf{x}) \frac{\partial}{\partial x_j} v(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_{\Omega} \gamma(\mathbf{x}) u(\mathbf{x}) v(\mathbf{x}) \mathrm{d}\mathbf{x},$$

for all functions $u \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$.

Clearly, $\mathcal{L}$ is a linear differential operator. The inverse operator $\mathcal{L}^{-1} : L^2(\Omega) \to L^2(\Omega)$ exists and it is a bounded linear compact operator. Any nonzero complex number $\mu \in \sigma(\mathcal{L}^{-1})$ must be an eigenvalue of $\mathcal{L}^{-1}$. Consequently, λ must be an eigenvalue of $\mathcal{L}$, where $\lambda\mu = 1$. Obviously

$$\int_\Omega v(\mathbf{x})\mathcal{L}u(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega u(\mathbf{x})\mathcal{L}v(\mathbf{x})\mathrm{d}\mathbf{x},$$

for all $u \in C^2(\Omega) \cap H_0^1(\Omega)$ and $v \in C^2(\Omega) \cap H_0^1(\Omega)$. Hence, $\mathcal{L}$ is a self-adjoint operator. Therefore, all eigenvalues of $\mathcal{L}$ must be real. Moreover

$$B[u, v] = B[v, u],$$

for all $u \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$.

Theorem 1.

(1) All eigenvalues of the differential operator $\mathcal{L}$ are real.

(2) The eigenvalues of the operator $\mathcal{L}$ are positive. They may be arranged in the following increasing way

$$\begin{aligned}&0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots\cdots \le \lambda_k \le \cdots\cdots\\&\lim_{k\to\infty} \lambda_k = \infty.\end{aligned}$$

(3) The dimension of the eigenspace $\mathcal{N}(\lambda_k I - \mathcal{L})$ is finite. That is, the number of linearly independent solutions of

$$\begin{aligned}-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}\psi\right]&\\ +\gamma\psi(\mathbf{x}) = \lambda_k\psi, &\quad \text{in} \quad \Omega,\\ \psi = 0, &\quad \text{on} \quad \partial\Omega,\end{aligned}$$

is finite.
(4) There exists an orthonormal basis

$$\mathcal{B} = \{\psi_1, \psi_2, \psi_3, \cdots\cdots, \psi_k, \cdots\cdots\} \subset H_0^1(\Omega),$$

for the vector space $L^2(\Omega)$, where ψ_k is an eigenfunction of the operator $\mathcal{L}$ associated with the eigenvalue λ_k:

$$\begin{aligned} -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}\psi_k\right] & \\ +\gamma(\mathbf{x})\psi_k = \lambda_k\psi_k, &\quad \text{in} \quad \Omega, \\ \psi_k = 0, &\quad \text{on} \quad \partial\Omega, \end{aligned}$$

for all $k = 1, 2, 3, \cdots\cdots$.

That is

$$\begin{aligned} \int_\Omega |\psi_k(\mathbf{x})|^2 \mathrm{d}\mathbf{x} &= 1, \\ \int_\Omega \psi_k(\mathbf{x})\psi_l(\mathbf{x})\mathrm{d}\mathbf{x} &= 0, \end{aligned}$$

for all $k = 1, 2, 3, \cdots\cdots$ and $l = 1, 2, 3, \cdots\cdots$, with $k \neq l$.

For any function $u \in L^2(\Omega)$, the series

$$u = \sum_{k=1}^{\infty} a_k\psi_k,$$

is convergent in $L^2(\Omega)$, where

$$a_k = \int_\Omega u(\mathbf{x})\psi_k(\mathbf{x})\mathrm{d}\mathbf{x},$$

where $k = 1, 2, 3, \cdots\cdots$.

Note: The eigenfunctions

$$\psi_k \in C^\infty(\Omega),$$

for all $k = 1, 2, 3, \cdots\cdots$, if

$$\alpha_{ij} \in C^{\infty}(\Omega), \qquad \gamma \in C^{\infty}(\Omega),$$

for all $i = 1, 2, 3, \cdots\cdots, n$ and $j = 1, 2, 3, \cdots\cdots, n$.

Definition: The eigenvalue $\lambda_1 > 0$ is called the principal eigenvalue of the operator $\mathcal{L}$.

The Variational Principle for the Principal Eigenvalue

(1) There holds

$$\lambda_1 = \min_{u \in H_0^1(\Omega)} \left\{ B[u, u] : \int_{\Omega} |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = 1 \right\}.$$

(2) There exists a positive eigenfunction $\psi_1 \in C^{\infty}(\Omega) \cap H_0^1(\Omega)$ associated with the eigenvalue λ_1, such that

$$\begin{aligned} -\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left[\alpha_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} \psi_1 \right] & \\ + \gamma(\mathbf{x}) \psi_1 = \lambda_1 \psi_1, & \quad \text{in} \quad \Omega, \\ \psi_1 > 0, & \quad \text{in} \quad \Omega, \\ \psi_1 = 0, & \quad \text{on} \quad \partial\Omega, \\ \int_{\Omega} |\psi_1(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = 1, & \\ \lambda_1 = B[\psi_1, \psi_1]. & \end{aligned}$$

(3) If the function $\psi \in H_0^1(\Omega)$ is a solution of the boundary value problem

$$-\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left[\alpha_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} \psi \right] + \gamma(\mathbf{x}) \psi = \lambda_1 \psi, \quad \text{in} \quad \Omega,$$

$$\psi = 0, \quad \text{on} \quad \partial\Omega,$$

then

$$\psi(\mathbf{x}) = c\psi_1(\mathbf{x}),$$

for all $\mathbf{x} \in \Omega$ and for some real constant $c \in \mathbb{R}$.

Therefore, without loss of generality, we may claim that the eigenfunction ψ corresponding to the first eigenvalue is positive.

Proof. Recall that the set

$$\mathcal{B} = \{\psi_1, \psi_2, \psi_3, \cdots\cdots, \psi_k \cdots\cdots\}$$

of eigenfunctions is an orthonormal basis of $L^2(\Omega)$, that is,

$$\begin{aligned} -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}\psi_k\right] \\ +\gamma(\mathbf{x})\psi_k = \lambda_k\psi_k, \quad \text{in} \quad \Omega, \\ \psi_k = 0, \quad \text{on} \quad \partial\Omega, \\ \int_\Omega |\psi_k(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = 1, \\ \int_\Omega \psi_k(\mathbf{x})\psi_l(\mathbf{x})\mathrm{d}\mathbf{x} = 0, \end{aligned}$$

for all $k = 1, 2, 3, \cdots\cdots$ and $l = 1, 2, 3, \cdots\cdots$. Thus we have

$$B[\psi_k, \psi_k] = \lambda_k, \qquad B[\psi_k, \psi_l] = 0,$$

for all $k = 1, 2, 3, \cdots\cdot$ and $l = 1, 2, 3, \cdots\cdots$, with $k \neq l$.

Obviously

$$\lambda_1 = B[\psi_1, \psi_1] \geq \min_{u \in H_0^1(\Omega)} \left\{ B[u, u] : \int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = 1 \right\}.$$

For all functions $u \in L^2(\Omega)$, with $\int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = 1$, we may write

$$
\begin{aligned}
&u(\mathbf{x}) = \sum_{k=1}^{\infty} a_k \psi_k(\mathbf{x}), \\
&\int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = \sum_{k=1}^{\infty} |a_k|^2 = 1, \\
&a_k = \int_\Omega u(\mathbf{x}) \psi_k(\mathbf{x}) \mathrm{d}\mathbf{x}, \qquad k = 1, 2, 3, \cdots\cdots
\end{aligned}
$$

Now we have the following computations

$$
\begin{aligned}
B[u,u] &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_k a_l B[\psi_k, \psi_l] \\
&= \sum_{k=1}^{\infty} \lambda_k |a_k|^2 \geq \lambda_1 \sum_{k=1}^{\infty} |a_k|^2 = \lambda_1.
\end{aligned}
$$

Note that $H_0^1(\Omega)$ is a subspace of $L^2(\Omega)$. Hence

$$
\min_{u \in H_0^1(\Omega)} \left\{ B[u,u] : \int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = 1 \right\} \geq \lambda_1.
$$

Now we have proved that

$$
\lambda_1 = \min_{u \in H_0^1(\Omega)} \left\{ B[u,u] : \int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = 1 \right\}.
$$

Now let

$$
u \in H_0^1(\Omega), \qquad \int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = 1.
$$

Let us prove that u is a weak solution of the boundary value problem

$$
\begin{aligned}
-\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left[\alpha_{ij}(\mathbf{x}) \frac{\partial}{\partial x_i} u \right] \\
+\gamma(\mathbf{x}) u = \lambda_1 u, \quad \text{in} \quad \Omega, \\
u = 0, \quad \text{on} \quad \partial\Omega,
\end{aligned}
$$

if and only if

$$\lambda_1 = B[u,u].$$

This is not hard, because we have obtained the equality

$$B[u,u] = \sum_{k=1}^{\infty} \lambda_k |a_k|^2.$$

Let u be a nontrivial weak solution of the boundary value problem

$$\begin{aligned} -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i} u\right] & \\ +\gamma(\mathbf{x})u = \lambda_1 u, &\quad \text{in} \quad \Omega, \\ u = 0, &\quad \text{on} \quad \partial\Omega. \end{aligned}$$

Let

$$\begin{aligned} u^+(\mathbf{x}) &= \max\{+u(\mathbf{x}), 0\}, \\ u^-(\mathbf{x}) &= \max\{-u(\mathbf{x}), 0\}, \end{aligned}$$

for all $\mathbf{x} \in \Omega$. Then

$$u(\mathbf{x}) = u^+(\mathbf{x}) - u^-(\mathbf{x}), \qquad u^+(\mathbf{x})u^-(\mathbf{x}) = 0,$$

and

$$\begin{aligned} &\frac{\partial}{\partial x_i} u(\mathbf{x}) = \frac{\partial}{\partial x_i} u^+(\mathbf{x}) - \frac{\partial}{\partial x_i} u^-(\mathbf{x}), \\ &\frac{\partial}{\partial x_i} u^+(\mathbf{x}) \frac{\partial}{\partial x_i} u^-(\mathbf{x}) = 0, \end{aligned}$$

for all $\mathbf{x} \in \Omega$. Moreover

$$\int_\Omega |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = \int_\Omega |u^+(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \int_\Omega |u^-(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = 1.$$

Now we have

$$\begin{aligned} B[u^+, u^-] &= \sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\Omega}\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u^+(\mathbf{x})\frac{\partial}{\partial x_j}u^-(\mathbf{x})\mathrm{d}\mathbf{x} \\ &+ \int_{\Omega}\gamma(\mathbf{x})u^+(\mathbf{x})u^-(\mathbf{x})\mathrm{d}\mathbf{x} = 0. \end{aligned}$$

The following estimates are very interesting and important. There hold the estimates

$$
\begin{aligned}
& \lambda_1 = B[u,u] \\
= \ & \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega \alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u(\mathbf{x})\frac{\partial}{\partial x_j}u(\mathbf{x})\mathrm{d}\mathbf{x} \\
+ \ & \int_\Omega \gamma(\mathbf{x})|u(\mathbf{x})|^2\mathrm{d}\mathbf{x} \\
= \ & \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega \alpha_{ij}(\mathbf{x})\left[\frac{\partial}{\partial x_i}u^+(\mathbf{x}) - \frac{\partial}{\partial x_i}u^-(\mathbf{x})\right]\left[\frac{\partial}{\partial x_j}u^+(\mathbf{x}) - \frac{\partial}{\partial x_i}u^-(\mathbf{x})\right]\mathrm{d}\mathbf{x} \\
+ \ & \int_\Omega \gamma(\mathbf{x})|u^+(\mathbf{x}) - u^-(\mathbf{x})|^2\mathrm{d}\mathbf{x} \\
= \ & \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega \alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u^+(\mathbf{x})\frac{\partial}{\partial x_j}u^+(\mathbf{x})\mathrm{d}\mathbf{x} \\
+ \ & \sum_{i=1}^{n}\sum_{j=1}^{n}\int_\Omega \alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u^-(\mathbf{x})\frac{\partial}{\partial x_j}u^-(\mathbf{x})\mathrm{d}\mathbf{x} \\
+ \ & \int_\Omega \gamma(\mathbf{x})|u^+(\mathbf{x})|^2\mathrm{d}\mathbf{x} \\
+ \ & \int_\Omega \gamma(\mathbf{x})|u^-(\mathbf{x})|^2\mathrm{d}\mathbf{x} \\
\geq \ & \lambda_1\int_\Omega |u^+(\mathbf{x})|^2\mathrm{d}\mathbf{x} \\
+ \ & \lambda_1\int_\Omega |u^-(\mathbf{x})|^2\mathrm{d}\mathbf{x} = \lambda_1.
\end{aligned}
$$

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\Omega}\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u(\mathbf{x})\frac{\partial}{\partial x_j}u(\mathbf{x})\mathrm{d}\mathbf{x}+\int_{\Omega}\gamma(\mathbf{x})\mathrm{d}\mathbf{x}\\
&\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\Omega}\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u(\mathbf{x})\frac{\partial}{\partial x_j}u(\mathbf{x})\mathrm{d}\mathbf{x}+\int_{\Omega}\gamma(\mathbf{x})\mathrm{d}\mathbf{x}\\
&\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\Omega}\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u(\mathbf{x})\frac{\partial}{\partial x_j}u(\mathbf{x})\mathrm{d}\mathbf{x}+\int_{\Omega}\gamma(\mathbf{x})\mathrm{d}\mathbf{x}\\
&B[u,u]B[u^+,u^+]B[u^-,u^-]+.
\end{aligned}$$

Then

$$\begin{aligned}
B[u^+,u^+] &= \lambda_1\int_{\Omega}|u^+(\mathbf{x})|^2\mathrm{d}\mathbf{x},\\
B[u^-,u^-] &= \lambda_1\int_{\Omega}|u^-(\mathbf{x})|^2\mathrm{d}\mathbf{x}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u^+(\mathbf{x})\right]&\\
+\gamma u^+(\mathbf{x}) &= \lambda_1 u^+(\mathbf{x}), \quad \text{in} \quad \Omega,\\
u^+ &= 0, \quad \text{on} \quad \partial\Omega,
\end{aligned}$$

and

$$\begin{aligned}
-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u^-(\mathbf{x})\right]&\\
+\gamma u^-(\mathbf{x})(\mathbf{x}) &= \lambda_1 u^-(\mathbf{x}), \quad \text{in} \quad \Omega,\\
u^- &= 0, \quad \text{on} \quad \partial\Omega.
\end{aligned}$$

Finally, by using the strong maximum principle, we find that

$$\begin{aligned}
\text{Either } u > 0, &\quad \text{in} \quad \Omega,\\
\text{or } u < 0, &\quad \text{in} \quad \Omega.
\end{aligned}$$

Let ϕ and ψ be nontrivial weak solutions of the boundary value problem

$$\begin{aligned} -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u\right] & \\ +\gamma(\mathbf{x})u = \lambda_1 u, & \\ u = 0, \quad \text{on} \quad \partial\Omega. & \end{aligned}$$

Note that

$$\int_\Omega \phi(\mathbf{x})\mathrm{d}\mathbf{x} \neq 0, \qquad \int_\Omega \psi(\mathbf{x})\mathrm{d}\mathbf{x} \neq 0.$$

There exists a real constant c, such that

$$\int_\Omega [\phi(\mathbf{x}) - c\psi(\mathbf{x})]\mathrm{d}\mathbf{x} = 0.$$

Note that $u = \phi - c\psi$ is also a weak solution of

$$\begin{aligned} -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[\alpha_{ij}(\mathbf{x})\frac{\partial}{\partial x_i}u\right] & \\ +\gamma(\mathbf{x})u = \lambda_1 u, & \qquad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. & \end{aligned}$$

Therefore, we get $\phi = c\psi$. The proof of Theorem is finished now. $\square$

$$\mathcal{L}u = -\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(\mathbf{x})\frac{\partial^2}{\partial x_i \partial x_j}u + \sum_{i=1}^{n}\beta_i(\mathbf{x})\frac{\partial}{\partial x_i}u + \gamma(\mathbf{x})u.$$

$(t^{-1/2}\eta), (t^{-1/2}\eta), (t^{-1/2}\eta), (t^{-1/2}\eta), \cdots\cdots\cdots\cdots, 1.$

$\lambda_1 \frac{\partial}{\partial x_i}\psi = 0$. With respect to the inner product, the set of eigenfunctions is an orthonormal basis of the space. $\psi_1, \psi_2, \psi_3, \cdots, \psi_k$

$$\begin{aligned}
&+\ \exp[\lambda(a_1x_1 + a_2x_2 + a_3x_3 + \cdots\cdots + a_nx_n)] \\
&+\ \exp[\lambda(a_1x_1 + a_2x_2 + a_3x_3 + \cdots\cdots + a_nx_n)] \\
&+\ \exp[\lambda(a_1x_1 + a_2x_2 + a_3x_3 + \cdots\cdots + a_nx_n)] \\
&+\ \exp[\lambda(a_1x_1 + a_2x_2 + a_3x_3 + \cdots\cdots + a_nx_n)] \\
&+\ \exp[\lambda](a_1x_1 + a_2x_2 + a_3x_3 + \cdots\cdots + a_nx_n) \\
&+\ \exp[\lambda(a_1x_1 + a_2x_2 + a_3x_3 + \cdots\cdots + a_nx_n)] \\
&+\ \exp[\lambda(a_1x_1 + a_2x_2 + a_3x_3 + \cdots\cdots + a_nx_n)] \\
&+\ \exp[\lambda(a_1x_1 + a_2x_2 + a_3x_3 + \cdots\cdots + a_nx_n)] \\
&+\ \exp[\lambda(a_1x_1 + a_2x_2 + a_3x_3 + \cdots\cdots + a_nx_n)] \\
&+\ \exp[\lambda(a_1x_1 + a_2x_2 + a_3x_3 + \cdots\cdots + a_nx_n)] \\
&+\ \exp[\lambda(a_1x_1 + a_2x_2 + a_3x_3 + \cdots\cdots + a_nx_n)]
\end{aligned}$$

Applications of the method of Fourier transformation to a hyperbolic equation

Theorem 1. Consider the Cauchy problems for the one-dimensional nonhomogeneous hyperbolic equation

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}u - \alpha^2 \frac{\partial^2}{\partial x^2}u = f(x,t),\\
&u(x,0) = u_0(x), \qquad \frac{\partial}{\partial t}u(x,0) = v_0(x).
\end{aligned}$$

In this problem, $\alpha > 0$ is a positive constant, the initial functions $u_0 \in C^2(\mathbb{R})$ and $v_0 \in C^1(\mathbb{R})$, the external force $f \in C^1(\mathbb{R} \times \mathbb{R}^+)$. There exists a unique global classical solution $u \in C^2(\mathbb{R} \times \mathbb{R}^+)$, given explicitly by

$$\begin{aligned}
u(x,t) \;=\;& \frac{1}{2}[u_0(x+\alpha t) + u_0(x-\alpha t)] + \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} v_0(\xi)\mathrm{d}\xi\\
&+\; \frac{1}{2\alpha}\int_0^t \int_{x+\alpha(t-\tau)}^{x+\alpha(t-\tau)} f(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau,
\end{aligned}$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

Proof. Performing the Fourier transformation to the initial value problems leads to

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\widehat{u}(\xi,t) + \alpha^2|\xi|^2\widehat{u}(\xi,t) = \widehat{f}(\xi,t),\\
&\widehat{u}(\xi,0) = \widehat{u}_0(\xi), \qquad \frac{\partial}{\partial t}\widehat{u}(\xi,0) = \widehat{v}_0(\xi).
\end{aligned}$$

Solving the initial value problems by using the method of variation of parameters, we have the explicit representation of the Fourier transformation

$$\begin{aligned}\widehat{u}(\xi,t) = \frac{1}{2}\left[\widehat{u}_0(\xi) + \frac{1}{\mathrm{i}\alpha\xi}\widehat{v}_0(\xi)\right]\exp(+\mathrm{i}t\alpha\xi)\\ + \ \frac{1}{2}\left[\widehat{u}_0(\xi) - \frac{1}{\mathrm{i}\alpha\xi}\widehat{v}_0(\xi)\right]\exp(-\mathrm{i}t\alpha\xi)\\ + \ \frac{1}{2\mathrm{i}\alpha\xi}\int_0^t \exp[+\mathrm{i}\alpha\xi(t-\tau)]\widehat{f}(\xi,\tau)\mathrm{d}\tau\\ - \ \frac{1}{2\mathrm{i}\alpha\xi}\int_0^t \exp[-\mathrm{i}\alpha\xi(t-\tau)]\widehat{f}(\xi,\tau)\mathrm{d}\tau,\end{aligned}$$

for all $(\xi, t) \in \mathbb{R} \times \mathbb{R}^+$.

Recall that

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp(+\mathrm{i}x\xi)\widehat{\phi}(\xi)\mathrm{d}\xi,$$

for all $\xi \in \mathbb{R}$, where the function $\phi \in \mathcal{S}(\mathbb{R})$. Now if we perform the inverse Fourier transformation to the representation and use some elementary property, then we obtain the solution representation

$$\begin{aligned}u(x,t) &= \frac{1}{2}[u_0(x-\alpha t) + u_0(x+\alpha t)] + \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} v_0(\eta)\mathrm{d}\eta\\ &+ \ \frac{1}{2\alpha}\int_0^t \int_{x-\alpha(t-\tau)}^{x+\alpha(t-\tau)} f(\eta,\tau)\mathrm{d}\eta\mathrm{d}\tau,\end{aligned}$$

for all $(x, t) \in \mathbb{R} \times \mathbb{R}^+$.

$$\frac{\partial}{\partial x}u, \frac{\partial}{\partial y}u, \widehat{u}(\xi,t)(t-)(t-)(t-\tau)(t-\tau)$$

Sobolev Spaces

Theorem 1. Let the positive integer $n \geq 2$, let the positive constant $p \geq 1$, such that $1 \leq p < n$. Then $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$, where $p^* > p$ and $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. Moreover, there holds the following estimate

$$\left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{p^*} \mathrm{d}\mathbf{x} \right\}^{\frac{1}{p^*}} \leq \frac{1}{2} \left\{ \int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{1}{p}},$$

for all functions $u \in C_0^\infty(\mathbb{R}^n)$.

Proof. First of all, by using fundamental theorem of calculus, it is easy to show that

$$|u(\mathbf{x})| \leq \frac{1}{2} \int_{\mathbb{R}} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}x_i,$$

for all functions $u \in C_0^\infty(\mathbb{R}^n)$, where $i = 1, 2, 3, \cdots, n$, the vector $\mathbf{x} = (x_1, x_2, x_3, \cdots, x_n)$. Now

$$|u(\mathbf{x})|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left\{ \frac{1}{2} \int_{\mathbb{R}} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}x_i \right\}^{\frac{1}{n-1}},$$

for all functions $u \in C_0^\infty(\mathbb{R}^n)$.

Let us integrate this inequality step by step with respect to the variables x_1, x_2, x_3, $\cdots$, x_n, respectively. In each step, we will use the general Hölder's inequality.

Step 1: The integration with respect to x_1 leads to the estimate

$$\begin{aligned}
&\int_{\mathbb{R}} |u(\mathbf{x})|^{\frac{n}{n-1}} \mathrm{d}x_1 \\
\leq\ & \left\{ \frac{1}{2} \int_{\mathbb{R}} \left| \frac{\partial}{\partial x_1} u(\mathbf{x}) \right| \mathrm{d}x_1 \right\}^{\frac{1}{n-1}} \\
\cdot\ & \prod_{i=2}^{n} \left\{ \frac{1}{2} \int_{\mathbb{R}^2} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_i \right\}^{\frac{1}{n-1}} .
\end{aligned}$$

Step 2: The integration with respect to x_2 leads to the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^2} |u(\mathbf{x})|^{\frac{n}{n-1}} \mathrm{d}x_1 \mathrm{d}x_2 \\
\leq\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^2} \left| \frac{\partial}{\partial x_1} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \right\}^{\frac{1}{n-1}} \\
\cdot\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^2} \left| \frac{\partial}{\partial x_2} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \right\}^{\frac{1}{n-1}} \\
\cdot\ & \prod_{i=3}^{n} \left\{ \frac{1}{2} \int_{\mathbb{R}^3} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_i \right\}^{\frac{1}{n-1}} .
\end{aligned}$$

Step 3: The integration with respect to x_3 leads to the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^3} |u(\mathbf{x})|^{\frac{n}{n-1}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\
\leq\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^3} \left| \frac{\partial}{\partial x_1} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \right\}^{\frac{1}{n-1}} \\
\cdot\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^3} \left| \frac{\partial}{\partial x_2} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \right\}^{\frac{1}{n-1}} \\
\cdot\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^3} \left| \frac{\partial}{\partial x_3} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \right\}^{\frac{1}{n-1}} \\
\cdot\ & \prod_{i=4}^{n} \left\{ \frac{1}{2} \int_{\mathbb{R}^4} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_i \right\}^{\frac{1}{n-1}} .
\end{aligned}$$

Step 4: The integration with respect to x_4 leads to the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^4} |u(\mathbf{x})|^{\frac{n}{n-1}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \\
\leq\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^4} \left| \frac{\partial}{\partial x_1} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \right\}^{\frac{1}{n-1}} \\
\cdot\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^4} \left| \frac{\partial}{\partial x_2} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \right\}^{\frac{1}{n-1}} \\
\cdot\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^4} \left| \frac{\partial}{\partial x_3} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \right\}^{\frac{1}{n-1}} \\
\cdot\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^4} \left| \frac{\partial}{\partial x_4} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \right\}^{\frac{1}{n-1}} \\
\cdot\ & \prod_{i=5}^{n} \left\{ \frac{1}{2} \int_{\mathbb{R}^5} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \mathrm{d}x_4 \mathrm{d}x_i \right\}^{\frac{1}{n-1}} .
\end{aligned}$$

Step 5: The integration with respect to x_k leads to the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^k} |u(\mathbf{x})|^{\frac{n}{n-1}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \cdots \mathrm{d}x_k \\
\leq\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^k} \left| \frac{\partial}{\partial x_1} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \cdots \mathrm{d}x_k \right\}^{\frac{n}{n-1}} \\
\cdot\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^k} \left| \frac{\partial}{\partial x_2} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \cdots \mathrm{d}x_k \right\}^{\frac{n}{n-1}} \\
\cdot\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^k} \left| \frac{\partial}{\partial x_3} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \cdots \mathrm{d}x_k \right\}^{\frac{n}{n-1}} \\
\cdot\ & \cdots\cdots \\
\cdot\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^k} \left| \frac{\partial}{\partial x_k} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \cdots \mathrm{d}x_k \right\}^{\frac{n}{n-1}} \\
\cdot\ & \cdots\cdots \\
\cdot\ & \left\{ \frac{1}{2} \int_{\mathbb{R}^{k+1}} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \cdots \mathrm{d}x_k \mathrm{d}x_i \right\}^{\frac{n}{n-1}} .
\end{aligned}$$

Step 6: The integration with respect to x_n leads to the estimate

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}} \mathrm{d}\mathbf{x} \\
\leq\ & \prod_{i=1}^{n} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n-1}} .
\end{aligned}$$

Now let $1 < p < n$ and define $r = \frac{(n-1)p}{n-p}$. Then $\frac{nr}{n-1} = \frac{np}{n-p}$ and $\frac{n}{n-1}\frac{r}{r-1} = \frac{p}{p-1}$. Let $w(\mathbf{x}) = |u(\mathbf{x})|^r$, where $u \in C_0^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n} |w(\mathbf{x})|^{\frac{n}{n-1}} \mathrm{d}\mathbf{x} \\
\leq\ & \prod_{i=1}^{n} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} w(\mathbf{x}) \right| \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n-1}} .
\end{aligned}$$

That is

$$\begin{aligned}&\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{nr}{n-1}} \mathrm{d}\mathbf{x}\\ \leq\ & \prod_{i=1}^{n} \left\{ \frac{r}{2} \int_{\mathbb{R}^n} |u(\mathbf{x})|^{r-1} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n-1}}.\end{aligned}$$

By using Hölder's inequality, we have the following estimate

$$\begin{aligned}&\int_{\mathbb{R}^n} |u(\mathbf{x})|^{r-1} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}\mathbf{x}\\ \leq\ & \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{nr}{n-1}} \mathrm{d}\mathbf{x} \right\}^{\frac{p-1}{p}}\\ \cdot\ & \left\{ \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right|^p \mathrm{d}\mathbf{x} \right\}^{\frac{1}{p}}.\end{aligned}$$

Now we have

$$\begin{aligned}&\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{nr}{n-1}} \mathrm{d}\mathbf{x}\\ \leq\ & \prod_{i=1}^{n} \left\{ \frac{r}{2} \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{nr}{n-1}} \mathrm{d}\mathbf{x} \right\}^{\frac{p-1}{(n-1)p}}\\ \cdot\ & \left\{ \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right|^p \mathrm{d}\mathbf{x} \right\}^{\frac{1}{(n-1)p}}.\end{aligned}$$

Simplifying, we have

$$\begin{aligned}&\left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{np}{n-p}} \mathrm{d}\mathbf{x} \right\}^{\frac{n-p}{(n-1)p}}\\ \leq\ & \prod_{i=1}^{n} \left\{ \frac{r}{2} \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right|^p \mathrm{d}\mathbf{x} \right\}^{\frac{1}{(n-1)p}}.\end{aligned}$$

Finally, we obtain the desired estimate

$$\begin{aligned} &\left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{np}{n-p}} \mathrm{d}\mathbf{x}\right\}^{\frac{n-p}{np}} \\ \leq\ & \frac{(n-1)p}{2(n-p)} \prod_{i=1}^{n} \left\{\int_{\mathbb{R}^n} \left|\frac{\partial}{\partial x_i} u(\mathbf{x})\right|^p \mathrm{d}\mathbf{x}\right\}^{\frac{1}{np}}, \end{aligned}$$

for all functions $u \in C_0^\infty(\mathbb{R}^n).r^{\frac{n}{n-1}} r^{\frac{n}{n-1}} 2^{\frac{n}{n-1}} 2^{\frac{n}{n-1}}$

Theorem 2. Let the positive integers $m \geq 1$ and $n \geq 2$, let the positive constant $p \geq 1$, such that $mp < n$. Consider the Sobolev space $W^{m,p}(\mathbb{R}^n)$.

Define the positive constants p^*, p^{**}, p^{***}, p^{****}, $\cdots$, $p^{****\cdots*}$ by

$$\begin{aligned} \frac{1}{p^*} &= \frac{1}{p} - \frac{1}{n}, \\ \frac{1}{p^{**}} &= \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}, \\ \frac{1}{p^{***}} &= \frac{1}{p^{**}} - \frac{1}{n} = \frac{1}{p} - \frac{3}{n}, \\ \frac{1}{p^{****}} &= \frac{1}{p^{***}} - \frac{1}{n} = \frac{1}{p} - \frac{4}{n}, \\ \cdots\cdots & \quad \cdots\cdots \\ \frac{1}{p^{***\cdots*}} &= \frac{1}{p^{**\cdots*}} - \frac{1}{n} = \frac{1}{p} - \frac{m}{n}. \end{aligned}$$

Then

$$\begin{aligned} & W^{m,p}(\mathbb{R}^n) \subset W^{m-1,p^*}(\mathbb{R}^n) \subset W^{m-2,p^{**}}(\mathbb{R}^n) \\ \subset\ & W^{m-3,p^{***}}(\mathbb{R}^n) \subset \cdots\cdots \subset L^{p^{***\cdots*}}(\mathbb{R}^n). \end{aligned}$$

Moreover, there hold the following estimates.

(1)

$$\left\{\int_{\mathbb{R}^n}\left|\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}}u(\mathbf{x})\right|^{p^*}\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p^*}}$$

$$\leq\ C_1\left\{\int_{\mathbb{R}^n}\left|\nabla\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}}u(\mathbf{x})\right|^{p}\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}},$$

for all $u \in W^{m,p}(\mathbb{R}^n)$, where $\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n \leq m-1$.

(2)

$$\left\{\int_{\mathbb{R}^n}\left|\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}}u(\mathbf{x})\right|^{p^{**}}\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p^{**}}}$$

$$\leq\ C_2\left\{\int_{\mathbb{R}^n}\left|\nabla\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}}u(\mathbf{x})\right|^{p^*}\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p^*}},$$

for all $u \in W^{m-1,p^*}(\mathbb{R}^n)$, where $\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n \leq m-2$.

(3)

$$\left\{\int_{\mathbb{R}^n}\left|\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}}u(\mathbf{x})\right|^{p^{***}}\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p^{***}}}$$

$$\leq\ C_3\left\{\int_{\mathbb{R}^n}\left|\nabla\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}}u(\mathbf{x})\right|^{p^{**}}\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p^{**}}},$$

for all $u \in W^{m-2,p^{**}}(\mathbb{R}^n)$, where $\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n \leq m-3$.

(4)

$$\left\{\int_{\mathbb{R}^n}\left|\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}}u(\mathbf{x})\right|^{p^{****}}\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p^{****}}}$$

$$\leq\ C_4\left\{\int_{\mathbb{R}^n}\left|\nabla\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}}u(\mathbf{x})\right|^{p^{***}}\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p^{***}}},$$

for all $u \in W^{m-3,p^{***}}(\mathbb{R}^n)$, where $\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n \leq m-4$.
(5)

$$\left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^{p^{***\cdots *}} \mathrm{d}\mathbf{x}\right\}^{\frac{1}{p^{***\cdots *}}}$$
$$\leq \; C_m \left\{\int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^{p^{**\cdots *}} \mathrm{d}\mathbf{x}\right\}^{\frac{1}{p^{**\cdots *}}},$$

for all $u \in W^{1,p^{**\cdots *}}(\mathbb{R}^n)$.

Therefore, we obtain the estimate

$$\left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^{p^{***\cdots *}} \mathrm{d}\mathbf{x}\right\}^{\frac{1}{p^{***\cdots *}}}$$
$$\leq \; C\left\{\sum_{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n=m} \int_{\mathbb{R}^n} \left|\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots \partial x_n^{\alpha_n}} u(\mathbf{x})\right|^p \mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}},$$

for all functions $u \in W^{m,p}(\mathbb{R}^n)$.

Lemma 1. There holds the following estimate

$$\int_{B(\mathbf{x}_0,R)} |u(\mathbf{x})-u(\mathbf{x}_0)|\mathrm{d}\mathbf{x} \leq \frac{1}{n}\mathbb{R}^n \int_{B(\mathbf{x}_0,R)} \frac{|\nabla u(\mathbf{x})|}{|\mathbf{x}-\mathbf{x}_0|^{n-1}}\mathrm{d}\mathbf{x},$$

for all functions $u \in C^1(\mathbb{R}^n)$, for all points $\mathbf{x}_0 \in \mathbb{R}^n$ and for all radius $R > 0$.

Lemma 1. Let $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ be positive constants, such that $\alpha > \beta > \gamma \geq 1$. There holds the following estimate

$$\int_{\mathbb{R}^n} |\phi(\mathbf{x})|^\beta \mathrm{d}\mathbf{x} \leq \left\{ \int_{\mathbb{R}^n} |\phi(\mathbf{x})|^\alpha \mathrm{d}\mathbf{x} \right\}^{\frac{\beta-\gamma}{\alpha-\gamma}} \left\{ \int_{\mathbb{R}^n} |\phi(\mathbf{x})|^\gamma \mathrm{d}\mathbf{x} \right\}^{\frac{\alpha-\beta}{\alpha-\gamma}},$$

for all functions $\phi \in L^\alpha(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n)$.

Proof. Note that $\alpha - \beta > 0$, $\beta - \gamma > 0$, $\alpha - \gamma > 0$ and $\alpha - \gamma = \alpha - \beta + \beta - \gamma$. Also note that

$$\frac{\beta-\gamma}{\alpha-\gamma} + \frac{\alpha-\beta}{\alpha-\gamma} = 1, \qquad \beta = \alpha\frac{\beta-\gamma}{\alpha-\gamma} + \gamma\frac{\alpha-\beta}{\alpha-\gamma}.$$

The proof followings from a simple application of Hölder's inequality. We have

$$\int_{\mathbb{R}^n} |\phi(\mathbf{x})|^\beta \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} |\phi(\mathbf{x})|^{\alpha\frac{\beta-\gamma}{\alpha-\gamma}+\gamma\frac{\alpha-\beta}{\alpha-\gamma}} \mathrm{d}\mathbf{x}$$
$$\leq \left\{ \int_{\mathbb{R}^n} |\phi(\mathbf{x})|^\alpha \mathrm{d}\mathbf{x} \right\}^{\frac{\beta-\gamma}{\alpha-\gamma}} \left\{ \int_{\mathbb{R}^n} |\phi(\mathbf{x})|^\gamma \mathrm{d}\mathbf{x} \right\}^{\frac{\alpha-\beta}{\alpha-\gamma}}.$$

Below there are four simple applications to this inequality.

Now let $p > n \geq 2$. By using the above estimate, we have

$$\int_{\mathbb{R}^n} |u(\mathbf{x})|^p \mathrm{d}\mathbf{x}$$
$$\leq \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}p} \mathrm{d}\mathbf{x} \right\}^{\frac{(n-1)(p-n)}{(p+1-n)n}} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^n \mathrm{d}\mathbf{x} \right\}^{\frac{p}{(p+1-n)n}}.$$

Note that $p > n > 1$. Now we have

$$(p-1)\frac{n}{n-1} > p > n, \qquad (p-1)\frac{n}{n-1} < \frac{n}{n-1}p.$$

Therefore, we have the following estimate

$$\begin{aligned}&\int_{\mathbb{R}^n}|u(\mathbf{x})|^{\frac{n}{n-1}(p-1)}\mathrm{d}\mathbf{x}\\ \leq\ &\left\{\int_{\mathbb{R}^n}|u(\mathbf{x})|^{\frac{n}{n-1}p}\mathrm{d}\mathbf{x}\right\}^{\frac{p-n}{p+1-n}}\left\{\int_{\mathbb{R}^n}|u(\mathbf{x})|^{n}\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p+1-n}}.\end{aligned}$$

Now let $\lambda > p > n$. Note that

$$\frac{n}{n-1}\lambda > \frac{p}{p-1}(\lambda-1) > p > n > 1.$$

We have the estimates

$$\begin{aligned}&\int_{\mathbb{R}^n}|u(\mathbf{x})|^{\lambda}\mathrm{d}\mathbf{x}\\ \leq\ &\left\{\int_{\mathbb{R}^n}|u(\mathbf{x})|^{\frac{n}{n-1}\lambda}\mathrm{d}\mathbf{x}\right\}^{\frac{(n-1)(\lambda-p)}{p+(\lambda-p)n}}\\ \cdot\ &\left\{\int_{\mathbb{R}^n}|u(\mathbf{x})|^{p}\mathrm{d}\mathbf{x}\right\}^{\frac{\lambda}{p+(\lambda-p)n}},\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n}|u(\mathbf{x})|^{\frac{p}{p-1}(\lambda-1)}\mathrm{d}\mathbf{x}\\ \leq\ &\left\{\int_{\mathbb{R}^n}|u(\mathbf{x})|^{\frac{n}{n-1}\lambda}\mathrm{d}\mathbf{x}\right\}^{\frac{(n-1)(\lambda-p)p}{(p-1)[p+(\lambda-p)n]}}\\ \cdot\ &\left\{\int_{\mathbb{R}^n}|u(\mathbf{x})|^{p}\mathrm{d}\mathbf{x}\right\}^{\frac{(\lambda-1)p-(\lambda-p)n}{(p-1)[p+(\lambda-p)n]}}.\end{aligned}$$

Lemma 2. Let the positive integer $n \geq 2$. Then

$$W^{1,n}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n),$$

for all positive constants $p > n$ (but not for the case $p = \infty$). Moreover, there exists a positive constant $C = C(n,p) > 0$, such that there holds the following estimate

$$\left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^p \mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}} \leq \left(\frac{1}{2}p\right)^{\frac{p-n}{p}} \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^n \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^n \mathrm{d}\mathbf{x}\right\}^{\frac{1}{n}},$$

for all functions $u \in W^{1,n}(\mathbb{R})$.

Proof. In the following well-known estimate

$$\int_{\mathbb{R}^n} |\phi(\mathbf{x})|^{\frac{n}{n-1}} \mathrm{d}\mathbf{x} \leq \prod_{i=1}^{n} \left\{\frac{1}{2}\int_{\mathbb{R}^n} \left|\frac{\partial}{\partial x_i}\phi(\mathbf{x})\right| \mathrm{d}\mathbf{x}\right\}^{\frac{1}{n-1}},$$

letting $\phi(\mathbf{x}) = |u(\mathbf{x})|^p$, where $u \in W^{1,n}(\mathbb{R}^n)$ and $p > n$ is a positive constant, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}p} \mathrm{d}\mathbf{x} \\
\leq\ & \prod_{i=1}^{n} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} [|u(\mathbf{x})|^p] \right| \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n-1}} \\
\leq\ & \prod_{i=1}^{n} \left\{ \frac{1}{2} p \int_{\mathbb{R}^n} |u(\mathbf{x})|^{p-1} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n-1}} \\
\leq\ & \left(\frac{1}{2} p \right)^{\frac{n}{n-1}} \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}(p-1)} \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n}} \\
\cdot\ & \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right|^n \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n(n-1)}} \\
\leq\ & \left(\frac{1}{2} p \right)^{\frac{n}{n-1}} \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^n \mathrm{d}\mathbf{x} \right\}^{\frac{1}{(p+1-n)n}} \\
\cdot\ & \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}p} \mathrm{d}\mathbf{x} \right\}^{\frac{p-n}{(p+1-n)n}} \\
\cdot\ & \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right|^n \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n(n-1)}} \\
\leq\ & \left(\frac{1}{2} p \right)^{\frac{n}{n-1}} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^n \mathrm{d}\mathbf{x} \right\}^{\frac{1}{p+1-n}} \\
\cdot\ & \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}p} \mathrm{d}\mathbf{x} \right\}^{\frac{p-n}{p+1-n}} \\
\cdot\ & \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right|^n \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n(n-1)}} .
\end{aligned}$$

Let us cancel out

$$\left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}p} \mathrm{d}\mathbf{x}\right\}^{\frac{p-n}{p+1-n}},$$

then we get the estimate

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}p} \mathrm{d}\mathbf{x}\right\}^{\frac{1}{p+1-n}} \\
\leq\ & \left(\frac{1}{2}p\right)^{\frac{n}{n-1}} \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^n \mathrm{d}\mathbf{x}\right\}^{\frac{1}{p+1-n}} \\
\cdot\ & \prod_{i=1}^{n} \left\{\int_{\mathbb{R}^n} \left|\frac{\partial}{\partial x_i} u(\mathbf{x})\right|^n \mathrm{d}\mathbf{x}\right\}^{\frac{1}{n(n-1)}}.
\end{aligned}$$

That is

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}p} \mathrm{d}\mathbf{x} \\
\leq\ & \left(\frac{1}{2}p\right)^{\frac{(p+1-n)n}{n-1}} \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^n \mathrm{d}\mathbf{x}\right\} \left\{\int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^n \mathrm{d}\mathbf{x}\right\}^{\frac{p+1-n}{n-1}}.
\end{aligned}$$

Finally, we obtain the desired estimate

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u(\mathbf{x})|^p \mathrm{d}\mathbf{x} \\
\leq\ & \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}p} \mathrm{d}\mathbf{x}\right\}^{\frac{(n-1)(p-n)}{(p+1-n)n}} \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^n \mathrm{d}\mathbf{x}\right\}^{\frac{p}{(p+1-n)n}} \\
\leq\ & \left(\frac{1}{2}p\right)^{p-n} \left\{\int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^n \mathrm{d}\mathbf{x}\right\}^{\frac{p-n}{n}} \\
\cdot\ & \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^n \mathrm{d}\mathbf{x}\right\}^{\frac{(n-1)(p-n)}{(p+1-n)n}} \\
\cdot\ & \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^n \mathrm{d}\mathbf{x}\right\}^{\frac{p}{(p+1-n)n}} \\
\leq\ & \left(\frac{1}{2}p\right)^{p-n} \left\{\int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^n \mathrm{d}\mathbf{x}\right\}^{\frac{p-n}{n}} \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^n \mathrm{d}\mathbf{x}\right\} \\
\leq\ & \left(\frac{1}{2}p\right)^{p-n} \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^n \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^n \mathrm{d}\mathbf{x}\right\}^{\frac{p}{n}}.
\end{aligned}$$

The proof of the Theorem is finished now.

Theorem . Let the positive integer $n \geq 2$, let the positive constant $p > n$. Then

$$W^{1,p}(\mathbb{R}^n) \subset L^{\lambda}(\mathbb{R}^n),$$

for all positive constants $\lambda > p > n$. Moreover, there exists a positive constant $C(\lambda, n, p) = (\frac{1}{2}\lambda)^{\frac{(\lambda-p)n}{p}} > 0$, such that there holds the following estimate

$$\left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\lambda} \mathrm{d}\mathbf{x}\right\}^{\frac{1}{\lambda}} \leq \left(\frac{1}{2}\lambda\right)^{\frac{(\lambda-p)n}{p}} \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^p \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^p \mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}},$$

for all functions $u \in W^{1,p}(\mathbb{R}^n)$.

Proof. In the well-known estimate

$$\int_{\mathbb{R}^n} |\phi(\mathbf{x})|^{\frac{n}{n-1}} \mathrm{d}\mathbf{x} \leq \prod_{i=1}^{n} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} \phi(\mathbf{x}) \right| \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n-1}},$$

letting $\phi(\mathbf{x}) = |u(\mathbf{x})|^{\lambda}$, where $u \in W^{1,p}(\mathbb{R}^n)$ and the positive constants $\lambda > p > n$, we have the following estimates

$$\begin{aligned}
&\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}\lambda} \mathrm{d}\mathbf{x} \\
\leq\ & \prod_{i=1}^{n} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} \left[|u(\mathbf{x})|^{\lambda} \right] \right| \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n-1}} \\
\leq\ & \prod_{i=1}^{n} \left\{ \frac{1}{2} \lambda \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\lambda-1} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right| \mathrm{d}\mathbf{x} \right\}^{\frac{1}{n-1}} \\
\leq\ & \left(\frac{1}{2} \lambda \right)^{\frac{n}{n-1}} \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{p}{p-1}(\lambda-1)} \mathrm{d}\mathbf{x} \right\}^{\frac{p-1}{(n-1)p}} \\
\cdot\ & \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{1}{(n-1)p}} \\
\leq\ & \left(\frac{1}{2} \lambda \right)^{\frac{n}{n-1}} \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{(\lambda-1)p-(\lambda-p)n}{(n-1)[p+(\lambda-p)n]p}} \\
\cdot\ & \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}\lambda} \mathrm{d}\mathbf{x} \right\}^{\frac{\lambda-p}{p+(\lambda-p)n}} \\
\cdot\ & \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{1}{(n-1)p}} \\
\leq\ & \left(\frac{1}{2} \lambda \right)^{\frac{n}{n-1}} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{(\lambda-1)p-(\lambda-p)n}{(n-1)[p+(\lambda-p)n]p} n} \\
\cdot\ & \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}\lambda} \mathrm{d}\mathbf{x} \right\}^{\frac{(\lambda-p)n}{p+(\lambda-p)n}} \\
\cdot\ & \prod_{i=1}^{n} \left\{ \int_{\mathbb{R}^n} \left| \frac{\partial}{\partial x_i} u(\mathbf{x}) \right|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{1}{(n-1)p}} .
\end{aligned}$$

Let us cancel out

$$\left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}\lambda} \mathrm{d}\mathbf{x}\right\}^{\frac{(\lambda-p)n}{p+(\lambda-p)n}},$$

in the above estimates, then we have

$$\begin{aligned} & \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}\lambda} \mathrm{d}\mathbf{x}\right\}^{\frac{p}{p+(\lambda-p)n}} \\ \leq \ & \left(\frac{1}{2}\lambda\right)^{\frac{n}{n-1}} \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^p \mathrm{d}\mathbf{x}\right\}^{\frac{(\lambda-1)p-(\lambda-p)n}{(n-1)[p+(\lambda-p)n]p}n} \\ \cdot \ & \prod_{i=1}^{n} \left\{\int_{\mathbb{R}^n} \left|\frac{\partial}{\partial x_i} u(\mathbf{x})\right|^p \mathrm{d}\mathbf{x}\right\}^{\frac{1}{(n-1)p}}. \end{aligned}$$

That is

$$\begin{aligned} & \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}\lambda} \mathrm{d}\mathbf{x} \\ \leq \ & \left(\frac{1}{2}\lambda\right)^{\frac{[p+(\lambda-p)n]n}{(n-1)p}} \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^p \mathrm{d}\mathbf{x}\right\}^{\frac{(\lambda-1)p-(\lambda-p)n}{(n-1)p^2}n} \\ \cdot \ & \left\{\int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^p \mathrm{d}\mathbf{x}\right\}^{\frac{p+(\lambda-p)n}{(n-1)p^2}n}. \end{aligned}$$

Finally, we obtained the following estimates

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\lambda} \mathrm{d}\mathbf{x} \\
\leq\ & \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}\lambda} \mathrm{d}\mathbf{x} \right\}^{\frac{(n-1)(\lambda-p)}{p+(\lambda-p)n}} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{\lambda}{p+(\lambda-p)n}} \\
\leq\ & \left(\frac{1}{2}\lambda \right)^{\frac{(\lambda-p)n}{p}} \left\{ \int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{(\lambda-p)n}{p^2}} \\
\cdot\ & \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{(\lambda-p)n}{p^2}} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{(\lambda-p)[(\lambda-1)p-(\lambda-p)n]}{[p+(\lambda-p)n]p^2} n} \\
\leq\ & \left(\frac{1}{2}\lambda \right)^{\frac{(\lambda-p)n}{p}} \left\{ \int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{(\lambda-p)n}{p^2}} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{\lambda\lambda}{}} \\
\leq\ & \left(\frac{1}{2}\lambda \right)^{\frac{(\lambda-p)n}{p}} \left\{ \int_{\mathbb{R}^n} |u(\mathbf{x})|^{p} \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^{p} \mathrm{d}\mathbf{x} \right\}^{\frac{\lambda}{p}} .
\end{aligned}
$$

The proof of the theorem is finished now. $\square$

Here is the second step. Now let $p > n$. Note that $nnnp(n-1)^2n^2p(1+)1+n^2(2)\int_{\mathbb{R}^n}\left|\frac{\partial}{\partial x_i}u(\mathbf{x})\right|^p \mathrm{d}\mathbf{x}.|\alpha, \cdot\cdot p\lambda$, C$np2u\phi()\beta111^n\sum_{i=1}^n CC^npn^2\sum_{i=1}^n 1nC\frac{1}{1+p-n}\cdot$, ()1n11+np1+ $+ 11p1+$, $2p-n(1+)C1+ + 1 + \cdot\{\}\, 1nn^2(1+)\cdots+\mathbf{x}$.arrive a $nn-1n(2p-n)$Now we have If we $\frac{n}{n-1}p\frac{p-n}{1+p-n}$, .That is There are two steps in the proof. Here is the first step. Then

Theorem 3. Let the positive integer $n \geq 1$ and let the positive constant $p > 1$, such that $p > n$. Then

$$W^{1,p}(\mathbb{R}^n) \subset C^{0,1-\frac{n}{p}}(\mathbb{R}^n).$$

$$W^{1,p}(\mathbb{R}^n) \subset C^{0,1-\frac{n}{p}}(\mathbb{R}^n).$$

There exists a positive constant $C > 0$, such that there holds the following estimate

$$\sup_{\mathbf{x}\in\mathbb{R}^n}|u(\mathbf{x})| + \sup_{\mathbf{x}\in\mathbb{R}^n,\mathbf{y}\in\mathbb{R}^n,\mathbf{x}\neq\mathbf{y}}\frac{|u(\mathbf{x})-u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}}$$
$$\leq C\left\{\int_{\mathbb{R}^n}|u(\mathbf{x})|^p\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n}|\nabla u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}},$$

for all functions $u \in W^{1,p}(\mathbb{R}^n)$.

Proof. Let $\mathbf{x}_0 \in \mathbb{R}^n$ be any point, let $R > 0$ be any positive constant. Consider the open ball $B(\mathbf{x}_0, R)$. First of all, by using the fundamental theorem of calculus, we have

$$u(\mathbf{x}_0 + s\mathbf{z}) - u(\mathbf{x}_0) = \int_0^s \mathbf{z}\cdot\nabla u(\mathbf{x}_0 + t\mathbf{z})\mathrm{d}t,$$

for all $\mathbf{z} \in \mathbb{R}^n$ and for all positive constant $s > 0$, where $|\mathbf{z}| = 1$. Thus

$$|u(\mathbf{x}_0 + s\mathbf{z}) - u(\mathbf{x}_0)| \leq \int_0^s |\nabla u(\mathbf{x}_0 + t\mathbf{z})|\mathrm{d}t.$$

Integrating this inequality over the unit sphere $S(\mathbf{0},1)$ yields

$$\int_{S(\mathbf{0},1)} |u(\mathbf{x}_0+s\mathbf{z})-u(\mathbf{x}_0)|\mathrm{d}S(\mathbf{z}) \le \int_0^s \left\{\int_{S(\mathbf{0},1)} |\nabla u(\mathbf{x}_0+t\mathbf{z})|\mathrm{d}S(\mathbf{z})\right\}\mathrm{d}t.$$

Let us make a change of variables on the right hand side. Let $\mathbf{x}=\mathbf{x}_0+t\mathbf{z}$, then $\mathrm{d}S(\mathbf{x})=t^{n-1}\mathrm{d}S(\mathbf{z})$. Now we have

$$\begin{aligned}\int_{S(\mathbf{0},1)} |u(\mathbf{x}_0+s\mathbf{z})-u(\mathbf{x}_0)|\mathrm{d}S(\mathbf{z}) &\le \int_0^s \left\{\int_{S(\mathbf{x}_0,t)} \frac{|\nabla u(\mathbf{x})|}{|\mathbf{x}-\mathbf{x}_0|^{n-1}}\mathrm{d}S(\mathbf{x})\right\}\mathrm{d}t \\ &\le \int_{B(\mathbf{x}_0,R)} \frac{|\nabla u(\mathbf{x})|}{|\mathbf{x}-\mathbf{x}_0|^{n-1}}\mathrm{d}\mathbf{x}.\end{aligned}$$

Now multiplying this inequality by s^{n-1} and integrating the result with respect to s over $[0,R]$, we get

$$\begin{aligned}&\int_0^R \left\{s^{n-1}\int_{S(\mathbf{0},1)} |u(\mathbf{x}_0+s\mathbf{z})-u(\mathbf{x}_0)|\mathrm{d}S(\mathbf{z})\right\}\mathrm{d}s \\ \le\ & \frac{1}{n}R^n \int_{B(\mathbf{x}_0,R)} \frac{|\nabla u(\mathbf{x})|}{|\mathbf{x}-\mathbf{x}_0|^{n-1}}\mathrm{d}\mathbf{x}.\end{aligned}$$

Therefore, we obtain the following elementary estimate

$$\int_{B(\mathbf{x}_0,R)} |u(\mathbf{x})-u(\mathbf{x}_0)|\mathrm{d}\mathbf{x} \le \frac{1}{n}R^n \int_{B(\mathbf{x}_0,R)} \frac{|\nabla u(\mathbf{x})|}{|\mathbf{x}-\mathbf{x}_0|^{n-1}}\mathrm{d}\mathbf{x},$$

for all balls $B(\mathbf{x}_0,R)\subset\mathbb{R}^n$.

Now let $\mathbf{x}_0\in\mathbb{R}^n$ and $\mathbf{y}_0\in\mathbb{R}^n$ be any two distinct points, let $\Omega=B(\mathbf{x}_0,R)\cap B(\mathbf{y}_0,R)$, where $R=|\mathbf{x}_0-\mathbf{y}_0|>0$. There hold the following estimates

$$
\begin{aligned}
&|u(\mathbf{x}_0)-u(\mathbf{y}_0)| = \left\{\int_\Omega |u(\mathbf{x}_0)-u(\mathbf{y}_0)|\mathrm{d}\mathbf{x}\right\} / \left\{\int_\Omega 1\mathrm{d}\mathbf{x}\right\}\\
\leq\ & \left\{\int_\Omega |u(\mathbf{x})-u(\mathbf{x}_0)|\mathrm{d}\mathbf{x}\right\} / \left\{\int_\Omega 1\mathrm{d}\mathbf{x}\right\}\\
+\ & \left\{\int_\Omega |u(\mathbf{x})-u(\mathbf{y}_0)|\mathrm{d}\mathbf{x}\right\} / \left\{\int_\Omega 1\mathrm{d}\mathbf{x}\right\}\\
\leq\ & C\left\{\int_{B(\mathbf{x}_0,R)} |u(\mathbf{x})-u(\mathbf{x}_0)|\mathrm{d}\mathbf{x}\right\} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\}\\
+\ & C\left\{\int_{B(\mathbf{y}_0,R)} |u(\mathbf{x})-u(\mathbf{y}_0)|\mathrm{d}\mathbf{x}\right\} / \left\{\int_{B(\mathbf{y}_0,R)} 1\mathrm{d}\mathbf{x}\right\}\\
\leq\ & C\left\{\frac{1}{n}R^n\int_{B(\mathbf{x}_0,R)} \frac{|\nabla u(\mathbf{x})|}{|\mathbf{x}-\mathbf{x}_0|^{n-1}}\mathrm{d}\mathbf{x}\right\} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\}\\
+\ & C\left\{\frac{1}{n}R^n\int_{B(\mathbf{y}_0,R)} \frac{|\nabla u(\mathbf{x})|}{|\mathbf{x}-\mathbf{y}_0|^{n-1}}\mathrm{d}\mathbf{x}\right\} / \left\{\int_{B(\mathbf{y}_0,R)} 1\mathrm{d}\mathbf{x}\right\}\\
\leq\ & C\left\{\int_{B(\mathbf{x}_0,R)} \frac{1}{|\mathbf{x}-\mathbf{x}_0|^{\frac{p}{p-1}(n-1)}}\mathrm{d}\mathbf{x}\right\}^{\frac{p-1}{p}} \left\{\int_{B(\mathbf{x}_0,R)} |\nabla u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}}\\
\cdot\ & \left\{\frac{1}{n}R^n\right\} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\}\\
+\ & C\left\{\int_{B(\mathbf{y}_0,R)} \frac{1}{|\mathbf{x}-\mathbf{y}_0|^{\frac{p}{p-1}(n-1)}}\mathrm{d}\mathbf{x}\right\}^{\frac{p-1}{p}} \left\{\int_{B(\mathbf{y}_0,R)} |\nabla u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}}\\
\cdot\ & \left\{\frac{1}{n}R^n\right\} / \left\{\int_{B(\mathbf{y}_0,R)} 1\mathrm{d}\mathbf{x}\right\}\\
\leq\ & CR^{1-\frac{n}{p}}\left\{\int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}}.
\end{aligned}
$$

That is, there holds the following estimate

$$\sup_{\mathbf{x}_0 \neq \mathbf{y}_0} \left\{ \frac{|u(\mathbf{x}_0) - u(\mathbf{y}_0)|}{|\mathbf{x}_0 - \mathbf{y}_0|^{1-n/p}} \right\} \leq C \left\{ \int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^p \mathrm{d}\mathbf{x} \right\}^{\frac{1}{p}} .$$

Next, let $\mathbf{x}_0 \in \mathbb{R}^n$ be any point and let $R > 0$ be any positive constant. Then we have the following estimates

$$\begin{aligned}
& |u(\mathbf{x}_0)| \\
= & \left\{\int_{B(\mathbf{x}_0,R)} |u(\mathbf{x}_0)|\mathrm{d}\mathbf{x}\right\} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\} \\
\leq & \left\{\int_{B(\mathbf{x}_0,R)} |u(\mathbf{x}) - u(\mathbf{x}_0) - u(\mathbf{x})|\mathrm{d}\mathbf{x}\right\} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\} \\
\leq & \left\{\int_{B(\mathbf{x}_0,R)} |u(\mathbf{x}) - u(\mathbf{x}_0)|\mathrm{d}\mathbf{x}\right\} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\} \\
+ & \left\{\int_{B(\mathbf{x}_0,R)} |u(\mathbf{x})|\mathrm{d}\mathbf{x}\right\} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\} \\
\leq & \left\{\frac{1}{n}R^n \int_{B(\mathbf{x}_0,R)} \frac{|\nabla u(\mathbf{x})|}{|\mathbf{x} - \mathbf{x}_0|^{n-1}}\mathrm{d}\mathbf{x}\right\} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\} \\
+ & \left\{\int_{B(\mathbf{x}_0,R)} |u(\mathbf{x})|\mathrm{d}\mathbf{x}\right\} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\} \\
\leq & \frac{1}{n}R^n \left\{\int_{B(\mathbf{x}_0,R)} \frac{1}{|\mathbf{x} - \mathbf{x}_0|^{\frac{p}{p-1}(n-1)}}\mathrm{d}\mathbf{x}\right\}^{\frac{p-1}{p}} \\
\cdot & \left\{\int_{B(\mathbf{x}_0,R)} |\nabla u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\} \\
+ & \left\{\int_{B(\mathbf{x}_0,R)} |u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}} \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\}^{\frac{p-1}{p}} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\} \\
\leq & CR^{n+1-\frac{n}{p}} \left\{\int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\} \\
+ & \left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}} / \left\{\int_{B(\mathbf{x}_0,R)} 1\mathrm{d}\mathbf{x}\right\}^{\frac{1}{p}} \\
\leq & C\left\{\int_{\mathbb{R}^n} |u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{1-\frac{n}{p}} \left\{\int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{n}{p}}.
\end{aligned}$$

In the last step, we let

$$(1) \quad R=\left\{\int_{\mathbb{R}^n}|u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{2n-p}{p^2}} \Big/ \left\{\int_{\mathbb{R}^n}|\nabla u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{2n-p}{p^2}}, \text{ if } p<2n,$$

$$(2) \quad R=\left\{\int_{\mathbb{R}^n}|\nabla u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{p-2n}{p^2}} \Big/ \left\{\int_{\mathbb{R}^n}|u(\mathbf{x})|^p\mathrm{d}\mathbf{x}\right\}^{\frac{p-2n}{p^2}}, \text{ if } p\geq 2n.$$

The proof of the theorem is finished now. □

Theorem 4. Let the positive integers $m\geq 1$ and $n\geq 1$, let the positive constant $p\geq 1$, such that $mp>n$. Then

$$W^{m,p}(\mathbb{R}^n)\subset C^{m-1-[\frac{n}{p}],1+[\frac{n}{p}]-\frac{n}{p}}(\mathbb{R}^n),$$

if $\frac{n}{p}$ is not an integer. And

$$W^{m,p}(\mathbb{R}^n)\subset C^{m-1-[\frac{n}{p}],\alpha}(\mathbb{R}^n),$$

if $\frac{n}{p}$ is an integer, where $0<\alpha<1$ is any positive constant.

Proof. Let $k=[\frac{n}{p}]$. Then $m\geq 1+k$ and $k<\frac{n}{p}<k+1$, if $\frac{n}{p}$ is not an integer.

Define the positive constants p^*, p^{**}, $\cdots$, $p^{***\cdots*}$ by

$$\begin{aligned}
\frac{1}{p^*} &= \frac{1}{p}-\frac{1}{n},\\
\frac{1}{p^{**}} &= \frac{1}{p^*}-\frac{1}{n}=\frac{1}{p}-\frac{2}{n},\\
\frac{1}{p^{***}} &= \frac{1}{p^{**}}-\frac{1}{n}=\frac{1}{p}-\frac{3}{n},\\
\cdots\cdots &\quad \cdots\cdots\\
\frac{1}{p^{***\cdots*}} &= \frac{1}{p^{**\cdots*}}-\frac{1}{n}=\frac{1}{p}-\frac{k}{n}.
\end{aligned}$$

By using the result of Theorem 1, we find that

$$\begin{aligned}
&W^{m,p}(\mathbb{R}^n)\subset W^{m-1,p^*}(\mathbb{R}^n)\subset W^{m-2,p^{**}}(\mathbb{R}^n)\\
\subset\ &W^{m-3,p^{***}}(\mathbb{R}^n)\subset\cdots\cdots\subset W^{m-k,p^{***\cdots*}}(\mathbb{R}^n).
\end{aligned}$$

In each of the inclusion step, there holds the estimate

$$\begin{aligned} & \left\{\int_{\mathbb{R}^n} \left| \frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}} u(\mathbf{x}) \right|^{p^{*\cdots*}} \mathrm{d}\mathbf{x} \right\}^{\frac{1}{p^{*\cdots*}}} \\ \leq \; & C \left\{\int_{\mathbb{R}^n} \left| \nabla \frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}} u(\mathbf{x}) \right|^{p^{*\cdots**}} \mathrm{d}\mathbf{x} \right\}^{\frac{1}{p^{*\cdots**}}}, \end{aligned}$$

for all $\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n \leq m-i$, where $i = 1, 2, 3, \cdots, k$.

Let

$$r = p^{*****\cdots*}.$$

Then

$$\frac{1}{r} = \frac{1}{p} - \frac{k}{n}, \qquad r > n.$$

Now

$$W^{m,p}(\mathbb{R}^n) \subset W^{m-k,r}(\mathbb{R}^n).$$

Recall that

$$W^{1,r}(\mathbb{R}^n) \subset C^{0,1-\frac{n}{r}}(\mathbb{R}^n).$$

Therefore, we have

$$\begin{aligned} & W^{m,p}(\mathbb{R}^n) \subset W^{m-k,r}(\mathbb{R}^n) \\ \subset \; & C^{m-k-1,1-\frac{n}{r}}(\mathbb{R}^n) = C^{m-1-[\frac{n}{p}],1+[\frac{n}{p}]-\frac{n}{p}}(\mathbb{R}^n). \end{aligned}$$

Now let us consider the case $\frac{n}{p}$ is an integer. As before, we have

$$\begin{aligned} & W^{m,p}(\mathbb{R}^n) \subset W^{m-1,p^*}(\mathbb{R}^n) \\ \subset \; & W^{m-2,p^{**}}(\mathbb{R}^n) \subset \cdots\cdots \subset W^{m-k,p^{****\cdots*}}(\mathbb{R}^n) \\ = \; & W^{m-k,n}(\mathbb{R}^n) \subset W^{m-k-1,\lambda}(\mathbb{R}^n) \\ = \; & W^{m-\frac{n}{p},\lambda}(\mathbb{R}^n) \subset C^{m-1-\frac{n}{p},1-\frac{n}{\lambda}}(\mathbb{R}^n) \\ = \; & C^{m-1-\frac{n}{p},\alpha}(\mathbb{R}^n), \end{aligned}$$

where

$$k = \frac{n}{p} - 1, \qquad p^{***\cdots*} = n.$$

$$\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}} u \in L^{\lambda}(\mathbb{R}^n),$$

$$\frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\cdots\partial x_n^{\alpha_n}} u \in C^{0,1-\frac{n}{\lambda}}(\mathbb{R}^n),$$

for all $\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n \leq m - 1 - k$ and for all $\lambda > n$.

Therefore, we have

$$u \in C^{m-1-\frac{n}{p},\alpha}(\mathbb{R}^n),$$

for all $0 < \alpha < 1$.

The Gagliardo-Nirenberg Sobolev's Interpolation Inequality.
Let $m \geq 1$, $n \geq 1$ and $k \geq 0$ be integers, such that $m > k$. Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $1 \leq q \leq \infty$. Let $k/m \leq \alpha < 1$ be a real constant, such that

$$\frac{n}{p} - k = \alpha(\frac{n}{r} - m) + (1-\alpha)\frac{n}{q}.$$

There holds the following Gagliardo-Nirenberg-Sobolev interpolation inequality

$$\begin{aligned} &\left\{ \sum_{\beta_1+\beta_2+\beta_3+\cdots+\beta_n=k} \int_{\mathbb{R}^n} \left| \frac{\partial^{\beta_1+\beta_2+\beta_3+\cdots+\beta_n}}{\partial_{x_1}^{\beta_1}\partial_{x_2}^{\beta_2}\partial_{x_3}^{\beta_3}\cdots\partial_{x_n}^{\beta_n}} f(\mathbf{x}) \right|^p \mathrm{d}\mathbf{x} \right\}^{\frac{1}{p}} \\ \leq\ & C \left\{ \sum_{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n=m} \int_{\mathbb{R}^n} \left| \frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\cdots\partial_{x_n}^{\alpha_n}} f(\mathbf{x}) \right|^r \mathrm{d}\mathbf{x} \right\}^{\alpha/r} \\ \cdot\ & \left\{ \int_{\mathbb{R}^n} |f(\mathbf{x})|^q \mathrm{d}\mathbf{x} \right\}^{(1-\alpha)/q}, \end{aligned}$$

for all functions $f \in W^{m,r}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$.

The only exception is that (1) if $mr < n$, $k = 0$ and $q = \infty$, we require that $f \to 0$, as $|\mathbf{x}| \to \infty$.

(2) $k/m \leq \alpha < 1$, if $1 < r < \infty$ and $m - k - \frac{n}{r} \geq 0$ is an integer.

Mathematics 435 - Introduction to Functional Analysis - 202

Mathematics 435 - Introduction to Functional Analysis - 202

Functional Analysis studies the structures and properties (of compact sets, closed sets, open sets, convex sets) of metric spaces, Banach spaces and Hilbert spaces and their adjoint spaces. Functional Analysis studies properties of bounded linear operators (including adjoint operator, compact operators, projection operators) and continuous linear functionals, such as eigenvalues, eigenvectors, spectrum, one-to-one, onto, boundedness of inverse operators, etc. There are many beautiful, powerful, elegant results, such as the Banach contraction mapping principle, open mapping theorem, closed graph theorem, uniform bounded theorem, Riesz representation theorem, Parseval's identity, etc. There are many important applications to partial differential equations, differential geometry, engineering, applied mathematics.

Instructor: Professor Linghai Zhang

Contact Information: liz5@lehigh.edu

Online Office Hours: Tuesday and Thursday, 8 PM - 9:30 PM and by appointments. The link of the ZOOM meeting is posted in the Coursesite.

Homework Assignments: There will be 6 homework assignments. Students will submit their solutions to the Coursesite before the deadline. 150 points.

Presentations: There will be 10 Presentations on Thursdays starting from the third week. Prepare your presentations in the

PDF documents or powerpoint. You may write down all important details as evidence. However, you will just talk about the most important ideas and steps. You do not have to talk about all the details. The presentation is worth 200 points

The Final Exam: Presentations of 45 minutes per student. 150 points.

Total Score: 500 points

A: 451 - 500

B: 401 - 450

C: 351 - 400

For the information in the Future

Chapter 1: Use about 2 weeks. Give about 1 presentation

Chapter 2: Use about 6 weeks. Give about 3 presentations

Chapter 3: Use about 6 weeks. Give about 3 presentations

Every student does 4 problems in any presentation.

There will be a Final Exam - Presentations. Every student will present 5 - 6 problems.

There will be three homework assignments. There will be 3 problems in each assignment.

Chapter 2: bounded linear operators (focus on projection operators, norm-preserving and onto operators, compact operators, general nilpotent operators) and unbounded linear operators (differential operators). Chapter 3: bounded linear operators (focus on projection operators, self-adjoint operators, unitary operators, Cayley transformation, Fourier transformation, Hilbert operator, compact operators) and unbounded linear operators (differential operators). Students may ask me for ideas and main steps.

Notations

$\mathbb{R}^n$: the standard Euclidean space,

$\mathcal{B}$: basis of the space $\mathbb{R}^n$,

$\|\mathbf{v}\|$: norm of a vector in $\mathbb{R}^n$, $\|\mathbf{v}\|^2 = \sum_{i=1}^{n} |v_i|^2$,

$\|A\|$: the norm of a square matrix, $\|A\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2$,

$\mathbf{u}, \mathbf{v}, \mathbf{w}$ represent vectors

$\|\mathbf{u}\|, \|\mathbf{v}\|, \|\mathbf{w}\|$ represent norms of vectors

$\mathcal{M}$: a metric space

$\mathcal{X}$: a normed linear space, a Banach space

$\mathcal{H}$: a vector space with inner product, a Hilbert space

$\mathcal{X}^*$: the adjoint space of $\mathcal{X}$

$\mathcal{H}^*$: the adjoint space of $\mathcal{H}$

$\mathcal{A}$: a bounded linear operator

$\mathcal{A}^*$: the adjoint operator of $\mathcal{A}$,

$\|\mathcal{A}\|$: the norm of bounded linear operator $\mathcal{A}$

$\|\mathcal{A}^*\|$: the norm of adjoint operator $\mathcal{A}^*$

$\mathcal{P}$: a projection operator

f : a continuous linear functional

$\|f\|$: the norm of continuous linear functional

$(\mathbf{u}, \mathbf{v})$: the inner product of two vectors $\mathbf{u}, \mathbf{v}$

$\Lambda = \{\lambda \in \text{ a set}\}$,

$\mathcal{S}$ a set in a space

Chapter 1: Metric Spaces

Chapter 1: Complete Metric Spaces

Definition and elementary properties

Examples of metric spaces: $\mathbb{R}^n$, $\mathbb{C}^n$, $M_n(\mathbb{R})$, $L^p(\mathbb{R})$, l^p.

Sequences, Bounded Sequences, Limits, Convergent Sequences

Bounded sets, Open sets, Closed Sets, Convex sets, Dense sets, Compact sets

Examples of compact sets

Mappings between metric spaces: Continuous mappings, Open mappings, Closed mappings, Contraction mappings

Complete metric spaces

Banach Contraction Mapping Principle, Applications

Summary

Chapter 2: Banach Spaces and Bounded Linear Operators

Section 1: Bounded Linear Operators

1.1 Boundedness and continuity of linear operators

1.2 The space $\mathcal{B}(\mathcal{X} \to \mathcal{Y})$

Section 2: Representation and Extension of Continuous Linear Functionals

2.1 Representation of continuous linear functionals

2.2 Extension of continuous linear functionals

2.3 Applications of the extension theorem

Section 3: Adjoint Space and Adjoint Operators

Definition 1. Let $\mathcal{M}$ be a nonempty set. $\mathcal{M}$ is called a metric space, if there exists a metric ρ, that is, ρ is a function from $\mathcal{M}\times\mathcal{M}$ to $\mathbb{R}$, such that

$$\begin{aligned} &(1) \quad \rho(\mathbf{u},\mathbf{v}) \ge 0, \\ &(2) \quad \rho(\mathbf{u},\mathbf{v}) = 0, \text{ if and only if } \mathbf{u}=\mathbf{v}, \\ &(3) \quad \rho(\mathbf{u},\mathbf{v}) \le \rho(\mathbf{u},\mathbf{w}) + \rho(\mathbf{v},\mathbf{w}), \end{aligned}$$

for all $\mathbf{u}\in\mathcal{M}$ and $\mathbf{v}\in\mathcal{M}$.

Example 1. The vector space $\mathbb{R}^n$ is a metric space, with the metric

$$\rho(\mathbf{x},\mathbf{y}) \overset{\text{def}}{=} \left\{ \sum_{i=1}^{n} |x_i - y_i|^2 \right\}^{1/2}.$$

Example 2. The vector space $M_n(\mathbb{R})$ is a metric space, with the metric

$$\rho(A,B) \overset{\text{def}}{=} \left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij} - b_{ij}|^2 \right\}^{1/2}.$$

Example 3. The vector space l^p is a metric space, with the metric

$$l^p \overset{\text{def}}{=} \left\{ (x_1, x_2, x_3, \cdots, x_n, \cdots) : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\},$$

$$\rho(\mathbf{x},\mathbf{y}) = \left\{ \sum_{n=1}^{\infty} |x_n - y_n|^p \right\}^{\frac{1}{p}}.$$

Example 4. The vector space $L^p(\mathbb{R}^n)$ is a metric space, with the metric

$$\rho(f,g) \overset{\text{def}}{=} \left\{ \int_{\mathbb{R}^n} |f(\mathbf{x}) - g(\mathbf{x})|^p \mathrm{d}\mathbf{x} \right\}^{\frac{1}{p}}$$

Definition 2. Let $\mathcal{M}$ be a metric space. The subset $\mathcal{S} \subset \mathcal{M}$ is called bounded, if there exists a point $\mathbf{u}_0 \in \mathcal{M}$ and a positive constant $C > 0$, such that

$$\rho(\mathbf{u}, \mathbf{u}_0) \leq C,$$

for all $\mathbf{u} \in \mathcal{S}$.

Definition 3. Let $\mathcal{M}$ be a metric space. The set $\mathcal{S} \subset \mathcal{M}$ is called open, if for every point $\mathbf{u}_0 \in \mathcal{S}$, there exists a positive number $\delta > 0$, such that

$$B(\mathbf{u}_0, \delta) \overset{\text{def}}{=} \{\mathbf{u} \in \mathcal{M} : \rho(\mathbf{u}, \mathbf{u}_0) < \delta\} \subset \mathcal{S}.$$

Definition 4. Let $\mathcal{M}$ be a metric space. Let $\{\mathbf{u}_n : n = 1, 2, 3, \cdots\cdots\} \subset \mathcal{S}$. If

$$\lim_{n \to \infty} \rho(\mathbf{u}_n, \mathbf{u}_0) = 0,$$

then we say the sequence $\mathbf{u}_n$ is convergent and we write the limit

$$\lim_{n \to \infty} \mathbf{u}_n = \mathbf{u}_0.$$

Definition 5. Let $\mathcal{M}$ be a metric space. The set $\mathcal{S} \subset \mathcal{M}$ is called closed, if the limit of any convergent sequence is also in $\mathcal{S}$, i.e. if

$$\lim_{n \to \infty} \rho(\mathbf{u}_n, \mathbf{u}_0) = 0,$$

then $\mathbf{u}_0 \in \mathcal{S}$.

Definition 6. Let $\mathcal{M}$ be a metric space. The set $\mathcal{S} \subset \mathcal{M}$ is called convex, if for any points $\mathbf{u}_0 \in \mathcal{S}$ and $\mathbf{v}_0 \in \mathcal{S}$, there holds

$$\lambda \mathbf{u}_0 + (1 - \lambda)\mathbf{v}_0 \in \mathcal{S},$$

for any number $0 \leq \lambda \leq 1$.

Definition 7. Let $\mathcal{M}$ be a metric space. The set $\mathcal{S} \subset \mathcal{M}$ is called dense in the metric space, if for every point $\mathbf{u}_0 \in \mathcal{M}$, there exists a sequence

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots\cdots, \mathbf{u}_n, \cdots\cdots\} \subset \mathcal{S},$$

such that

$$\lim_{n\to\infty} \rho(\mathbf{u}_n, \mathbf{u}_0) = 0.$$

Definition 8. Let $\mathcal{M}$ be a metric space. The set $\mathcal{S} \subset \mathcal{M}$ is called compact, if every bounded sequence has a convergent subsequence.

Theorem 1. Let $\mathcal{M}$ be a metric space. Then

$$\rho(\mathbf{u}, \mathbf{v}) = \rho(\mathbf{v}, \mathbf{u}),$$

for all $\mathbf{u} \in \mathcal{S}$ and $\mathbf{v} \in \mathcal{S}$.

Theorem 2. Let

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots\cdots, \mathbf{u}_n \cdots\cdots\}$$

be a convergence sequence in the metric space $\mathcal{M}$. Then the limit is unique.

Theorem 3. Let $\{\mathbf{u}_n : n = 1, 2, 3, \cdots\}$ and $\{\mathbf{v}_n : n = 1, 2, 3, \cdots\}$ be convergent sequences in the metric space $\mathcal{M}$, that is,

$$\lim_{n\to\infty} \rho(\mathbf{u}_n, \mathbf{u}_0) = 0, \qquad \lim_{n\to\infty} \rho(\mathbf{v}_n, \mathbf{v}_0) = 0,$$

then

$$\lim_{n\to\infty} \rho(\mathbf{u}_n, \mathbf{v}_n) = \rho(\mathbf{u}_0, \mathbf{v}_0).$$

Definition 9. Let $\mathcal{S} \subset \mathcal{M}$ and $\mathcal{T} \subset \mathcal{M}$ be subsets, let $\mathbf{u}_0 \in \mathcal{M}$. Define the distance between $\mathbf{u}_0$ and $\mathcal{S}$ by

$$\rho(\mathbf{u}_0, \mathcal{S}) \overset{\text{def}}{=} \inf_{\mathbf{u}\in\mathcal{S}} \rho(\mathbf{u}_0, \mathbf{u}).$$

Define the distance between $\mathcal{S}$ and $\mathcal{T}$ by

$$\rho(\mathcal{S},\mathcal{T}) \overset{\text{def}}{=} \inf_{\mathbf{u}\in\mathcal{S},\mathbf{v}\in\mathcal{T}} \rho(\mathbf{u},\mathbf{v}).$$

Theorem . Let

$$\{\mathcal{S}_\lambda, \lambda \in \Lambda\}$$

be a family of open sets. Then

$$\bigcup_{\lambda\in\Lambda} \mathcal{S}_\lambda$$

is also open.

Let

$$\{\mathcal{S}_\lambda : \lambda \in \Lambda\}$$

be a family of closed sets. Then

$$\bigcap_{\lambda\in\Lambda} \mathcal{S}_\lambda$$

is closed.

Definition 11. Let $(\mathcal{M},\rho_1)$ and $(\mathcal{N},\rho_2)$ be metric spaces. The mapping $\mathcal{A}:(\mathcal{M},\rho_1)\to(\mathcal{N},\rho_2)$ is called continuous, if for any point $\mathbf{u}_0\in\mathcal{M}$ and for any positive constant $\varepsilon>0$, i.e. for any ball $B(\mathcal{A}(\mathbf{u}_0),\varepsilon)\subset\mathcal{N}$, there exists another positive constant $\delta>0$, i.e. there is another ball $B(\mathbf{u}_0,\delta)\subset\mathcal{M}$, such that

$$\mathcal{A}(B(\mathbf{u}_0,\delta))\subset B(\mathcal{A}(\mathbf{u}_0),\varepsilon).$$

Definition 12. Let $(\mathcal{M},\rho_1)$ and $(\mathcal{N},\rho_2)$ be metric spaces. The mapping $\mathcal{A}:(\mathcal{M},\rho_1)\to(\mathcal{N},\rho_2)$ is called open, if for any $\mathbf{u}_0\in\mathcal{M}$ and for any $\delta>0$, the image

$$\mathcal{A}(B(\mathbf{u}_0,\delta))\subset\mathcal{N}$$

is open in the space $(\mathcal{N},\rho_2)$.

Definition 13. Let $(\mathcal{M},\rho_1)$ and $(\mathcal{N},\rho_2)$ be metric spaces. The mapping $\mathcal{A}:(\mathcal{M},\rho_1)\to(\mathcal{N},\rho_2)$ is called closed, if the graph

$$\mathbf{G}(\mathcal{A})\overset{\text{def}}{=}\{(\mathbf{u},\mathcal{A}(\mathbf{u})):\mathbf{u}\in\mathcal{M}\}$$

is closed in the metric space $(\mathcal{M},\rho_1)\times(\mathcal{N},\rho_2)$, where the metric is defined by

$$\rho(\mathbf{u},\mathbf{v})=\left\{[\rho_1(\mathbf{u},\mathbf{v})]^2+[\rho_2(\mathbf{u},\mathbf{v})]^2\right\}^{1/2}.$$

Theorem (Baire) Any complete metric space is of the second category.

Proof. Suppose that $(\mathcal{X},\rho)$ is a complete metric space and that $\mathcal{X}$ is of the first category. Let

$$\mathcal{X}=\bigcup_{n=1}^{\infty}\mathcal{S}_n.$$

Suppose that all $\mathcal{S}_n$ are sparse sets. That is, for every set $\mathcal{S}_n$, for any closed ball $S(\mathbf{y}_n,\rho_n)$, there exists another closed ball $S(\mathbf{z}_n,\sigma_n)$,

such that

$$S(\mathbf{z}_n, \sigma_n) \subset S(\mathbf{y}_n, \rho_n), \qquad S(\mathbf{z}_n, \sigma_n) \cap \mathcal{S}_n = \emptyset.$$

Let $S(\mathbf{x}, R)$ be any closed ball. Then there exists a closed ball $S(\mathbf{x}_1, R_1)$, such that

$$S(\mathbf{x}_1, R_1) \subset S(\mathbf{x}, R), \qquad S(\mathbf{x}_1, R_1) \cap \mathcal{S}_1 = \emptyset.$$

Similarly, there exists a closed ball $S(\mathbf{x}_2, R_2)$, such that

$$S(\mathbf{x}_2, R_2) \subset S(\mathbf{x}_1, R_1), \qquad S(\mathbf{x}_2, R_2) \cap \mathcal{S}_2 = \emptyset.$$

Then there exists a closed ball $S(\mathbf{x}_3, R_3)$, such that

$$S(\mathbf{x}_3, R_3) \subset S(\mathbf{x}_2, R_2), \qquad S(\mathbf{x}_3, R_3) \cap \mathcal{S}_3 = \emptyset.$$

Repeating this process, there exists a sequence of closed balls $S(\mathbf{x}_n, R_n)$, such that

$$S(\mathbf{x}_n, R_n) \subset S(\mathbf{x}_{n-1}, R_{n-1}), \qquad S(\mathbf{x}_n, R_n) \cap \mathcal{S}_n = \emptyset,$$

where $n = 2, 3, 4, \cdots$. Without loss of generality, we may let the radius $0 < R_n < \frac{1}{2^n}$. Overall, we obtain

$$S(\mathbf{x}_1, R_1) \supset S(\mathbf{x}_2, R_2) \supset S(\mathbf{x}_3, R_3) \supset \cdots \supset S(\mathbf{x}_n, R_n) \supset \cdots, \\ S(\mathbf{x}_n, R_n) \cap \mathcal{S}_n = \emptyset.$$

Based on Theorem , there exists a unique point $\mathbf{x}_0$:

$$\mathbf{x}_0 \in \bigcap_{n=1}^{\infty} S(\mathbf{x}_n, R_n).$$

However, $\mathbf{x}_0 \notin \mathcal{S}_n$, for all $n = 1, 2, 3, \cdots$. Hence

$$\mathbf{x}_0 \notin \mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{S}_n.$$

This is a contradiction. Therefore, there exists a set $\mathcal{S}_n$ and there exists an open ball $B(\mathbf{y}_0, R)$, such that

$$\mathcal{S}_n \text{ is dense in } S(\mathbf{y}_0, R).$$

The proof is finished now. $\square$

Definition 14. Let $(\mathcal{M}, \rho_1)$ and $(\mathcal{N}, \rho_2)$ be metric spaces. The mapping $\mathcal{A} : (\mathcal{M}, \rho_1) \to (\mathcal{N}, \rho_2)$ is called a topological mapping, if $\mathcal{A}$ is one-to-one and onto, and both $\mathcal{A}$ and $\mathcal{A}^{-1}$ are continuous mappings.

Definition 15. Let $(\mathcal{M}, \rho_1)$ and $(\mathcal{N}, \rho_2)$ be metric spaces. The mapping $\mathcal{A} : (\mathcal{M}, \rho_1) \to (\mathcal{N}, \rho_2)$ is called distance-preserving, if

$$\rho_2(\mathcal{A}(\mathbf{u}), \mathcal{A}(\mathbf{v})) = \rho_1(\mathbf{u}, \mathbf{v}),$$

for all $\mathbf{u} \in \mathcal{M}$ and $\mathbf{v} \in \mathcal{M}$. Additionally, if $\mathcal{A}$ is onto, then it is called an isomorphism.

Definition 16. Let $(\mathcal{M}, \rho)$ be a metric space. The mapping $\mathcal{A}(\mathcal{M}, \rho) \to (\mathcal{M}, \rho)$ is called a contraction, if there exists a positive constant $0 \leq \alpha < 1$, such that

$$\rho(\mathcal{A}(\mathbf{u}), \mathcal{A}(\mathbf{v})) \leq \alpha \rho(\mathbf{u}, \mathbf{v}),$$

for all $\mathbf{u} \in \mathcal{M}$ and $\mathbf{v} \in \mathcal{M}$.

Banach Contraction Mapping Principle

Theorem 1. Let $(\mathcal{M}, \rho)$ be a complete metric space and let $\mathcal{A} : (\mathcal{M}, \rho) \to (\mathcal{M}, \rho)$ be a contraction mapping. Then there exists a unique fixed point $\mathbf{v}_0 \in \mathcal{M}$, such that

$$\mathcal{A}(\mathbf{v}_0) = \mathbf{v}_0.$$

The key point in the proof is to show that the sequence

$$\{\mathbf{u}_0, \mathcal{A}\mathbf{u}_0, \mathcal{A}^2\mathbf{u}_0, \mathcal{A}^3\mathbf{u}_0, \cdots\cdots, \mathcal{A}^n\mathbf{u}_0, \mathcal{A}^{n+1}\mathbf{u}_0, \cdots\cdots\}$$

is convergent, because there holds the following estimate

$$\rho(\mathcal{A}^n\mathbf{u}_0, \mathcal{A}^{n+1}\mathbf{u}_0) \leq \alpha^n \rho(\mathbf{u}_0, \mathcal{A}\mathbf{u}_0),$$

for all $n \geq 1$.

Theorem 2. Let $(\mathcal{M}, \rho)$ be a complete metric space, let $\mathcal{A} : (\mathcal{M}, \rho) \to (\mathcal{M}, \rho)$ be a mapping, such that $\mathcal{A}^n$ is a contraction mapping, where $n > 1$ is a sufficiently large positive integer. Then there exists a unique point $\mathbf{u}_0 \in \mathcal{M}$, such that

$$\mathcal{A}(\mathbf{u}_0) = \mathbf{u}_0.$$

The key point in the proof is to show that

$$\mathcal{A}^n(\mathbf{u}_0) = \mathbf{u}_0 \Rightarrow \mathcal{A}(\mathbf{u}_0) = \mathbf{u}_0.$$

Note that

$$\rho(\mathbf{u}_0, \mathcal{A}(\mathbf{u}_0)) = \rho(\mathcal{A}^n(\mathbf{u}_0), \mathcal{A}^{n+1}(\mathbf{u}_0)) \leq \alpha \rho(\mathbf{u}_0, \mathcal{A}(\mathbf{u}_0)).$$

Therefore

$$\mathcal{A}(\mathbf{u}_0) = \mathbf{u}_0.$$

Example 1. Let the continuous function $f = f(x, y)$ be defined in

$$\mathcal{R} = \{(x, y) : a \leq x \leq b, \qquad y \in \mathbb{R}\},$$

such that its partial derivative with respect to y exists. Let $C_1 > 0$ and $C_2 > 0$ be positive constants, such that

$$C_1 \leq \frac{\partial}{\partial y} f(x, y) \leq C_2,$$

for all $(x, y) \in \mathcal{R}$. Then there exists a unique solution $y = \phi(x) \in C[a, b]$ to the equation

$$f(x, y) = 0.$$

The key point in the proof is to define the mapping

$$\mathcal{A} : C[a,b] \to C[a,b],$$

$$\mathcal{A}(\phi) = \phi - \frac{1}{C_2} f(x,\phi).$$

$$\mathcal{A}(\phi) - \mathcal{A}(\psi) = \Big[\phi - \frac{1}{C_2} f(x,\phi)\Big] - \Big[\psi - \frac{1}{C_2} f($$

$$= \ (\phi - \psi)\left\{1 - \frac{1}{C_2}\frac{\partial}{\partial y} f(x,)\right\},$$

$$|\mathcal{A}(\phi) - \mathcal{A}(\psi)|$$

$$\leq |\phi - \psi|(1 - C_1/C_2).$$

The estimate shows that the mapping is a contraction.

Example 2. Let $f = f(x)$ be a continuous function defined on $[a,b]$, let $K = K(x,y)$ be a continuous function defined on $[a,b] \times [a,b]$, such that

$$\mathcal{K} \overset{\text{def}}{=} \sup_{[a,b]} \int_a^b |K(x,y)| \mathrm{d}y < \infty.$$

Then there exists a unique solution $y = \phi(x) \in C[a,b]$ to the integral equation

$$\phi(x) = f(x) + \lambda \int_a^b K(x,y)\phi(y)\mathrm{d}y,$$

as long as the constant λ satisfies the condition $\mathcal{K}|\lambda| < 1$.
Let

$$\mathcal{A}(\phi) = f(x) + \lambda \int_a^b K(x,y)\phi(y)\mathrm{d}y.$$

The key point in the proof is to establish the estimate

$$|\mathcal{A}(\phi) - \mathcal{A}(\psi)| \leq \mathcal{K}|\lambda| \|\phi - \psi\|.$$

Example 3. Let $f = f(x)$ be a continuous function defined on $[a, b]$, let $K = K(x, y)$ be a continuous function defined on

$$R = \{(x, y) : a \leq x \leq b, \qquad a \leq y \leq x\}.$$

Let

$$\mathcal{K} = \max_{\mathcal{R}} |K(x, y)|.$$

Then, for each constant $\lambda \in \mathbb{R}$, there exists a unique solution $y = \phi(x)$ to the equation

$$\phi = f(x) + \lambda \int_a^x K(x, y)\phi(y)\mathrm{d}y.$$

Let

$$\mathcal{A}(\phi) = f(x) + \lambda \int_a^x K(x, y)\phi(y)\mathrm{d}y.$$

Then

$$\mathcal{A}^{n+1}(\phi) = f(x) + \lambda \int_a^x K(x, y)[\mathcal{A}^n(\phi)](y)\mathrm{d}y.$$

Obviously, we see that

$$\mathcal{A}^{n+1}(\phi) - \mathcal{A}^{n+1}(\psi) = \lambda \int_a^x K(x, y)[\mathcal{A}^n(\phi) - \mathcal{A}^n(\psi)]\mathrm{d}y.$$

The key point in the proof is to show the estimate

$$|\mathcal{A}^{n+1}(\phi) - \mathcal{A}^{n+1}(\psi)| \leq \frac{[(b-a)\mathcal{K}|\lambda|]^{n+1}}{(n+1)!}\|\phi - \psi\|.$$

Example 4. Let the real vector valued, continuous function $\mathbf{f} = \mathbf{f}(\mathbf{u}, t)$ be defined on

$$\mathcal{R} = \mathbb{R}^n \times \mathbb{R}.$$

Given the constant vector $\mathbf{u}_0$, let

$$\|\mathbf{f}(\mathbf{u}_0, t)\| \leq C_0,$$

for all $t \in \mathbb{R}$, where $C_0 > 0$ is a positive constant. Let $L > 0$ be a positive constant, such that

$$|\mathbf{f}(\mathbf{u}, t) - \mathbf{f}(\mathbf{v}, t)| \leq L|\mathbf{u} - \mathbf{v}|,$$

for all $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{v} \in \mathbb{R}^n$ and for all $t \in \mathbb{R}$. Consider the initial value problem for the first order system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \mathbf{f}(\mathbf{u}, t), \qquad \mathbf{u}(t_0) = \mathbf{u}_0.$$

The function $\mathbf{u} = \mathbf{u}(t) \in C^1(\mathbb{R})$ is a solution to the initial value problem, if and only if it is a solution to the integral equation

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{u}(\tau), \tau)\mathrm{d}\tau.$$

There exists a unique continuously differentiable solution on $\mathbb{R}$. Let $t_0 < T < \infty$ be a fixed positive constant. Define the operator $\mathcal{A}$ on $C[t_0, T]$ and iterate for infinitely many times

$$\begin{aligned}
\mathcal{A}(\mathbf{u}) &= \mathbf{u}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{u}(\tau), \tau)\mathrm{d}\tau, \\
\mathcal{A}^2(\mathbf{u}) &= \mathbf{u}_0 + \int_{t_0}^{t} \mathbf{f}(\mathcal{A}(\mathbf{u}(\tau)), \tau)\mathrm{d}\tau, \\
\mathcal{A}^3(\mathbf{u}) &= \mathbf{u}_0 + \int_{t_0}^{t} \mathbf{f}(\mathcal{A}^2(\mathbf{u}(\tau)), \tau)\mathrm{d}\tau, \\
\cdots\cdots & \quad \cdots\cdots \\
\mathcal{A}^n(\mathbf{u}) &= \mathbf{u}_0 + \int_{t_0}^{t} \mathbf{f}(\mathcal{A}^{n-1}(\mathbf{u}(\tau)), \tau)\mathrm{d}\tau, \\
\mathcal{A}^{n+1}(\mathbf{u}) &= \mathbf{u}_0 + \int_{t_0}^{t} \mathbf{f}(\mathcal{A}^{n}(\mathbf{u}(\tau)), \tau)\mathrm{d}\tau, \\
\cdots\cdots & \quad \cdots\cdots
\end{aligned}$$

The key point is to establish the estimates

$$\begin{aligned}
& \|\mathcal{A}^{n+1}(\mathbf{u}) - \mathcal{A}^{n+1}(\mathbf{v})\| \\
\leq\ & L \int_{t_0}^{t} \|\mathcal{A}^{n}(\mathbf{u}) - \mathcal{A}^{n}(\mathbf{v})\| \mathrm{d}\tau \\
\leq\ & L^2 \int_{t_0}^{t} \|\mathcal{A}^{n-1}(\mathbf{u}) - \mathcal{A}^{n-1}(\mathbf{v})\| \mathrm{d}\tau \\
\leq\ & L^n \int_{t_0}^{t} \|\mathcal{A}(\mathbf{u}) - \mathcal{A}(\mathbf{v})\| \mathrm{d}\tau \\
\leq\ & L^{n+1} \int_{t_0}^{t} \|\mathbf{u}(\tau) - \mathbf{v}(\tau)\| \mathrm{d}\tau \\
\leq\ & \frac{(LT)^{n+1}}{(n+1)!} \|\mathbf{u} - \mathbf{v}\|.
\end{aligned}$$

The estimates are true for all $n \geq 1$ and for all $t_0 < t < T$.

Note that

$$\lim_{n\to\infty} \frac{(LT)^{n+1}}{(n+1)!} = 0.$$

Therefore, for each fixed $T > t_0$ and for all sufficiently large n, the mapping $\mathcal{A}^{n+1}$ is a contraction. There exists a unique fixed point $\mathbf{u} = \mathbf{u}(t)$, such that $\mathcal{A}(\mathbf{u}) = \mathbf{u}$, that is

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{u}(\tau), \tau) \mathrm{d}\tau,$$

for all t with $t_0 < t < T$. Note that T is arbitrary, namely, the solution exists on (t_0, ∞). Very similarly, the solution exists on $(-\infty, t_0)$. Therefore, there exists a unique continuously differentiable solution on $\mathbb{R}$.

Complete Metric Spaces

Definition 1. Let $(\mathcal{M}, \rho)$ be a metric space, let $\{\mathbf{u}_n : n = 1, 2, 3, \cdots\} \subset \mathcal{M}$ be a sequence of points. We call it a fundamental sequence or a Cauchy sequence, if for every positive constant $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$, such that

$$\rho(\mathbf{u}_m, \mathbf{u}_n) < \varepsilon,$$

for all integers $m > N$ and $n > N$. The metric space $(\mathcal{M}, \rho)$ is called complete if every Cauchy sequence is convergent in $(\mathcal{M}, \rho)$.

Examples of Complete Metric Spaces: Let Ω be a subset of $\mathbb{R}^n$. Let $1 \leq p < \infty$ and $m > 0$ be positive constants.

$$\begin{array}{ll} \mathbb{R}^n, & C[a,b] \\ l^p(\mathbb{R}), & l^\infty(\mathbb{R}) \\ L^p(\mathbb{R}^n), & L^p(\Omega) \\ L^\infty(\mathbb{R}^n), & L^\infty(\Omega) \end{array}$$

$$H^m(\mathbb{R}^n) \overset{\text{def}}{=} \left\{ \phi \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^{m/2} |\widehat{\phi}(\xi)|^2 \mathrm{d}\xi < \infty \right\}.$$

Theorem 1. Let $(\mathcal{M}, \rho)$ be a complete metric space. Let

$$\begin{aligned} &\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset \cdots \supset \mathcal{S}_n \supset \cdots, \\ &\mathcal{S}_n = \{\mathbf{u} \in \mathcal{M} : \rho(\mathbf{u}, \mathbf{u}_n) \leq \varepsilon_n\}, \\ &\lim_{n \to \infty} \varepsilon_n = 0. \end{aligned}$$

Then there exists a unique point $\mathbf{u}_0 \in \mathcal{M}$, such that

$$\mathbf{u}_0 \in \bigcap_{n=1}^{\infty} \mathcal{S}_n.$$

The key point in the proof is that the sequence

$$\{\mathbf{u}_n : n = 1, 2, 3, \cdots\}$$

is a Cauchy sequence. It is convergent in $(\mathcal{M}, \rho)$.

Theorem 2. Let $(\mathcal{M}, \rho)$ be a metric space. If for every sequence of closed balls

$$\begin{aligned}
&\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset \cdots \supset \mathcal{S}_n \cdots, \\
&\mathcal{S}_n = \{\mathbf{u} \in \mathcal{M} : \rho(\mathbf{u}, \mathbf{u}_n) \leq \varepsilon_n\}, \\
&\lim_{n\to\infty} \varepsilon_n = 0,
\end{aligned}$$

there exists a unique point $\mathbf{u}_0$, such that

$$\mathbf{u}_0 \in \bigcap_{n=1}^{n} \mathcal{S}_n,$$

then $(\mathcal{M}, \rho)$ is a complete metric space.

Definition 2. Let $(\mathcal{M}, \rho)$ be a metric space. If there exists a complete metric space $(\mathcal{M}_1, \rho_1)$, and there exists a distance-preserving isomorphism $\mathcal{A} : (\mathcal{M}, \rho) \to (\mathcal{N}_1, \rho_1)$, then we say $(\mathcal{M}_1, \rho_1)$ is a completion of the metric space $(\mathcal{M}, \rho)$, where $\mathcal{N}_1 \subset \mathcal{M}_1$ is a dense subset, i.e

$$\mathcal{M}_1 = \overline{\mathcal{N}}_1.$$

Theorem 3. For any metric space $(\mathcal{M}, \rho)$, there exists a completion $(\mathcal{M}_1, \rho_1)$.

Proof. There are several steps in the proof of the theorem.

The First Step: Define the space $\mathcal{M}_1$ by

$$\mathcal{M}_1 = \{\Lambda : \Lambda = \{\mathbf{u}_n\} \text{ is a Cauchy sequence in } \mathcal{M}\}.$$

Define the metric ρ_1 by

$$\rho_1(\Lambda, \Gamma) = \lim_{n\to\infty} \rho(\mathbf{u}_n, \mathbf{v}_n),$$

for any two Cauchy sequences $\Lambda = \{\mathbf{u}_n\}$ and $\Gamma = \{\mathbf{v}_n\}$. The above definition makes sense for the following reasons. Note that

$$|\rho(\mathbf{u}_m, \mathbf{v}_m) - \rho(\mathbf{u}_n, \mathbf{v}_n)| \leq \rho(\mathbf{u}_m, \mathbf{u}_n) + \rho(\mathbf{v}_m, \mathbf{v}_n),$$

for all positive integers $m \geq 1$ and $n \geq 1$. Hence $\{\rho(\mathbf{u}_n, \mathbf{v}_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Therefore, the limit

$$\lim_{n\to\infty} \rho(\mathbf{u}_n, \mathbf{v}_n)$$

exists. If two Cauchy sequences $\Lambda = \{\mathbf{u}_n\}$ and $\Gamma = \{\mathbf{v}_n\}$ in $(\mathcal{M}, \rho)$ satisfy

$$\lim_{n\to\infty} \rho(\mathbf{u}_n, \mathbf{v}_n) = 0,$$

then we say $\{\mathbf{u}_n\} = \{\mathbf{v}_n\}$.

Moreover, if $\{\mathbf{u}_n\} = \{\tilde{\mathbf{u}}_n\}$ and $\{\mathbf{v}_n\} = \{\tilde{\mathbf{v}}_n\}$, then

$$|\rho(\mathbf{u}_n, \mathbf{v}_n) - \rho(\tilde{\mathbf{u}}_n, \tilde{\mathbf{v}}_n)| \leq \rho(\mathbf{u}_n, \tilde{\mathbf{u}}_n) + \rho(\mathbf{v}_n, \tilde{\mathbf{v}}_n),$$

for all positive integers $n \geq 1$. Now we have

$$\lim_{n\to\infty} \rho(\mathbf{u}_n, \mathbf{v}_n) = \lim_{n\to\infty} \rho(\tilde{\mathbf{u}}_n, \tilde{\mathbf{v}}_n).$$

Therefore, the definition of the metric ρ_1 is independent of the choices of the Cauchy sequences. Overall, $(\mathcal{M}_1, \rho_1)$ is a metric space.

The Second Step: Define

$$\mathcal{N}_1 = \{\Lambda(\mathbf{u}) : \Lambda(\mathbf{u}) = (\mathbf{u}, \mathbf{u}, \mathbf{u}, \cdots, \mathbf{u}, \cdots), \mathbf{u} \in \mathcal{M}\}.$$

Then $\mathcal{N}_1$ is dense in the metric space $(\mathcal{M}_1, \rho_1)$.

In fact, for any Cauchy sequence

$$\Lambda = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n, \cdots) \in \mathcal{M}_1,$$

let us define the following sequences

$$
\begin{aligned}
&\Lambda(\mathbf{u}_1) = (\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_1, \cdots, \mathbf{u}_1, \cdots),\\
&\Lambda(\mathbf{u}_2) = (\mathbf{u}_2, \mathbf{u}_2, \mathbf{u}_2, \cdots, \mathbf{u}_2, \cdots),\\
&\Lambda(\mathbf{u}_3) = (\mathbf{u}_3, \mathbf{u}_3, \mathbf{u}_3, \cdots, \mathbf{u}_3, \cdots),\\
&\cdots\cdots\\
&\Lambda(\mathbf{u}_n) = (\mathbf{u}_n, \mathbf{u}_n, \mathbf{u}_n, \cdots, \mathbf{u}_n, \cdots),\\
&\cdots\cdots,
\end{aligned}
$$

in $\mathcal{N}_1$. Then

$$\rho_1(\Lambda, \Lambda(\mathbf{u}_n)) = \lim_{m\to\infty} \rho(\mathbf{u}_m, \mathbf{u}_n) < \varepsilon.$$

The Third Step: The metric space $(\mathcal{M}_1, \rho_1)$ is complete. Let

$$\{\Lambda_1, \Lambda_2, \Lambda_3, \cdots, \Lambda_n, \cdots\}$$

be any Cauchy sequence in the metric space $(\mathcal{M}_1, \rho_1)$.

For any positive constant $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$, for any positive integers $m > N$ and $n > N$, there holds the following estimate

$$\rho_1(\Lambda_m, \Lambda_n) < \frac{1}{3}\varepsilon.$$

Recall that $\mathcal{N}_1$ is dense in $\mathcal{M}_1$. Therefore, there exists a sequence

$$
\begin{aligned}
&\Lambda(\mathbf{u}_1) = (\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_1, \cdots, \mathbf{u}_1, \cdots),\\
&\Lambda(\mathbf{u}_2) = (\mathbf{u}_2, \mathbf{u}_2, \mathbf{u}_2, \cdots, \mathbf{u}_2, \cdots),\\
&\Lambda(\mathbf{u}_3) = (\mathbf{u}_3, \mathbf{u}_3, \mathbf{u}_3, \cdots, \mathbf{u}_3, \cdots),\\
&\cdots\cdots,\\
&\Lambda(\mathbf{u}_n) = (\mathbf{u}_n, \mathbf{u}_n, \mathbf{u}_n, \cdots, \mathbf{u}_n, \cdots),\\
&\cdots\cdots
\end{aligned}
$$

in $\mathcal{N}_1$, such that

$$\rho_1(\Lambda_m, \Lambda(\mathbf{u}_m)) < \frac{1}{3}\varepsilon.$$

Now we have the following estimates

$$\begin{aligned}
&\rho(\mathbf{u}_m, \mathbf{u}_n) = \rho_1(\Lambda(\mathbf{u}_m), \Lambda(\mathbf{u}_n)) \\
\leq\ & \rho_1(\Lambda(\mathbf{u}_m), \Lambda_m) + \rho_1(\Lambda_m, \Lambda_n) + \rho_1(\Lambda_n, \Lambda(\mathbf{u}_n)) \\
<\ & \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon.
\end{aligned}$$

Let

$$\Lambda_0 = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n, \cdots).$$

Hence $\Lambda_0 \in \mathcal{M}_1$. Moreover,

$$\rho_1(\Lambda_n, \Lambda_0) \leq \rho_1(\Lambda_n, \Lambda(\mathbf{u}_n)) + \rho_1(\Lambda(\mathbf{u}_n), \Lambda_0) < \varepsilon,$$

where

$$\rho_1(\Lambda(\mathbf{u}_n), \Lambda_0) = \lim_{k\to\infty} \rho(\mathbf{u}_n, \mathbf{u}_k) < \varepsilon,$$

for all sufficiently large $n \gg 1$. The proof of the theorem is finished.

Theorem . Let $(\mathcal{M}_1, \rho_1)$ and $(\mathcal{M}_2, \rho_2)$ be two completions of the metric space $(\mathcal{M}, \rho)$. There must exist a distance-preserving isomorphism $\mathcal{A} : (\mathcal{M}_1, \rho_1) \to (\mathcal{M}_2, \rho_2)$, such that

$$\mathcal{A}(\mathbf{u}) = \mathbf{u},$$

for all $\mathbf{u} \in \mathcal{M}$.

Theorem. A complete metric space is of the second category.

Let us begin here!

Compact Sets in Metric Spaces

Definition 1. Let $(\mathcal{M}, \rho)$ be a metric space. The subset $\mathcal{S} \subset \mathcal{M}$ is called compact, if there exists a convergent subsequence $\{\mathbf{u}_{n_1}, \mathbf{u}_{n_2}, \mathbf{u}_{n_3}, \cdots, \mathbf{u}_{n_k}, \cdots\}$ in any sequence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n, \cdots\} \subset \mathcal{S}$, such that

$$\lim_{k\to\infty} \mathbf{u}_{n_k} = \mathbf{u}_0 \in \mathcal{S}.$$

The set $\mathcal{S} \subset \mathcal{M}$ is called pre-compact, if exists a convergent subsequence $\{\mathbf{u}_{n_1}, \mathbf{u}_{n_2}, \mathbf{u}_{n_3}, \cdots, \mathbf{u}_{n_k} \cdots\}$ in any sequence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n, \cdots\}$, such that

$$\lim_{k\to\infty} \mathbf{u}_{n_k} = \mathbf{u}_0 \in \mathcal{M}.$$

Here the limit $\mathbf{u}_0$ is not necessarily in $\mathcal{S}$.

Elementary Properties of Compact Sets:

(1) $\mathcal{A}$ is compact, if there are finitely many points.

(2) $\cup_{k=1}^{m} \mathcal{A}_k$ is compact, if every $\mathcal{A}_k$ is compact.

(3) $\cap_{\lambda\in\Lambda} \mathcal{A}_\lambda$ is compact, where $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ is a family of compact sets.

(4) $\overline{\mathcal{A}} = \mathcal{A}$, if $\mathcal{A}$ is compact.

(5) $(\mathcal{M}, \rho)$ is a complete metric space, if $\mathcal{M}$ is compact.

Theorem 1. Any bounded closed subset in $\mathbb{R}^n$ is compact.

Definition 2. Let $(\mathcal{M}, \rho)$ be a metric space and let $\mathcal{S} \subset \mathcal{M}$ be a set. We call $\mathcal{S}$ totally bounded, if for every positive constant $\varepsilon > 0$, there exist finitely many points $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_m\} \subset \mathcal{S}$, such that

$$\mathcal{S} \subset \bigcup_{k=1}^{m} B(\mathbf{u}_k, \varepsilon).$$

Theorem 2. Let $\mathcal{S}$ be a set of the metric space $(\mathcal{M}, \rho)$. Then $\mathcal{S}$ is totally bounded, if and only if every sequence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n, \cdots\} \subset \mathcal{S}$ contains a Cauchy subsequence $\{\mathbf{u}_{n_1}, \mathbf{u}_{n_2}, \mathbf{u}_{n_3}, \cdots, \mathbf{u}_{n_k}, \cdots\}$.

Theorem 3. (1) Any pre-compact set $\mathcal{S}$ in any metric space $(\mathcal{M}, \rho)$ is totally bounded.

(2) Any totally bounded set $\mathcal{S}$ in any complete metric space $(\mathcal{M}, \rho)$ is pre-compact.

(3) Any totally bounded set $\mathcal{S}$ is bounded. Any compact set $\mathcal{S}$ is bounded.

Theorem 4. Let $(\mathcal{M}, \rho)$ be a metric space. If every totally bounded set $\mathcal{S}$ is pre-compact, then $(\mathcal{M}, \rho)$ is a complete metric space.

Theorem 5. Let $(\mathcal{M}, \rho)$ be a metric space. Let $\mathcal{S}$ be a totally bounded set or a pre-compact set. Then there exists a subset $\mathcal{R}$, at most countable, $\mathcal{R}$ is dense in $\mathcal{S}$.

Let $\mathcal{S}$ be a set in the metric space $C[a, b]$, where $-\infty < a < b < \infty$. We call $\mathcal{S}$ uniformly continuous, if for every positive constant $\varepsilon > 0$, there exists a positive constant $\delta = \delta(\varepsilon) > 0$, such that

$$|f(x) - f(y)| < \varepsilon,$$

for all $f \in \mathcal{S}$ and for all $x, y \in [a, b]$, such that $|x - y| < \delta$.

Theorem 6. Any bounded, uniformly continuous set in $C[a, b]$ is compact.

Theorem 7. Any compact set in the metric space $C[a, b]$ must be bounded and uniformly continuous.

Theorem 8. Let $\mathcal{S}$ be a set in the metric space l^p. Then $\mathcal{S}$ is compact, if and only if $\mathcal{S}$ is bounded, and for every positive constant $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$, such that there holds

$$\sum_{n=N+1}^{\infty} |x_n|^p < \varepsilon^p,$$

for any sequence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n, \cdots\} \subset \mathcal{S}$.

Theorem 9. Let $\mathcal{S}$ be a compact set in the metric space $(\mathcal{M}, \rho)$. If there exists a family of open sets $\mathcal{O} = \{\mathcal{O}_\lambda : \lambda \in \Lambda\}$, which covers $\mathcal{S}$, that is

$$\mathcal{S} \subset \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda,$$

then there must be finitely many open sets, such that

$$\mathcal{S} \subset \bigcup_{i=1}^{n} \mathcal{O}_{\lambda_i}.$$

For any $\mathbf{u} \in \mathcal{S}$, there exists an open set $\mathcal{O}_\lambda$ and there exists a positive constant $\delta > 0$, such that

$$B(\mathbf{u}, \delta) \subset \mathcal{O}_\lambda.$$

Let

$$\delta(\mathbf{u}) = \sup_{B(\mathbf{u},\delta) \subset \mathcal{O}_\lambda} \delta.$$

Define the Lebesgue's number

$$\delta_+ L_N = \inf_{\mathbf{u} \in \mathcal{S}} \delta(\mathbf{u}).$$

We claim that $\delta_+ > 0$.

There exists a sequence of points

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n, \cdots\},$$

such that

$$\delta_+ \leq \delta(\mathbf{u}_n) < \delta_+ + \frac{1}{2^n}.$$

There exists a convergent subsequence

$$\{\mathbf{u}_{n_1}, \mathbf{u}_{n_2}, \mathbf{u}_{n_3}, \cdots, \mathbf{u}_{n_k}, \cdots\} \subset \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n, \cdots\},$$

and there exists a point $\mathbf{u}_0 \in \mathcal{S}$, such that

$$\lim_{k\to\infty} \rho(\mathbf{u}_{n_k}, \mathbf{u}_0) = 0.$$

There is a positive constant $\delta_0 > 0$ and there exists an open set $\mathcal{O}_{\lambda_0}$, such that

$$B(\mathbf{u}_0, \delta_0) \subset \mathcal{O}_{\lambda_0}.$$

$$\rho(\mathbf{u}_{n_k}, \mathbf{u}_0) < \frac{1}{2}\delta_0$$

Note that

$$B(\mathbf{u}_{n_k}, \frac{1}{2}\delta_0) \subset B(\mathbf{u}_0, \delta_0) \subset \mathcal{O}_{\lambda_0}.$$

There exist finitely many points $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n \in \mathcal{S}$

$$\bigcup_{k=1}^{n} B(\mathbf{u}_k, \frac{1}{2}\delta_0) \subset \mathcal{S}.$$

$$\mathcal{S} \subset \bigcup_{k=1}^{n} B(\mathbf{u}_k, \frac{1}{2}\delta_0) \subset \bigcup_{k=1}^{n} \mathcal{O}_{\lambda_k}.$$

$\lambda\lambda$

Theorem 10. Let $\mathcal{S}$ be a set in the metric space $(\mathcal{M}, \rho)$. If there are finitely many open sets in every open cover $\mathcal{O} = \{\mathcal{O}_\lambda : \lambda \in \Lambda\}$ of $\mathcal{S}$, such that

$$\mathcal{S} \subset \bigcup_{i=1}^{n} \mathcal{O}_{\lambda_i},$$

then $\mathcal{S}$ is compact. On the other hand, for any positive constant $\varepsilon > 0$,

$$\{B(\mathbf{w}, \varepsilon) : \mathbf{w} \in \mathcal{S}\}$$

is an open cover of $\mathcal{S}$. There must be finitely many open balls $B(\mathbf{w}_k, \varepsilon)$, such that

$$\mathcal{S} \subset \bigcup_{k=1}^{m} B(\mathbf{w}_k, \varepsilon).$$

Let

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n, \cdots\}$$

be any sequence in $\mathcal{S}$. There must be an open ball $B(\mathbf{w}_k, \varepsilon)$, so that

$$\{\mathbf{u}_{n_1}, \mathbf{u}_{n_2}, \mathbf{u}_{n_3}, \cdots, \mathbf{u}_{n_k}, \cdots\} \subset B(\mathbf{w}_k, \varepsilon).$$

Iterating this idea for infinitely many times, each time letting $\varepsilon = \frac{1}{2^p}$, we find a convergent subsequence

$$\{\mathbf{u}_{n_1}, \mathbf{u}_{n_2}, \mathbf{u}_{n_3}, \cdots, \mathbf{u}_{n_k}, \cdots\}$$

The limit

$$\mathbf{u}_0 = \lim_{k \to \infty} \mathbf{u}_{n_k} \in \mathcal{S}.$$

Otherwise, without loss of generality, we may assume that all points in the sequence $\{\mathbf{u}_n\}$ are distinct. One cannot find a finite open set in the open cover

$$\{B(\mathbf{u}, \varepsilon) : \mathbf{u} \in \mathcal{S}\} \cup \{B(\mathbf{u}_k), \varepsilon_k\},$$
$$B(\mathbf{u}_{n_k}, \varepsilon_k) \cap B(\mathbf{u}_{n_l}, \varepsilon_l) = \emptyset, k \neq l,$$

to cover the subsequence $\{\mathbf{u}_{n_1}, \mathbf{u}_{n_2}, \mathbf{u}_{n_3}, \cdots, \}$.

Theorem 11. Let $\mathcal{S}$ be a compact set in the metric space $(\mathcal{M}, \rho)$. Let $\mathcal{A} : (\mathcal{M}, \rho) \to (\mathcal{M}, \rho)$ be a continuous mapping. Then $\mathcal{A}(\mathcal{S})$ is also compact.

Theorem 12. Let $\mathbf{Y}$ be a closed subspace in the normed linear space $(\mathbf{X}, \rho)$, and $\mathbf{X} \neq \mathbf{Y}$. Then for any positive constant $0 < \varepsilon < 1$, there exists a unit vector $\mathbf{u}_0$, such that

$$\rho(\mathbf{u}_0, \mathbf{Y}) > \varepsilon, \qquad \|\mathbf{u}_0\| = 1.$$

Let $\mathbf{u}_1 \in \mathbf{X}$, but $\mathbf{u}_1 \notin \mathbf{Y}$. Then

$$\rho_1 \overset{\text{def}}{=} \rho(\mathbf{u}_1, \mathbf{Y}) > 0.$$

There exists a vector $\mathbf{u}_2 \in \mathbf{Y}$, such that

$$\|\mathbf{u}_1 - \mathbf{u}_2\| < \frac{\rho_1}{\varepsilon}.$$

Let

$$\mathbf{u}_0 = \frac{\mathbf{u}_1 - \mathbf{u}_2}{\|\mathbf{u}_1 - \mathbf{u}_2\|}.$$

Now for any $\mathbf{u} \in \mathbf{Y}$,

$$\mathbf{u}_2 + \|\mathbf{u}_1 - \mathbf{u}_2\|\mathbf{u} \in \mathbf{Y}.$$

Hence we have the following estimates

$$\|\mathbf{u} - \mathbf{u}_0\| = \frac{1}{\|\mathbf{u}_1 - \mathbf{u}_2\|}\|(\mathbf{u}_2 + \|\mathbf{u}_1 - \mathbf{u}_2\|\mathbf{u}) - \mathbf{u}_1\| > \varepsilon.$$

Since $\mathcal{S}$ is compact, $\mathcal{S}$ is totally bounded. For any positive constant $\varepsilon > 0$, there exists finitely many points $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_p, \} \subset \mathcal{S}$, such that

$$\mathcal{S} \subset \bigcup_{k=1}^{p} B(\mathbf{z}_k, \varepsilon).$$

For any linear combination of finitely many vectors

$$\sum_{k=1}^{m} \alpha_k \mathbf{u}_k,$$

where

$$\alpha_k > 0, \qquad \mathbf{u}_k \in \mathcal{S}, \qquad \sum_{k=1}^{m} \alpha_k,$$

without loss of generality, we may assume that

$$\begin{aligned}
&\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_{m_1} \in B(\mathbf{z}_1, \varepsilon),\\
&\mathbf{u}_{m_1+1}, \mathbf{u}_{m_1+2}, \mathbf{u}_{m_1+3}, \cdots, \mathbf{u}_{m_2} \in B(\mathbf{z}_2, \varepsilon),\\
&\mathbf{u}_{m_2+1}, \mathbf{u}_{m_2+2}, \mathbf{u}_{m_2+3}, \cdots, \mathbf{u}_{m_3} \in B(\mathbf{z}_3, \varepsilon),\\
&\cdots\cdots\\
&\mathbf{u}_{m_{p-1}+1}, \mathbf{u}_{m_{p-1}+2}, \mathbf{u}_{m_{p-1}+3}, \cdots, \mathbf{u}_{m_p} \in B(\mathbf{z}_p, \varepsilon).
\end{aligned}$$

Define

$$\begin{aligned}
&\beta_1 = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{m_1},\\
&\beta_2 = \alpha_{m_1+1} + \alpha_{m_1} + \alpha_{m_1} + \cdots + \alpha_{m_2},\\
&\beta_3 = \alpha_{m_2+1} + \alpha_{m_2} + \alpha_{m_2} + \cdots + \alpha_{m_3},\\
&\cdots\cdots\\
&\beta_p = \alpha_{m_{p-1}+1} + \alpha_{m_{p-1}+2} + \alpha_{m_{p-1}+3} + \cdots + \alpha_{m_p}.
\end{aligned}$$

Then

$$\sum_{i=1}^{p} \beta_i = 1.$$

Moreover, we have the following estimates

$$\begin{aligned} &\left\| \sum_{i=1}^{p} \beta_i \mathbf{z}_i - \sum_{k=1}^{m} \alpha_k \mathbf{u}_k \right\| \\ = &\left\| \sum_{i=1}^{p} \left(\sum \beta_0 \right) \mathbf{z}_i - \sum_{k=1}^{m} \alpha_k \mathbf{u}_k \right\| \\ &\qquad\qquad\qquad\qquad \| \| \; \| \| \end{aligned}$$

The convex set

$$\left\{ \sum_{k=1}^{m} \beta_i \mathbf{z}_i \right\}$$

is a bounded subset in a finite-dimensional space. It is totally bounded. Therefore, the set

$$\left\{ \sum_{k=1}^{m} \alpha_k \mathbf{u}_k \right\}$$

is totally bounded.

$$\begin{aligned} \mathcal{A}(\mathcal{S}) &\subset \mathcal{S}, \\ h(\mathcal{A}(\mathcal{S})) &\subset \mathcal{S}. \end{aligned}$$

$$\mathcal{A} : h(\mathcal{S}) \to h(\mathcal{S}),$$

is a continuous mapping. For any positive constant $\varepsilon > 0$, there exist finitely many $(\rho\rho)(\rho)\rho)\rho)(\rho)(\rho) \in \rho)\rho)\rho)\rho)(\rho)$

Chapter 2: Banach Spaces and Bounded Linear Operators

Section 1: Bounded Linear Operators

1.1 Boundedness and Continuity of Linear Operators

Definition 1. Let $\mathcal{X}$ be a vector space, let $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ be a function. $\|\cdot\|$ is called a norm on $\mathcal{X}$, if

$$\begin{aligned}
&\|\mathbf{x}\| \geq 0, \qquad \mathbf{x} \in \mathcal{X}, \\
&\|\mathbf{x}\| = 0 \text{ if } \mathbf{x} = \mathbf{0}, \qquad \|\mathbf{x}\| > 0 \text{ if } \mathbf{x} \neq \mathbf{0}, \\
&\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|, \\
&\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|,
\end{aligned}$$

for any vectors $\mathbf{x} \in \mathbf{X}$ and $\mathbf{y} \in \mathbf{X}$ and for any constants $\alpha \in \mathbb{R}$. The space $(\mathcal{X}, \|\cdot\|)$ is called a normed linear space. If the space $(\mathcal{X}, \|\cdot\|)$ is complete, then it is called a Banach space.

Definition 2. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces and let $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ be a mapping. $\mathcal{A}$ is called a linear operator, if

$$\begin{aligned}
& \mathcal{A}(\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_3 + \cdots + \alpha_n \mathbf{x}_n) \\
= \; & \alpha_1 \mathcal{A}\mathbf{x}_1 + \alpha_2 \mathcal{A}\mathbf{x}_2 + \alpha_3 \mathcal{A}\mathbf{x}_3 + \cdots + \alpha_n \mathcal{A}\mathbf{x}_n,
\end{aligned}$$

for any vectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\cdots$, $\mathbf{x}_n$ in $\mathcal{X}$, for any real constants α_1, α_2, α_3, $\cdots$, $\alpha_n \in \mathbb{R}$, and for any positive integer $n \geq 1$.

Definition 3. Let $\mathcal{A} : (\mathcal{X}, \|\cdot\|) \to (\mathcal{Y}, \|\cdot\|)$ be a linear operator. The mapping $\mathcal{A}$ is called bounded, if it maps any bounded subset in $\mathcal{X}$ to a bounded subset in $\mathcal{Y}$.

Definition 4. The linear operator $\mathcal{A} : (\mathcal{X}, \|\cdot\|) \to (\mathcal{Y}, \|\cdot\|)$ is called continuous at the point $\mathbf{x}_0 \in \mathcal{X}$, if

$$\lim_{n\to\infty} \|\mathcal{A}\mathbf{x}_n - \mathcal{A}\mathbf{x}_0\| = 0,$$

for any convergent sequence $\{\mathbf{x}_n\} \subset \mathcal{X}$:

$$\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}_0\| = 0.$$

Theorem 1. Let $\mathcal{A} : (\mathcal{X}, \|\cdot\|) \to (\mathcal{Y}, \|\cdot\|)$ be a linear operator. If $\mathcal{A}$ is continuous at some point $\mathbf{x}_0 \in \mathcal{X}$, then it is continuous everywhere in $\mathcal{X}$.

Theorem 2. Let $\mathcal{A} : (\mathcal{X}, \|\cdot\|) \to (\mathcal{Y}, \|\cdot\|)$ be a linear operator. Then $\mathcal{A}$ is bounded, if and only if there exists a positive constant $C > 0$, such that

$$\|\mathcal{A}\mathbf{x}\| \le C\|\mathbf{x}\|,$$

for any point $\mathbf{x} \in \mathcal{X}$.

Definition 5. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces, let

$$\mathcal{A} : \mathcal{X} \to \mathcal{Y}$$

be a bounded linear operator. Define the norm of $\mathcal{A}$ by

$$\begin{aligned} \|\mathcal{A}\| &= \sup_{\mathbf{x}\in\mathcal{X}, \|\mathbf{x}\|=1} \|\mathcal{A}\mathbf{x}\| \\ &= \sup_{\mathbf{x}\in\mathcal{X}, \mathbf{x}\neq\mathbf{0}} \frac{\|\mathcal{A}\mathbf{x}\|}{\|\mathbf{x}\|}. \end{aligned}$$

Theorem 3. Let

$$\mathcal{A} : (\mathcal{X}, \|\cdot\|) \to (\mathcal{Y}, \|\cdot\|)$$

be a linear operator. Then $\mathcal{A}$ is bounded, if and only if it is continuous.

1.2 The Space $\mathbb{B}(\mathcal{X} \to \mathcal{Y})$

Definition 6. Let $n \geq 1$ be any positive integer, let

$$\mathcal{A}_k : (\mathcal{X}, \|\cdot\|) \to (\mathcal{Y}, \|\cdot\|)$$

be bounded linear operators, let α_1, α_2, α_3, $\cdots$, α_n be real constants. Define the new operator $\mathcal{A}$ by

$$\begin{aligned} \mathcal{A} &= \alpha_1\mathcal{A}_1 + \alpha_2\mathcal{A}_2 + \alpha_3\mathcal{A}_3 + \cdots + \alpha_n\mathcal{A}_n : \\ \mathcal{A}\mathbf{x} &= (\alpha_1\mathcal{A}_1 + \alpha_2\mathcal{A}_2 + \alpha_3\mathcal{A}_3 + \alpha_n\mathcal{A}_n)(\mathbf{x}) \\ &= \alpha_1\mathcal{A}_1\mathbf{x} + \alpha_2\mathcal{A}_2\mathbf{x} + \alpha_3\mathcal{A}_3\mathbf{x} + \cdots + \alpha_n\mathcal{A}_n\mathbf{x}, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{X}$.

Theorem 4. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces. Then

$$\mathbb{B}(\mathcal{X} \to \mathcal{Y}) \overset{\text{def}}{=} \{\mathcal{A} : \mathcal{A} : \mathcal{X} \to \mathcal{Y} \text{ is a bounded linear operator }\},$$

is a normed linear space. If $\mathcal{Y}$ is a Banach space, then $\mathbb{B}(\mathcal{X} \to \mathcal{Y})$ is a Banach space.

Definition 7. The functional $f : (\mathcal{X}, \|\cdot\|) \to \mathbb{R}$ is called linear and continuous, if $f : \mathcal{X} \to \mathbb{R}$ is a bounded linear operator.

Theorem 5. The linear functional $f : \mathcal{X} \to \mathbb{R}$ is continuous, if and only if the null space of f:

$$\mathcal{N}(f) = \{\mathbf{x} \in \mathcal{X} : f(\mathbf{x}) = 0\},$$

is closed in $\mathbf{X}$.

Theorem 6. The dual space

$$\mathcal{X}^* \overset{\text{def}}{=} \mathbb{B}(\mathcal{X} \to \mathbb{R}),$$

is a Banach space.

Theorem 7. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space, let $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ be a bounded linear operator. Then there exists the following limit

$$\lim_{n\to\infty} \left\{ \|\mathcal{A}^n\|^{1/n} \right\} = \inf_{n\geq 1} \left\{ \|\mathcal{A}^n\|^{1/n} \right\}.$$

Section 2: The Representations and Extensions of Continuous Linear Functionals

2.1 Representations of Continuous Linear Functions

Definition 1. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces, let $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ be a linear mapping. The mapping $\mathcal{A}$ is called a norm-preserving isomorphism, if $\mathcal{A}$ is one-to-one, onto, such that

$$\|\mathcal{A}\mathbf{x}\| = \|\mathbf{x}\|,$$

for all $\mathbf{x} \in \mathcal{X}$. In this case we say the spaces $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic. Roughly speaking, we may regard $\mathcal{A}\mathbf{x}$ and $\mathbf{x}$ as the same vector.

Theorem 1. The normed linear spaces $(l^1)^*$ and l^∞ are isomorphic.

Proof. Define the mapping $\mathcal{A} : (l^1)^* \to l^\infty$ by

$$\mathcal{A}(f) = (f(\mathbf{e}_1), f(\mathbf{e}_2), f(\mathbf{e}_3), \cdots, f(\mathbf{e}_n), \cdots),$$

for all $f \in (l^1)^*$, where

$$\begin{aligned}
&\mathbf{e}_1 = (1, 0, 0, 0, \cdots, 0, \cdots) \in l^1,\\
&\mathbf{e}_2 = (0, 1, 0, 0, \cdots, 0, \cdots) \in l^1,\\
&\mathbf{e}_3 = (0, 0, 1, 0, \cdots, 0, \cdots) \in l^1,\\
&\cdots\cdots\\
&\mathbf{e}_n = (0, 0, 0, 0, \cdots, 0, 1, 0, \cdots) \in l^1,\\
&\cdots\cdots
\end{aligned}$$

Then the mapping

$$\mathcal{A} : (l^1)^* \to l^\infty$$

is one-to-one, onto and

$$\|\mathcal{A}f\| = \|f\|,$$

for all $f \in (l^1)^*$.

For any vector

$$\mathbf{x} = (x_1, x_2, x_3, \cdots, x_n, \cdots) \in l^1,$$

we know that

$$\sum_{n=1}^{\infty} |x_n| < \infty,$$

$$\mathbf{x} = \sum_{n=1}^{\infty} x_n \mathbf{e}_n.$$

Moreover

$$f(\mathbf{x}) = \sum_{n=1}^{\infty} x_n f(\mathbf{e}_n).$$

Obviously

$$\begin{aligned}
&|f(\mathbf{e}_n)| \le \|f\|, \qquad n \ge 1 \\
&\sup_{n\ge 1} |f(\mathbf{e}_n)| \le \|f\| \\
&\|\mathcal{A}f\| \le \|f\|
\end{aligned}$$

On the other hand

$$\begin{aligned}
&|f(\mathbf{x})| \leq \sum_{n=1}^{\infty} |x_n||f(\mathbf{e}_n)| \\
&|f(\mathbf{x})| \leq \sup_{n\geq 1} |f(\mathbf{e}_n)| \sum_{n=1}^{\infty} |x_n| \\
&\|f\| \leq \|\mathcal{A}f\| \\
&\|\mathcal{A}f\| = \|f\|
\end{aligned}$$

Theorem 2. The normed linear spaces $(l^p)^*$ and l^q are isomorphic, where $p > 1$ and $q > 1$ are positive constants, such that $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Define the mapping $\mathcal{A} : (l^p)^* \to l^q$ by

$$\mathcal{A}(f) = (f(\mathbf{e}_1), f(\mathbf{e}_2), f(\mathbf{e}_3), \cdots, f(\mathbf{e}_n), \cdots),$$

for all $f \in (l^p)^*$, where

$$\begin{aligned}
&\mathbf{e}_1 = (1, 0, 0, 0, \cdots, 0, \cdots) \in l^p, \\
&\mathbf{e}_2 = (0, 1, 0, 0, \cdots, 0, \cdots) \in l^p, \\
&\mathbf{e}_3 = (0, 0, 1, 0, \cdots, 0, \cdots) \in l^p, \\
&\cdots\cdots \\
&\mathbf{e}_n = (0, 0, 0, 0, \cdots, 0, 1, 0, \cdots) \in l^p, \\
&\cdots\cdots
\end{aligned}$$

First of all, the mapping is well defined. Let

$$\begin{aligned}
\mathbf{x}_n = \big(&|f(\mathbf{e}_1)|^{q-1} \exp[-\mathrm{i} \arg f(\mathbf{e}_1)], |f(\mathbf{e}_2)|^{q-1} \exp[-\mathrm{i} \arg f(\mathbf{e}_2)], \\
&|f(\mathbf{e}_3)|^{q-1} \exp[-\mathrm{i} \arg f(\mathbf{e}_3)], \cdots, |f(\mathbf{e}_n)|^{q-1} \exp[-\mathrm{i} \arg f(\mathbf{e}_n)], 0, \cdots \big),
\end{aligned}$$

for all positive integers $n \geq 1$. Then $\mathbf{x}_n \in l^p$. Moreover

$$\|\mathbf{x}_n\|_{l^p}^p = \sum_{k=1}^{n} |f(\mathbf{e}_k)|^{p(q-1)} = \sum_{k=1}^{n} |f(\mathbf{e}_k)|^q.$$

$$|f(\mathbf{x}_n)| = \sum_{k=1}^{n} |f(\mathbf{e}_k)|^q \leq \|f\| \|\mathbf{x}_n\|_{l^p}$$

$$\sum_{k=1}^{n} |f(\mathbf{e}_k)|^q = |f(\mathbf{x}_n)| \leq \|f\| \|\mathbf{x}_n\|_{l^p} = \|f\| \left\{ \sum_{k=1}^{n} |f(\mathbf{e}_k)|^q \right\}^{1/p}$$

$$\left\{ \sum_{k=1}^{n} |f(\mathbf{e}_k)|^q \right\}^{1/q} \leq \|f\|, \qquad n \geq 1$$

Therefore

$$\left\{ \sum_{n=1}^{\infty} |f(\mathbf{e}_n)|^q \right\}^{1/q} \leq \|f\| < \infty, \qquad \|\mathcal{A}f\| \leq \|f\|.$$

Note that for any $\mathbf{x} \in l^p$,

$$|f(\mathbf{x})| = \left| \sum_{n=1}^{\infty} x_n f(\mathbf{e}_n) \right| \leq \left\{ \sum_{n=1}^{\infty} |x_n|^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} |f(\mathbf{e}_n)|^q \right\}^{1/q}$$

$$\|f\| \leq \|\mathcal{A}f\|$$

$$\|\mathcal{A}f\| \leq \|f\|$$

$$\|\mathcal{A}f\| = \|f\|$$

For any $\mathbf{x} \in l^p$ and $\mathbf{y} \in l^q$,

$$\mathbf{x} = (x_1, x_2, x_3, \cdots, x_n, \cdots),$$
$$\mathbf{y} = (y_1, y_2, y_3, \cdots, y_n, \cdots),$$

let us define the linear functional $f \in (l^p)^*$:

$$f(\mathbf{x}) = \sum_{n=1}^{\infty} x_n y_n.$$

Then

$$\mathcal{A}f = (y_1, y_2, y_3, \cdots, y_n, \cdots).$$

Thus the mapping $\mathcal{A} : (l^p)^* \to l^q$ is both one-to-one and onto.

Theorem 3. Either let $p > 1$ and $q > 1$ be positive constants, such that $\frac{1}{p} + \frac{1}{q} = 1$, or let $p = 1$ and $q = \infty$. Then the following statements are true.

(1) The normed linear spaces $[L^p(\mathbb{R})]^*$ and $L^q(\mathbb{R})$ are isomorphic.

(2) The normed linear spaces $[L^p(\mathbb{R}^n)]^*$ and $L^q(\mathbb{R}^n)$ are isomorphic.

(3) The normed linear spaces $(L^p[a, b])^*$ and $L^q[a, b]$ are isomorphic.

(4) The normed linear spaces $[L^p(\Omega)]^*$ and $L^q(\Omega)$ are isomorphic, where $\Omega \subset \mathbb{R}^n$ is a bounded open domain.

2.2 Extensions of Continuous Linear Functionals

Definition 1. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces. Let $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator and let $\mathcal{A}_0 : \mathcal{E} \to \mathcal{Y}$ be a bounded linear operator, where $\mathcal{E}$ is a subspace of $\mathcal{X}$. We say $\mathcal{A}$ is an extension of $\mathcal{A}_0$, if

$$\begin{aligned} &(1) \quad \|\mathcal{A}\| = \|\mathcal{A}_0\|, \\ &(2) \quad \mathcal{A}\mathbf{x} = \mathcal{A}_0\mathbf{x}, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{E}$.

Theorem 4. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let $(\mathcal{Y}, \|\cdot\|)$ be a Banach space. Let $\mathcal{A}_0 : \mathcal{E} \to \mathcal{Y}$ be a bounded linear operator, where $\mathcal{E}$ is a dense subspace of $\mathcal{X}$. Then there exists a unique bounded linear operator $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$, such that

$$\begin{aligned} &(1) \quad \|\mathcal{A}\| = \|\mathcal{A}_0\|, \\ &(2) \quad \mathcal{A}\mathbf{x} = \mathcal{A}_0\mathbf{x}, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{E}$.

Proof. For any point $\mathbf{x}_0 \in \mathcal{X}$, there exist a sequence of points

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n, \cdots\} \subset \mathcal{E},$$

such that

$$\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}_0\| = 0.$$

Now $\{\mathcal{A}_0\mathbf{x}_n\}$ is a Cauchy sequence in the Banach space $\mathcal{Y}$. Define the operator $\mathcal{A}$:

$$\mathcal{A}\mathbf{x}_0 = \lim_{n\to\infty} \mathcal{A}_0\mathbf{x}_n.$$

Theorem 5. Let $(\mathcal{X}, \|\cdot\|)$ be a real normed linear space, let $\mathcal{E} \subset \mathcal{X}$ be a subspace and let $f_0 : \mathcal{E} \to \mathbb{R}$ be a real continuous linear functional. For any vector $\mathbf{x}_0 \in \mathcal{X} - \mathcal{E}$, let

$$\mathcal{F} = \text{ span } \{\mathbf{x}_0, \mathcal{E}\}.$$

Then there exists a continuous linear functional $f : \mathcal{F} \to \mathbb{R}$, such that

$$\begin{aligned} &(1) \quad \|f\| = \|f_0\|, \\ &(2) \quad f(\mathbf{x}) = f_0(\mathbf{x}), \end{aligned}$$

for all $\mathbf{x} \in \mathcal{E}$.

Proof. Let the real constant λ_0 satisfy the conditions

$$\sup_{\mathbf{x}\in\mathcal{E}} \Big\{ f_0(\mathbf{x}) - \|f_0\|\|\mathbf{x} - \mathbf{x}_0\| \Big\} \le \lambda_0 \le \inf_{\mathbf{x}\in\mathcal{E}} \Big\{ \|f_0\|\|\mathbf{x}_0 + \mathbf{x}\| - f_0(\mathbf{x}) \Big\}.$$

For any point $r\mathbf{x}_0 + \mathbf{y} \in \text{span}\,\{\mathbf{x}_0, \mathcal{E}\}$, we define the linear functional f:

$$f(r\mathbf{x}_0 + \mathbf{y}) = r\lambda_0 + f_0(\mathbf{y}),$$

for all $\mathbf{y} \in \mathcal{E}$ and for all $r \in \mathbb{R}$. Then f is a continuous linear function on $\mathcal{F}$. The above conditions guarantee that

$$|r\lambda_0 + f_0(\mathbf{y})| \le \|f_0\|\|r\mathbf{x}_0 + \mathbf{y}\|,$$

for all real constants $r \in \mathbb{R}$ and for all $\mathbf{y} \in \mathcal{E}$.

Theorem 6. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let $\mathcal{E} \subset \mathcal{X}$ be a subspace. Suppose that there exist finitely many, at most countable many points

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n, \cdots$$

in $\mathcal{X}$, such that $\mathbf{x}_1 \notin \mathcal{E}$,

$$\mathbf{x}_{n+1} \notin \text{ span } \left\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n, \mathcal{E}\right\},$$

for all positive integers $n \geq 1$. Moreover, suppose that

$$\mathcal{X} = \overline{\text{ span } \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n, \cdots, \mathcal{E}\}}.$$

For any continuous linear function $f_0 : \mathcal{E} \to \mathbb{R}$, there exists a continuous linear functional $f : \mathcal{X} \to \mathbb{R}$, such that

$$\begin{aligned} &(1) \quad \|f\| = \|f_0\|, \\ &(2) \quad f(\mathbf{x}) = f_0(\mathbf{x}), \end{aligned}$$

for all $\mathbf{x} \in \mathcal{E}$.

Proof. By using induction and Theorem 4.

2.3 Applications of the Extension Theorem

Theorem 7. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let $\mathcal{E} \subset \mathcal{X}$ be a subspace, let $\mathbf{x}_0 \in \mathcal{X}$, such that $\rho_0 = \rho(\mathbf{x}_0, \mathcal{E}) > 0$. Then there exists a continuous linear functional $f : \mathcal{X} \to \mathbb{R}$, such that

$$\begin{aligned} &(1) \quad \|f\| = 1, \\ &(2) \quad f(\mathbf{x}_0) = \rho(\mathbf{x}_0, \mathcal{X}), \\ &(3) \quad f(\mathbf{x}) = 0, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{E}$.

The key point in the proof is to define the subspace

$$\mathcal{F}_0 = \text{ span } \left\{\mathbf{x}_0, \mathcal{E}\right\}.$$

and define the functional

$$f_0(\lambda \mathbf{x}_0 + \mathbf{y}) = \lambda \rho(\mathbf{x}_0, \mathcal{E}),$$

for all $\lambda \in \mathbb{R}$ and all $\mathbf{y} \in \mathcal{E}$. Note that

$$\begin{aligned} &f_0(\mathbf{x}_0) = \rho(\mathbf{x}_0, \mathcal{E}), \\ &f(\mathbf{y}) = 0, \end{aligned}$$

for all $\mathbf{y} \in \mathcal{E}$. Moreover

$$f_0(\lambda \mathbf{x}_0 + \mathbf{y}) = \lambda \rho(\mathbf{x}_0, \mathcal{E}) \leq \|\lambda \mathbf{x}_0 + \mathbf{y}\|.$$

Then $\|f\| = 1$ follows from $\|f\| \leq 1$ and $\|f\| \geq 1$.

Theorem 8. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let $\mathbf{x}_0 \in \mathcal{X}$ and $\mathbf{x}_0 \neq \mathbf{0}$. Then there exists a continuous linear functional $f \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f\| = 1, \\ &(2) \quad f(\mathbf{x}_0) = \|\mathbf{x}_0\|. \end{aligned}$$

Theorem 9. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let $\mathbf{x}_0 \in \mathcal{X}$. Then

$$\|\mathbf{x}_0\| = \sup_{f \in \mathcal{X}^*, \|f\|=1} |f(\mathbf{x}_0)|.$$

Theorem 10.

Theorem 11. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. Then

$$\begin{aligned} &\dim(\mathcal{X}) < \infty, \ \text{if and only if} \ \dim(\mathcal{X}^*) < \infty. \\ &\dim(\mathcal{X}) = \dim(\mathcal{X}^*) < \infty. \end{aligned}$$

Proof. The main idea is to apply Theorem in Chapter 1, Section 6 and to use the extension theorems for continuous linear functionals in Chapter 2, Section 2. This is the first half of the proof. Let the dimension

$$1 \leq \dim(\mathcal{X}) = n < \infty.$$

Let $\mathbf{x}_1 \in \mathcal{X}$, $\|\mathbf{x}_1\| = 1$. Then there exists a continuous linear functional $f_1 \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f_1\| = 1, \\ &(2) \quad f(\mathbf{x}_1) = \|\mathbf{x}_1\| = 1. \end{aligned}$$

Let

$$\mathcal{E}_1 = \text{ span } \{\mathbf{x}_1\}.$$

Then there exists a point $\mathbf{x}_2 \in \mathcal{X}$, $\|\mathbf{x}_2\| = 1$, such that

$$\rho(\mathbf{x}_2, \mathcal{E}_1) > \frac{1}{2}.$$

Moreover, there exists a continuous linear functional $f_2 \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f_2\| = 1, \\ &(2) \quad f_2(\mathbf{x}_2) = \rho(\mathbf{x}_2, \mathcal{E}_1) > \frac{1}{2}, \\ &(3) \quad f_2(\mathbf{x}) = 0, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{E}_1$.

Let

$$\mathcal{E}_2 = \text{ span } \{\mathbf{x}_1, \mathbf{x}_2\}.$$

Then there exists a point $\mathbf{x}_3 \in \mathcal{X}$, $\|\mathbf{x}_3\| = 1$, such that

$$\rho(\mathbf{x}_3, \mathcal{E}_2) > \frac{1}{2}.$$

Furthermore, there exists a continuous linear functional $f_3 \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f_3\| = 1, \\ &(2) \quad f_3(\mathbf{x}_3) = \rho(\mathbf{x}_3, \mathcal{E}_2) > \frac{1}{2}, \\ &(3) \quad f_3(\mathbf{x}) = 0, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{E}_2$. We repeat this process for finitely many times and we get the following points, closed subspaces and continuous linear functionals:

$$\begin{aligned} &\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n \in \mathcal{X}, \|\mathbf{x}_k\| = 1, \\ &\mathcal{E}_k = \text{ span } \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_k, \} \\ &f_k \in \mathcal{X}^*, \\ &\|f_k\| = 1, \\ &f_k(\mathbf{x}_k) = \rho(\mathbf{x}_k, \mathcal{E}_{k-1}) > \frac{1}{2}, \\ &f_k(\mathbf{x}) = 0, \end{aligned}$$

for all points $\mathbf{x} \in \mathcal{E}_{k-1}$, where $k = 1, 2, 3, \cdots, n$.

Now we may claim that

$$\mathcal{B}(\mathcal{X}) \stackrel{\text{def}}{=} \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n\}$$

is a basis of the normed linear space $\mathcal{X}$, and that

$$\mathcal{B}(\mathcal{X}^*) \stackrel{\text{def}}{=} \{f_1, f_2, f_3, \cdots, f_n\}$$

is a basis of the dual space $\mathcal{X}^*$.

In fact, let

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \cdots + \alpha_n\mathbf{x}_n = \mathbf{0}.$$

Note that

$$\begin{aligned}
&\mathbf{x}_n \notin \mathcal{E}_{n-1}, \qquad \to \alpha_n = 0,\\
&\mathbf{x}_{n-1} \notin \mathcal{E}_{n-2}, \qquad \to \alpha_{n-1} = 0,\\
&\cdots\cdots\\
&\mathbf{x}_3 \notin \mathcal{E}_2, \qquad \to \alpha_3 = 0,\\
&\mathbf{x}_2 \notin \mathcal{E}_1, \qquad \to \alpha_2 = 0.
\end{aligned}$$

Therefore

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0.$$

Now let

$$\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \cdots + \alpha_n f_n = 0.$$

Note that

$$f_1(\mathbf{x}_1) = \|\mathbf{x}_1\| = 1, f_2(\mathbf{x}_1) = f_3(\mathbf{x}_1) = \cdots = f_{n-1}(\mathbf{x}_1) = f_n(\mathbf{x}_1) = 0.$$

Plugging $\mathbf{x} = \mathbf{x}_1$ leads to $\alpha_1 = 0$. Similarly, plugging $\mathbf{x} = \mathbf{x}_2$ leads to $\alpha_2 = 0$, plugging $\mathbf{x} = \mathbf{x}_3$ leads to $\alpha_3 = 0$, $\cdots$, plugging $\mathbf{x} = \mathbf{x}_n$ leads to $\alpha_n = 0$. Therefore f_1, f_2, f_3, $\cdots$, f_n are linearly independent.

Now any continuous linear functional $f \in \mathcal{X}^*$ may be written as the linear combination of f_1, f_2, f_3, $\cdots$, f_n. In fact, let

$$f(\mathbf{x}) = \alpha_1 f_1(\mathbf{x}) + \alpha_2 f_2(\mathbf{x}) + \alpha_3 f_3(\mathbf{x}) + \cdots + \alpha_n f_n(\mathbf{x}),$$

for all $\mathbf{x} \in \mathcal{X}$.

To find the values of α_k, where $k = 1, 2, 3, \cdots, n$, we let $\mathbf{x} = \mathbf{x}_1$,

$\mathbf{x} = \mathbf{x}_2$, $\mathbf{x} = \mathbf{x}_3$, $\cdots$, $\mathbf{x} = \mathbf{x}_n$, respectively, to get

$$\begin{aligned}
&\alpha_1 f_1(\mathbf{x}_1) = f(\mathbf{x}_1),\\
&\alpha_1 f_1(\mathbf{x}_2) + \alpha_2 f_2(\mathbf{x}_2) = f(\mathbf{x}_2),\\
&\alpha_1 f_1(\mathbf{x}_3) + \alpha_2 f_2(\mathbf{x}_3) + \alpha_3 f_3(\mathbf{x}_3) = f(\mathbf{x}_3),\\
&\cdots\cdots\\
&\alpha_1 f_1(\mathbf{x}_n) + \alpha_2 f_2(\mathbf{x}_n) + \alpha_3 f_3(\mathbf{x}_n) + \cdots + \alpha_n f_n(\mathbf{x}_n) = f(\mathbf{x}_n).
\end{aligned}$$

Intuitively, it is easy to solve these equations one by one to get

$$\begin{aligned}
&\alpha_1 = \frac{f(\mathbf{x}_1)}{f_1(\mathbf{x}_1)},\\
&\alpha_2 = \frac{f(\mathbf{x}_2)}{f_2(\mathbf{x}_2)} - \alpha_1 \frac{f_1(\mathbf{x}_2)}{f_2(\mathbf{x}_2)},\\
&\alpha_3 = \frac{f(\mathbf{x}_3)}{f_3(\mathbf{x}_3)} - \alpha_1 \frac{f_1(\mathbf{x}_3)}{f_3(\mathbf{x}_3)} - \alpha_2 \frac{f_2(\mathbf{x}_3)}{f_3(\mathbf{x}_3)},\\
&\cdots\cdots\\
&\alpha_n = \frac{f(\mathbf{x}_n)}{f_n(\mathbf{x}_n)} - \alpha_1 \frac{f_1(\mathbf{x}_n)}{f_n(\mathbf{x}_n)} - \alpha_2 \frac{f_2(\mathbf{x}_n)}{f_n(\mathbf{x}_n)} - \alpha_3 \frac{f_3(\mathbf{x}_n)}{f_n(\mathbf{x}_n)} - \cdots - \alpha_{n-1} \frac{f_{n-1}(\mathbf{x}_n)}{f_n(\mathbf{x}_n)}.
\end{aligned}$$

Now for any point

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + \cdots + c_n\mathbf{x}_n \in \mathcal{X},$$

we have

$$\begin{aligned}
&f\left(\sum_{k=1}^{n} c_k \mathbf{x}_k\right) = \sum_{k=1}^{n} c_k f(\mathbf{x}_k)\\
&= \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_n \end{pmatrix} \begin{pmatrix} f_1(\mathbf{x}_1) & f_2(\mathbf{x}_1) & f_3(\mathbf{x}_1) & \cdots & f_n(\mathbf{x}_1) \\ f_1(\mathbf{x}_2) & f_2(\mathbf{x}_2) & f_3(\mathbf{x}_2) & \cdots & f_n(\mathbf{x}_2) \\ f_1(\mathbf{x}_3) & f_2(\mathbf{x}_3) & f_3(\mathbf{x}_3) & \cdots & f_n(\mathbf{x}_3) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_1(\mathbf{x}_n) & f_2(\mathbf{x}_n) & f_3(\mathbf{x}_n) & \cdots & f_n(\mathbf{x}_n) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \cdots \\ \alpha_n \end{pmatrix}.
\end{aligned}$$

Therefore, the dimension

$$\dim(\mathcal{X}) = \dim(\mathcal{X}^*) = n < \infty.$$

Below is the second half of the proof. Let the dimension

$$1 \leq \dim(\mathcal{X}^*) = m < \infty.$$

From the first half of the proof, we know that the dimension

$$\dim(\mathcal{X}) < \infty.$$

Moreover, if the dimension

$$\dim(\mathcal{X}) = n < \infty,$$

then

$$\dim(\mathcal{X}^*) = \dim(\mathcal{X}) = n.$$

Overall

$$\dim(\mathcal{X}) = \dim(\mathcal{X}^*) = m = n < \infty.$$

The proof of the theorem is finished. $\square$

Theorem 12. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. Then

$$\begin{aligned} &\dim \mathcal{X} = \infty, \text{ if and only if } \dim(\mathcal{X}^*) = \infty. \\ &\dim(\mathcal{X}) = \dim(\mathcal{X}^*) = \infty. \end{aligned}$$

Proof. The main idea is to apply Theorem in Chapter 1 Section 6 and to use the extension theorems for continuous linear functionals in Chapter 2, Section 2. First of all, let the dimension $\dim \mathcal{X} = \infty$. Let $\mathbf{x}_1 \in \mathcal{X}$, $\|\mathbf{x}_1\| = 1$. Then there exists a continuous linear functional $f_1 \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f_1\| = 1, \\ &(2) \quad f_1(\mathbf{x}_1) = \|\mathbf{x}_1\| = 1. \end{aligned}$$

Let

$$\mathcal{E}_1 = \text{ span } \{\mathbf{x}_1\}.$$

There exists a point $\mathbf{x}_2 \in \mathcal{X}$, $\|\mathbf{x}_2\| = 1$, such that

$$\rho(\mathbf{x}_2, \mathcal{E}_1) > \frac{1}{2}.$$

Moreover, there exists a continuous linear functional $f_2 \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f_2\| = 1, \\ &(2) \quad f_2(\mathbf{x}_2) = \rho(\mathbf{x}_2, \mathcal{E}_1) > \frac{1}{2}, \\ &(3) \quad f_2(\mathbf{x}) = 0, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{E}_1$.

Let

$$\mathcal{E}_2 = \text{ span } \{\mathbf{x}_1, \mathbf{x}_2\}.$$

Then there exists a point $\mathbf{x}_3 \in \mathcal{X}$, $\|\mathbf{x}_3\| = 1$, such that

$$\rho(\mathbf{x}_3, \mathcal{E}_2) > \frac{1}{2}.$$

Furthermore, there exists a continuous linear functional $f_3 \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f_3\| = 1, \\ &(2) \quad f_3(\mathbf{x}_3) = \rho(\mathbf{x}_3, \mathcal{E}_2) > \frac{1}{2}, \\ &(3) \quad f_3(\mathbf{x}) = 0, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{E}_2$.

Let

$$\mathcal{E}_3 = \text{ span } \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}.$$

Let us continue this process. There exists a point $\mathbf{x}_n \in \mathcal{X}$, $\|\mathbf{x}_n\| = 1$, such that

$$\rho(\mathbf{x}_n, \mathcal{E}_{n-1}) > \frac{1}{2}.$$

There exists a continuous linear functional $f_n \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f_n\| = 1, \\ &(2) \quad f_n(\mathbf{x}_n) = \rho(\mathbf{x}_n, \mathcal{E}_{n-1}) > \frac{1}{2}, \\ &(3) \quad f_n(\mathbf{x}) = 0, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{E}_{n-1}$.

If we continue this process forever, then we obtain a sequence of points, a sequence of closed subspaces and a sequence of continuous linear functionals.

Now we may claim that

$$\mathcal{B}(\mathcal{X}) \overset{\text{def}}{=} \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n, \cdots\},$$

is a linearly independent subset of $\mathcal{X}$, and that

$$\mathcal{B}(\mathcal{X}^*) \overset{\text{def}}{=} \{f_1, f_2, f_3, \cdots, f_n, \cdots\},$$

is a linearly independent subset in $\mathcal{X}^*$.

In fact, let

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \cdots + \alpha_n\mathbf{x}_n = \mathbf{0}.$$

Note that

$$\begin{aligned} &\mathbf{x}_n \notin \mathcal{E}_{n-1}, \qquad \to \alpha_n = 0, \\ &\mathbf{x}_{n-1} \notin \mathcal{E}_{n-2}, \qquad \to \alpha_{n=1} = 0, \\ &\cdots\cdots \\ &\mathbf{x}_3 \notin \mathcal{E}_2, \qquad \to \alpha_3 = 0, \\ &\mathbf{x}_2 \notin \mathcal{E}_1, \qquad \to \alpha_2 = 0. \end{aligned}$$

Overall, we have the trivial solution

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0.$$

Now let

$$\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \cdots + \alpha_n f_n = 0.$$

Note that

$$f(\mathbf{x}_1) = \|\mathbf{x}_1\| = 1, f_2(\mathbf{x}_1) = f_3(\mathbf{x}_1) = \cdots = f_n(\mathbf{x}_1) = 0.$$

Letting $\mathbf{x} = \mathbf{x}_1$ leads to $\alpha_1 = 0$. Similarly, letting $\mathbf{x} = \mathbf{x}_2$ leads to $\alpha_2 = 0$, letting $\mathbf{x} = \mathbf{x}_3$ leads to $\alpha_3 = 0$, $\cdots$, letting $\mathbf{x} = \mathbf{x}_n$ leads to $\alpha_n = 0$. Overall, we have the trivial solution

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0.$$

Therefore, the dimension

$$\dim(\mathcal{X}^*) = \infty.$$

Secondly, let

$$\dim(\mathcal{X}^*) = \infty.$$

Claim: The dimension

$$\dim(\mathcal{X}) = \infty.$$

Otherwise, if the dimension $\dim(\mathcal{X}) < \infty$, then $\dim(\mathcal{X}^*) < \infty$. This is a contradiction with Theorem 11.

Very similarly to the above analysis, there exists a sequence of continuous linear functionals, which are linearly independent in $\mathcal{X}^{**}$:

$$\begin{aligned}
&g_1, g_2, g_3, \cdots, g_n, \cdots \in \mathcal{X}^{**},\\
&\|g_1\| = \|g_2\| = \|g_3\| = \cdots = \|g_n\| = \cdots = 1,\\
&\rho(g_{n+1}, \mathcal{F}_n) > \frac{1}{2},\\
&\mathcal{F}_n = \text{ span } \{g_1, g_2, g_3, \cdots, g_n\}.
\end{aligned}$$

If the space $\mathcal{X}$ is self-reflexive, that is

$$\mathcal{X} \cong \mathcal{X}^{**},$$

then there exists a sequence of points

$$\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n, \cdots \in \mathcal{X},$$

such that

$$g_n = \mathcal{A}\mathbf{x}_n = \mathbf{x}_n^{**},$$

for all $n = 1, 2, 3, \cdots$.

Let

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \cdots + \alpha_n\mathbf{x}_n = \mathbf{0}.$$

Performing the operator $\mathcal{A}$ to this equation, we have

$$\alpha_1 g_1 + \alpha_2 g_2 + \alpha_3 g_3 + \cdots + \alpha_n g_n = 0.$$

Since g_1, g_2, g_3, $\cdots$, g_n are linearly independent, we see that

$$\alpha_1 = \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0.$$

Therefore

$$\dim \mathcal{X} = \infty.$$

The proof of the theorem is finished now. $\square$

Section 3: Dual Spaces and Adjoint Operators

Definition 1. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. Let

$$\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n, \cdots \in \mathcal{X}.$$

(1) We say the sequence $\{\mathbf{x}_n\}$ converges strongly to $\mathbf{x}_0$, if

$$\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}_0\| = 0.$$

We write

$$\lim_{n\to\infty} \mathbf{x}_n \overset{\text{strongly}}{=} \mathbf{x}_0.$$

(2) We say the sequence $\{\mathbf{x}_n\}$ converges weakly to $\mathbf{x}_0$, if

$$\lim_{n\to\infty} |f(\mathbf{x}_n) - f(\mathbf{x}_0)| = 0,$$

for all $f \in \mathcal{X}^*$. We write

$$\lim_{n\to\infty} \mathbf{x}_n \overset{\text{weakly}}{=} \mathbf{x}_0.$$

Definition 2. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. Let

$$f_0, f_1, f_2, f_3, \cdots, f_n, \cdots \in \mathcal{X}^*.$$

(1) We say the sequence of continuous linear functionals $\{f_n\}$ converges strongly to f_0, if

$$\lim_{n\to\infty} \|f_n - f_0\| = 0.$$

We write

$$\lim_{n\to\infty} f_n \overset{\text{strongly}}{=} f_0.$$

(2) We say the sequence of continuous linear functionals $\{f_n\}$ converges weakly$*$ to f_0, if

$$\lim_{n\to\infty} |f_n(\mathbf{x}) - f_0(\mathbf{x})| = 0,$$

for all points $\mathbf{x} \in \mathcal{X}$. We write

$$\lim_{n\to\infty} f_n \overset{\text{weakly}*}{=} f_0.$$

Definition 3. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces. Let

$$\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \cdots, \mathcal{A}_n, \cdots \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}).$$

(1) We say the sequence $\{\mathcal{A}_n\}$ converges uniformly to $\mathcal{A}_0$, if

$$\lim_{n\to\infty} \|\mathcal{A}_n - \mathcal{A}_0\| = 0.$$

We write

$$\lim_{n\to\infty} \mathcal{A}_n \overset{\text{uniformly}}{=} \mathcal{A}_0.$$

(2) We say the sequence $\{\mathcal{A}_n\}$ converges strongly to $\mathcal{A}_0$, if

$$\lim_{n\to\infty} \|\mathcal{A}_n \mathbf{x} - \mathcal{A}_0 \mathbf{x}\| = 0,$$

for any point $\mathbf{x} \in \mathcal{X}$.

We write

$$\lim_{n\to\infty} \mathcal{A}_n \overset{\text{strongly}}{=} \mathcal{A}_0.$$

(3) We say the sequence $\{\mathcal{A}_n\}$ converges weakly to $\mathcal{A}_0$, if

$$\lim_{n\to\infty} |f(\mathcal{A}_n \mathbf{x}) - f(\mathcal{A}_0 \mathbf{x})| = 0,$$

for any $\mathbf{x} \in \mathcal{X}$ and for any $f \in \mathcal{Y}^*$.

We write

$$\lim_{n\to\infty} \mathcal{A}_n \overset{\text{weakly}}{=} \mathcal{A}_0.$$

Definition 4. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. Let

$$\mathcal{S} \subset \mathcal{X}^*.$$

(1) We say $\mathcal{S}$ is a strongly compact subset, if in any sequence $\{f_n\} \subset \mathcal{S}$, there exists a strongly convergent subsequence $\{f_{n_k}\}$, that is

$$\lim_{k\to\infty} \|f_{n_k} - f_0\| = 0.$$

(2) We say $\mathcal{S}$ is a weakly$*$ compact subset, if in any sequence $\{f_n\} \subset \mathcal{S}$, there exists a weakly$*$ convergent subsequence $\{f_{n_k}\}$, that is

$$\lim_{k\to\infty} |f_{n_k}(\mathbf{x}) - f_0(\mathbf{x})| = 0,$$

for all $\mathbf{x} \in \mathcal{X}$.

Definition 5. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. For any $\mathbf{x} \in \mathcal{X}$, we define the continuous linear functional $\mathbf{x}^{**} \in \mathcal{X}^{**}$:

$$\mathbf{x}^{**}(f) = f(\mathbf{x}),$$

for all $f \in \mathcal{X}^*$.

Definition 6. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. We call the mapping

$$\mathcal{A} : \mathcal{X} \to \mathcal{X}^{**}, \qquad \mathbf{x} \to \mathbf{x}^{**} = \mathcal{A}\mathbf{x}$$

a natural embedding.

Definition 7. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. The space $(\mathcal{X}, \|\cdot\|)$ is called self-reflexive, if the normed linear spaces $\mathcal{X}^{**}$ and $\mathcal{X}$ are isomorphic. We write

$$\mathcal{X} \cong \mathcal{X}^{**}.$$

Theorem 1. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. The mapping $\mathcal{A} : \mathcal{X} \to \mathcal{X}^{**}$ defined by $\mathbf{x}^{**} = \mathcal{A}\mathbf{x}$ is linear and norm-preserving. That is

$$\begin{aligned} (1) \qquad & (\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \cdots + \alpha_n\mathbf{x}_n)^{**} \\ = \quad & \alpha_1\mathbf{x}_1^{**} + \alpha_2\mathbf{x}_2^{**} + \alpha_3\mathbf{x}_3^{**} + \cdots + \alpha_n\mathbf{x}_n^{**}, \\ (2) \qquad & \|\mathbf{x}^{**}\| = \|\mathbf{x}\|, \end{aligned}$$

for any constants $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n \in \mathbb{R}$, and for any points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\cdots$, $\mathbf{x}_n \in \mathcal{X}$, where $n \geq 1$ is any positive integer.

The proof follows from the definition of $\mathbf{x}^{**}$ and Theorem 8 in Section 2.

Theorem 2. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. Suppose that there exists a sequence of continuous linear functionals

$$\{f_1, f_2, f_3, \cdots, f_n, \cdots\},$$

which is dense in $\mathcal{X}^*$. Then there exists a sequence of points

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n, \cdots\},$$

which is dense in $\mathcal{X}$.

Proof. Without loss of generality, let $\|f_n\| = 1$, otherwise replace f_n with $\frac{1}{\|f_n\|} f_n$. There exists a point $\mathbf{x}_n \in \mathcal{X}$, such that $\|\mathbf{x}_n\| = 1$ and $|f_n(\mathbf{x}_n)| > \frac{1}{2}$, for all positive integers $n = 1, 2, 3, \cdots$. Define

$$\mathcal{E} = \overline{\text{span}\left\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n, \cdots\right\}}.$$

Claim: The normed linear space $\mathcal{X} = \mathcal{E}$.

In fact, if there exists a point $\mathbf{x}_0 \in \mathcal{X}$, but $\mathbf{x}_0 \notin \mathcal{E}$, then there exists a continuous linear functional $f_0 \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f_0\| = 1, \\ &(2) \quad f(\mathbf{x}_0) = \rho(\mathbf{x}_0, \mathcal{E}) > 0, \\ &(3) \quad f_0(\mathbf{x}) = 0, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{E}$. On the other hand, we have the following computations and estimates

$$\|f_n - f_0\| \geq |f_n(\mathbf{x}_n) - f_0(\mathbf{x}_n)| = |f_n(\mathbf{x}_n)| > \frac{1}{2},$$

for all positive integers $n \geq 1$. This contradicts the fact that

$$\{f_1, f_2, f_3, \cdots, f_n, \cdots\}$$

is dense in the unit sphere of $\mathcal{X}^*$. Therefore, the normed linear space $\mathcal{X} = \mathcal{E}$.

Theorem 3. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, and let $(\mathcal{Y}, \|\cdot\|)$ be a Banach space. Let

$$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \cdots, \mathcal{A}_n, \cdots \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}).$$

There is a positive constant $C > 0$, such that

$$\|\mathcal{A}_n\| \le C,$$

for all positive integers $n \ge 1$. Suppose that there exists the following limit

$$\lim_{n\to\infty} \mathcal{A}_n \mathbf{x}$$

for every point $\mathbf{x} \in \mathcal{S}$, where $\mathcal{S}$ is dense in $\mathcal{X}$. Then there exists a bounded linear operator

$$\mathcal{A}_0 \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),$$

such that

$$\lim_{n\to\infty} \|\mathcal{A}_n \mathbf{x} - \mathcal{A}_0 \mathbf{x}\| = 0,$$

for all $\mathbf{x} \in \mathcal{X}$. Moreover

$$\|\mathcal{A}_0\| \le \liminf_{n\to\infty} \|\mathcal{A}_n\|.$$

Theorem 4. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces. Let

$$\begin{aligned} &\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \cdots, \mathcal{A}_n, \cdots \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}), \\ &f_0, f_1, f_2, f_3, \cdots, f_n, \cdots \in \mathcal{X}^*. \end{aligned}$$

If

$$\lim_{n\to\infty} \mathcal{A}_n \overset{\text{weakly}}{=} \mathcal{A}_0,$$

then the limit $\mathcal{A}_0$ is unique. If

$$\lim_{n\to\infty} f_n \overset{\text{weakly}*}{=} f_0,$$

then the limit f_0 is unique.

Definition 8. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. A set $\mathcal{S}$ in $\mathcal{X}^*$ is called bounded, if there exists positive constant $C > 0$, such that

$$\|f\| \leq C,$$

for all $f \in \mathcal{S}$.

Theorem 5. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. Suppose that there exists a sequence of points

$$\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n, \cdots\}$$

such that $\mathcal{S}$ is dense in $\mathcal{X}$. Then any bounded subset $\mathcal{T} \subset \mathcal{X}^*$ is weakly$*$ compact.

Proof. Let $\mathbf{x}_0 \in \mathcal{X}$. Without loss of generality, suppose that

$$\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}_0\| = 0.$$

Let

$$f_1, f_2, f_3, \cdots, f_n, \cdots$$

be any sequence in the bounded subset $\mathcal{T}$.

Consider the sequence

$$f_1(\mathbf{x}_1), f_2(\mathbf{x}_1), f_3(\mathbf{x}_1), \cdots, f_n(\mathbf{x}_1), \cdots$$

This is a bounded sequence in $\mathbb{R}$. There exists a convergent subsequence, say

$$f_{1n_1}(\mathbf{x}_1), f_{1n_2}(\mathbf{x}_1), f_{1n_3}(\mathbf{x}_1), \cdots, f_{1n_k}(\mathbf{x}_1), \cdots$$

Consider the sequence

$$f_{1n_1}(\mathbf{x}_2), f_{1n_2}(\mathbf{x}_2), f_{1n_3}(\mathbf{x}_2), \cdots, f_{1n_k}(\mathbf{x}_2), \cdots$$

This is a bounded sequence in $\mathbb{R}$. There exists a convergence subsequence, say

$$f_{2n_1}(\mathbf{x}_2), f_{2n_2}(\mathbf{x}_2), f_{2n_3}(\mathbf{x}_2), \cdots, f_{2n_k}(\mathbf{x}_2), \cdots$$

Consider the sequence

$$f_{2n_1}(\mathbf{x}_3), f_{2n_2}(\mathbf{x}_3), f_{2n_3}(\mathbf{x}_3), \cdots, f_{2n_k}(\mathbf{x}_3), \cdots$$

This is a bounded sequence in $\mathbb{R}$. There exists a convergence subsequence, say

$$f_{3n_1}(\mathbf{x}_3), f_{3n_2}(\mathbf{x}_3), f_{3n_3}(\mathbf{x}_3), \cdots, f_{3n_k}(\mathbf{x}_3), \cdots$$

Repeat this procedure for infinitely many times, we obtain convergent subsequences

$$f_{mn_1}(\mathbf{x}_m), f_{mn_2}(\mathbf{x}_m), f_{mn_3}(\mathbf{x}_m), \cdots, f_{mn_k}(\mathbf{x}_m), \cdots,$$

where $m = 1, 2, 3, \cdots\cdots$.

Let us consider the subsequences

$$f_{1n_1}(\mathbf{x}), f_{2n_2}(\mathbf{x}), f_{3n_3}(\mathbf{x}), \cdots, f_{mn_m}(\mathbf{x}), \cdots$$

and

$$f_{1n_1}(\mathbf{x}_1), f_{2n_2}(\mathbf{x}_2), f_{3n_3}(\mathbf{x}_3), \cdots, f_{mn_m}(\mathbf{x}_m), \cdots$$

Define f_0 by

$$f_0(\mathbf{x}) = \lim_{m\to\infty} f_{mn_m}(\mathbf{x}).$$

Then

$$f_0 \in \mathcal{X}^*.$$

Definition 9. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces. Let $\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y})$. If there exists an operator $\mathcal{A}^* : \mathcal{Y}^* \to \mathcal{X}^*$, such that

$$(\mathcal{A}^* g)(\mathbf{x}) = g(\mathcal{A}\mathbf{x}),$$

for all $\mathbf{x} \in \mathcal{X}$ and for all $g \in \mathcal{Y}^*$, then we call $\mathcal{A}^*$ the adjoint operator of $\mathcal{A}$.

Theorem 6. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces. For any bounded linear operator

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),$$

there exists a unique bounded linear operator

$$\mathcal{A}^* \in \mathbb{B}(\mathcal{Y}^* \to \mathcal{X}^*),$$

such that

$$(\mathcal{A}^* g)(\mathbf{x}) = g(\mathcal{A}\mathbf{x}),$$

for all $\mathbf{x} \in \mathcal{X}$ and all $g \in \mathcal{Y}^*$.

The mapping $\Phi : \mathcal{A} \to \mathcal{A}^*$

$$\Phi : \mathbb{B}(\mathcal{X} \to \mathcal{Y}) \to \mathbb{B}(\mathcal{Y}^* \to \mathcal{X}^*),$$

is a linear, norm-preserving operator. That is

$$\begin{aligned} (1) \qquad & (\alpha_1\mathcal{A}_1 + \alpha_2\mathcal{A}_2 + \alpha_3\mathcal{A}_3 + \cdots + \alpha_n\mathcal{A}_n)^* \\ = \; & \alpha_1\mathcal{A}_1^* + \alpha_2\mathcal{A}_2^* + \alpha_3\mathcal{A}_3^* + \cdots + \alpha_n\mathcal{A}_n^*, \\ (2) \qquad & \|\mathcal{A}^*\| = \|\mathcal{A}\|, \end{aligned}$$

for all bounded linear operators $\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\cdots$, $\mathcal{A}_n$, for all constants α_1, α_2, α_3, $\cdots$, $\alpha_n \in \mathbb{R}$, where $n \geq 1$ is any positive integer.

Note that

$$\begin{aligned} & \mathcal{A}^{**}:\mathcal{X}^{**}\to\mathcal{Y}^{**}, \\ & (\mathcal{A}^{**}\mathbf{x}^{**})(f) = \mathbf{x}^{**}(\mathcal{A}^{*}f) = (\mathcal{A}^{*}f)(\mathbf{x}) \\ = \; & f(\mathcal{A}\mathbf{x}) = (\mathcal{A}\mathbf{x})^{**}(f), \qquad f\in\mathbf{Y}^{*}, \\ & (\mathcal{A}\mathbf{x})^{**} = \mathcal{A}^{**}\mathbf{x}^{**}. \end{aligned}$$

Theorem 7. Let $(\mathcal{X},\|\cdot\|)$ and $(\mathcal{Y},\|\cdot\|)$ be normed linear spaces. Let $\mathcal{A}\in\mathcal{B}(\mathcal{X}\to\mathcal{Y})$. Then the bounded linear operator

$$\mathcal{A}^{**}:\mathcal{X}^{**}\to\mathcal{Y}^{**}$$

is an extension of the bounded linear operator

$$\mathcal{A}:\mathcal{X}\to\mathcal{Y}.$$

Moreover

$$\|\mathcal{A}\| = \|\mathcal{A}^{*}\| = \|\mathcal{A}^{**}\|.$$

Note that

$$\begin{aligned} \mathcal{X} &\subset \mathcal{X}^{**} \\ \mathcal{Y} &\subset \mathcal{Y}^{**}. \end{aligned}$$

Section 4: The Inverse Operator Theorem and the Uniform Boundedness Theorem

Definition 1. Let $(\mathcal{X},\|\cdot\|)$ and $(\mathcal{Y},\|\cdot\|)$ be normed linear spaces. Let

$$\mathcal{A}\in\mathbb{B}(\mathcal{X}\to\mathcal{Y}),$$

be a bounded linear operator. The operator $\mathcal{A}$ is called *regular*, if it is one-to-one and onto, and the inverse operator $\mathcal{A}^{-1}$ is a bounded linear operator. Namely

$$\mathcal{A}^{-1} \in \mathbb{B}(\mathcal{Y} \to \mathcal{X}).$$

Theorem 1. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces, let the bounded linear operator

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}).$$

Then $\mathcal{A}$ is regular, if and only if there exists a bounded linear operator

$$\mathcal{B} \in \mathbb{B}(\mathcal{Y} \to \mathcal{X}),$$

such that

$$\mathcal{B}\mathcal{A} = \mathcal{I}_{\mathcal{X}}, \qquad \mathcal{A}\mathcal{B} = \mathcal{I}_{\mathcal{Y}}.$$

Theorem 2. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces, let the bounded linear operator

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),$$

be regular. Then

$$\mathcal{A}^* \in \mathbb{B}(\mathcal{Y}^* \to \mathcal{X}^*),$$

is regular as well, and

$$(\mathcal{A}^*)^{-1} = (\mathcal{A}^{-1})^* \in \mathbb{B}(\mathcal{X}^* \to \mathcal{Y}^*).$$

Theorem 3. Let $(\mathcal{X}, \|\cdot\|)$, $(\mathcal{Y}, \|\cdot\|)$ and $(\mathcal{Z}, \|\cdot\|)$ be normed linear spaces. If both

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),$$

and

$$\mathcal{B} \in \mathbb{B}(\mathcal{Y} \to \mathcal{Z}),$$

are regular bounded linear operators, then

$$\mathcal{B}\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Z}),$$

is also a regular bounded linear operator. Moreover

$$(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1} \in \mathbb{B}(\mathcal{Z} \to \mathcal{X}).$$

Question: Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces, and let the bounded linear operator

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),$$

be both one-to-one and onto. Then the inverse operator $\mathcal{A}^{-1} : \mathcal{Y} \to \mathcal{X}$ exists. *Question*: Is $\mathcal{A}^{-1}$ a bounded linear operator? Is

$$\mathcal{A}^{-1} \in \mathbb{B}(\mathcal{Y} \to \mathcal{X}),$$

true?

Theorem 4. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be Banach spaces, let the bounded linear operator

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),$$

such that $\mathcal{A}\mathcal{X} = \mathcal{Y}$. Then for any positive constant $R > 0$, there exists another positive constant $\delta > 0$, such that

$$\mathcal{A}S(\mathbf{0}, R) \text{ is dense in } B(\mathbf{0}, R\delta).$$

Proof. There exists a positive integer $N \geq 1$ and there exists an open ball $B(\mathbf{y}_0, R_0)$, with center $\mathbf{y}_0$ and radius $R_0 > 0$, such that

$$\mathcal{A}S(\mathbf{0}, N) \text{ is dense in } B(\mathbf{y}_0, R_0).$$

Let $\delta = \frac{R_0}{N}$. Then

$$\mathcal{A}S(\mathbf{0}, R) \text{ is dense in } B(\mathbf{0}, R\delta).$$

Theorem 5. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be Banach spaces, let bounded linear operator

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),$$

satisfy $\mathcal{A}\mathcal{X} = \mathcal{Y}$. Then there exists a positive constant $\varepsilon = \frac{1}{2}\delta > 0$, such that

$$B(\mathbf{0}, \varepsilon) \subset \mathcal{A}S(\mathbf{0}, 1).$$

Proof. We have proved that there exists a positive constant $\delta > 0$, such that the image of the closed ball $S(\mathbf{0}, R)$:

$$\mathcal{A}S(\mathbf{0}, R),$$

is dense in the open ball $B(\mathbf{0}, R\delta)$, for any positive constant $R > 0$.

Now let $\varepsilon = \frac{1}{2}\delta$ and let $\mathbf{y}_0 \in B(\mathbf{0}, \varepsilon)$ be any point. By using the above result again and again, we find a sequence $\{\mathbf{x}_n\}$, such that

$$\mathbf{x}_n \in S\left(\mathbf{0}, \frac{1}{2^n}\right),$$
$$\|\mathbf{y}_0 - \mathcal{A}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \cdots + \mathbf{x}_n)\| < \frac{1}{2^n}\varepsilon,$$

for all positive integers $n \geq 1$. It is easy to see that both

$$\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \cdots + \mathbf{x}_n,$$
$$\mathcal{A}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \cdots + \mathbf{x}_n),$$

are convergent. Moreover

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \mathbf{x}_n = \mathbf{x}_0, \\
&\|\mathbf{x}_0\| \le \sum_{n=1}^{\infty} \frac{1}{2^n} \le 1, \\
&\mathbf{y}_0 = \mathcal{A}\mathbf{x}_0.
\end{aligned}
$$

The proof of the theroem is finished now. $\square$

Theorem 6. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be Banach spaces, let the bounded linear operator

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),$$

be one-to-one and onto. Then the inverse operator

$$\mathcal{A}^{-1} \in \mathbb{B}(\mathcal{Y} \to \mathcal{X}).$$

Proof. The proof follows from Theorem 5. $\square$

Theorem 7. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be Banach spaces, let the bounded linear operator

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),$$

satisfy $\mathcal{A}\mathcal{X} = \mathcal{Y}$. Then $\mathcal{A}$ is an open mapping.

Proof. Let $\mathcal{S}$ be any open set in $\mathcal{X}$. For any point $\mathbf{x}_0 \in \mathcal{S}$, there exists a positive constant $\delta > 0$, such that

$$\mathbf{x}_0 + B(\mathbf{0}, \delta) = B(\mathbf{x}_0, \delta) \subset \mathcal{S}.$$

It is easy to find that

$$B(\mathbf{0}, \frac{1}{2}\delta\varepsilon) \subset \mathcal{A}S(\mathbf{0}, \frac{1}{2}\delta).$$

Therefore

$$\mathcal{A}\mathbf{x}_0 + B(\mathbf{0}, \frac{1}{2}\delta\varepsilon) \subset \mathcal{A}\mathbf{x}_0 + \mathcal{A}S(\mathbf{0}, \frac{1}{2}\delta\varepsilon) = \mathcal{A}S(\mathbf{x}_0, \frac{1}{2}\delta\varepsilon).$$

The proof of the theorem is finished. □

Definition 2. Let $(\mathcal{X}, \|\cdot\|_1)$ and $(\mathcal{X}, \|\cdot\|_2)$ be normed linear spaces. If there exists a positive constant $C > 0$, such that

$$\|\mathbf{x}\|_2 \leq C\|\mathbf{x}\|_1,$$

for all $\mathbf{x} \in \mathcal{X}$, then we say $\|\cdot\|_1$ is stronger than $\|\cdot\|_1$, we also say $\|\cdot\|_2$ is weaker than $\|\cdot\|_1$, we say $\|\cdot\|_2$ is continuous with respect to $\|\cdot\|_1$.

Theorem 8. Let $(\mathcal{X}, \|\cdot\|_1)$ and $(\mathcal{X}, \|\cdot\|_2)$ be normed linear spaces. The following statements are true.

(1) The dual spaces

$$(\mathcal{X}, \|\cdot\|_1)^* = (\mathcal{X}, \|\cdot\|_2)^*,$$

and there are two positive constants $C_2 > C_1 > 0$, such that

$$C_1\|f\|_2 \leq \|f\|_1 \leq C_2\|f\|_2,$$

for all $f \in (\mathcal{X}, \|\cdot\|_1)^*$, if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

(2) The dual spaces

$$(\mathcal{X}, \|\cdot\|_2)^* \subset (\mathcal{X}, \|\cdot\|_1)^*,$$

and there exists a positive constant $C > 0$, such that

$$\|f\|_1 \leq C\|f\|_2,$$

for all $f \in (\mathcal{X}, \|\cdot\|_2)^*$, if $\|\cdot\|_1$ is strongewr than $\|\cdot\|_2$.

(3) $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$, if and only if $\|\cdot\|_2$ is a continuous function on the space $(\mathcal{X}, \|\cdot\|_1)$.

Theorem 9. Let $(\mathcal{X}, \|\cdot\|_1)$ and $(\mathcal{X}, \|\cdot\|_2)$ be Banach spaces. Suppose that there exists a positive constant $C > 0$, such that

$$\|\mathbf{x}\|_2 \leq C\|\mathbf{x}\|_1,$$

for all $\mathbf{x} \in \mathcal{X}$. Then there are two positive constants $C_2 > C_1 > 0$, such that

$$C_1\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq C_2\|\mathbf{x}\|_2,$$

for all $\mathbf{x} \in \mathcal{X}$.

Theorem 10. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be Banach spaces. Let

$$\mathcal{A} : \mathcal{X} \to \mathcal{Y},$$

be a linear closed operator. Then

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}).$$

Proof. The main idea is to apply the inverse operator theorem. Let us define another space $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ and define the norm

$$\|(\mathbf{x}, \mathbf{y})\|_0 = (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^{1/2},$$

for all $(\mathbf{x}, \mathbf{y}) \in \mathcal{X} \times \mathcal{Y}$.

Now it is not difficult to show that

$$(\mathcal{X} \times \mathcal{Y}, \|\cdot\|_0)$$

is a Banach space.

Consider

$$\mathcal{G} \overset{\text{def}}{=} \{(\mathbf{x}, \mathcal{A}\mathbf{x}) : \mathbf{x} \in \mathcal{X}\}.$$

This is a closed subspace of $\mathcal{X} \times \mathcal{Y}$. It is also a Banach space.

Consider the mapping

$$\mathcal{T} : (\mathcal{G}, \|\cdot\|_0) \to (\mathcal{X}, \|\cdot\|),$$

defined by

$$\mathcal{T}(\mathbf{x}, \mathcal{A}\mathbf{x}) = \mathbf{x},$$

for all $\mathbf{x} \in \mathcal{X}$. Clearly, $\mathcal{T}$ is one-to-one and onto. $\mathcal{T}$ is linear and bounded. Based on the inverse operator theorem, $\mathcal{T}^{-1}$ is a bounded operator. Therefore, there exists a positive constant $C > 0$, such that

$$\|\mathcal{A}\mathbf{x}\| \leq C\|\mathbf{x}\|,$$

for all $\mathbf{x} \in \mathcal{X}$.

Theorem 11. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and let $(\mathcal{Y}, \|\cdot\|)$ be a normed linear space. Let

$$\left\{\mathcal{A}_\lambda \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}) : \lambda \in \Lambda\right\},$$

be a family of bounded linear operators. If

$$\sup_{\lambda\Lambda} \|\mathcal{A}_\lambda \mathbf{x}\| < \infty,$$

for every point $\mathbf{x} \in \mathcal{X}$, then

$$\sup_{\lambda\in\Lambda} \|\mathcal{A}_\lambda\| < \infty.$$

Proof. The main idea of the proof is to apply the inverse operator theorem. Let us define another norm, $\|\cdot\|_1$, on $\mathcal{X}$:

$$\|\mathbf{x}\|_1 = \|\mathbf{x}\| + \sup_{\lambda\in\Lambda} \|\mathcal{A}_\lambda \mathbf{x}\|,$$

for all $\mathbf{x} \in \mathcal{X}$.

Claim 1: The space

$$(\mathcal{X}, \|\cdot\|_1)$$

is a normed linear space.

Claim 2:

$$(\mathcal{X}, \|\cdot\|_1)$$

is a Banach space.

Claim 3: The identity operator

$$\mathcal{A} = \mathcal{I} : (\mathcal{X}, \|\cdot\|_1) \to (\mathcal{X}, \|\cdot\|),$$

is one-to-one, onto, linear and bounded, because $\|\mathbf{x}\| \leq \|\mathbf{x}\|_1$, for all $\mathbf{x} \in \mathcal{X}$. Based on the inverse operator theorem, the inverse operator

$$\mathcal{A}^{-1} = \mathcal{I} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

There exists a positive constant $C > 0$, such that

$$\|\mathbf{x}\|_1 \leq C\|\mathbf{x}\|,$$

for all $\mathbf{x} \in \mathcal{X}$.

Theorem 12. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and let $(\mathcal{Y}, \|\cdot\|)$ be a normed linear space. Let

$$\mathcal{A}_n \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}), \qquad n \geq 1.$$

Suppose that there exists the following limit

$$\lim_{n\to\infty} \mathcal{A}_n\mathbf{x},$$

for every vector $\mathbf{x} \in \mathcal{X}$. Then there exists a bounded linear operator

$$\mathcal{A}_0 \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),$$

such that

$$\lim_{n\to\infty} \|\mathcal{A}_n\mathbf{x} - \mathcal{A}_0\mathbf{x}\| = 0,$$

for all $\mathbf{x} \in \mathcal{X}$, and

$$\|\mathcal{A}_0\| \leq \liminf_{n\to\infty} \|\mathcal{A}_n\|.$$

Definition 3. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let

$$\mathcal{S} \subset \mathcal{X}, \qquad \mathcal{T} \subset \mathcal{Y}^*.$$

The set $\mathcal{S}$ is called strongly bounded, if there exists a positive constant $C > 0$, such that

$$\|\mathbf{x}\| \leq C,$$

for all $\mathbf{x} \in \mathcal{S}$.

The set is called weakly bounded, if there exists a positive constant $C > 0$, such that

$$|f(\mathbf{x})| \leq C,$$

for all $f \in \mathcal{X}^*$.

The set $\mathcal{T}$ is called strongly bounded, if there exists a positive constant $C > 0$, such that

$$\|f\| \leq C,$$

for all $f \in \mathcal{T}$.

The set $\mathcal{T}$ is called weakly$*$ bounded, if there exists a positive constant $C = C(\mathbf{x}) > 0$, such that

$$|f(\mathbf{x})| \leq C(\mathbf{x}),$$

for all $f \in \mathcal{T}$.

Theorem 13. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space. Then

Any weakly$*$ bounded set in $\mathcal{X}^*$ is strongly bounded.

Any weakly bounded set in $\mathcal{X}^*$ is strongly bounded.

Any weakly$*$ compact set in $\mathcal{X}^*$ is strongly bounded.

Any weakly bounded set in $\mathcal{X}$ is strongly bounded.

Section 5: The Spectrum of Bounded Linear Operators and Invariant Subspaces of Linear Operators

Definition 1. Let $\mathcal{X}$ be a vector space and let $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ be a linear operator. If there exists a constant $\lambda \in \mathbb{C}$ and there exists a nonzero vector $\mathbf{x}$, such that

$$\mathcal{A}\mathbf{x} = \lambda\mathbf{x},$$

then we call λ an eigenvalue of the operator $\mathcal{A}$ and we call $\mathbf{x}$ an eigenvector of $\mathcal{A}$ associated with the eigenvalue λ. We define the eigenspace of $\mathcal{A}$ by

$$\mathcal{E}_\lambda = \{\mathbf{x} \in \mathcal{X} : \mathcal{A}\mathbf{x} = \lambda\mathbf{x}\}.$$

We call the dimension, $\dim(\mathcal{E}_\lambda)$, the geometric multiplicity of the eigenvalue λ.

Example 1. Let $\mathcal{X} = L^2(\mathbb{R}^n)$, let the linear operator

$$\mathcal{A} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

be defined by

$$(\mathcal{A}\phi)(\xi) = \int_{\mathbb{R}^n} \exp(-\mathrm{i}\mathbf{x} \cdot \xi)\phi(\mathbf{x})\mathrm{d}\mathbf{x}.$$

This is the Fourier transformation operator. Clearly, $\lambda = (2\pi)^{n/2}$ is an eigenvalue and $\phi(\mathbf{x}) = \exp(-\frac{1}{2}|\mathbf{x}|^2)$ is an eigenfunction of $\mathcal{A}$.

Example 2. Let $\mathcal{X} = C(0, \infty)$, the collection of all continuous functions. The Laplace transformation is a linear operator

$$[\mathcal{L}f](\lambda) = \int_0^\infty \exp(-\lambda t)\phi(t)\mathrm{d}t,$$

There exists an eigenvalue $\lambda_0 = \sqrt{\pi}$. The eigenfunction is

$$\phi(t) = \frac{1}{t^{1/2}}, \text{ for all } t > 0.$$

Definition 2. Let $(\mathcal{X}, \|\cdot\|)$ be a complex normed linear space, let

$$\mathcal{A} : \mathcal{X} \to \mathcal{X},$$

be a linear operator. Let $\lambda \in \mathbb{C}$ be a complex constant. If the operator $\lambda\mathcal{I} - \mathcal{A}$ is one-to-one, onto, and the inverse operator

$$(\lambda\mathcal{I} - \mathcal{A})^{-1}$$

is bounded, that is, $\lambda\mathcal{I} - \mathcal{A}$ is a regular operator, then we call λ a regular point of the linear operator $\mathcal{A}$. We define the resolvent set

$$\begin{aligned}\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} :\ & \text{the operator } \lambda\mathcal{I} - \mathcal{A} \text{ is one-to-one and onto,} \\ & \text{and the inverse operator } (\lambda\mathcal{I} - \mathcal{A})^{-1} \text{ is bounded}\}.\end{aligned}$$

We define the spectrum of the operator $\mathcal{A}$ by

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} : \lambda \notin \rho(\mathcal{A})\}.$$

We define the point spectrum (eigenvalues), $\sigma_p(\mathcal{A})$, and the continuous spectrum, $\sigma_c(\mathcal{A})$, by

$$\begin{aligned}\sigma_p(\mathcal{A}) &= \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } \mathcal{A}\}, \\ \sigma_c(\mathcal{A}) &= \{\lambda \in \mathbb{C} : \lambda \text{ is not an eigenvalue of } \mathcal{A}, \\ &\quad \text{either } \lambda\mathcal{I} - \mathcal{A} \text{ is not onto, that is, } (\lambda\mathcal{I} - \mathcal{A})\mathcal{X} \neq \mathcal{X}, \\ &\quad \text{or } (\lambda\mathcal{I} - \mathcal{A})^{-1} \text{ is onto, but it is unbounded}\}.\end{aligned}$$

Theorem 1. Let $(\mathcal{X}, \|\cdot\|)$ be a complex normed linear space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

(1) The complex constant $\lambda \in \rho(\mathcal{A})$, if and only if there exists a bounded linear operator

$$\mathcal{B} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}),$$

such that

$$(\lambda\mathcal{I} - \mathcal{A})\mathcal{B} = \mathcal{B}(\lambda\mathcal{I} - \mathcal{A}) = \mathcal{I},$$

if and only if there exists a unique solution to the equation

$$(\lambda\mathcal{I} - \mathcal{A})\mathbf{x} = \mathbf{y},$$

for any $\mathbf{y} \in \mathcal{X}$. Moreover, there exists a positive constant $C > 0$, such that

$$\|\mathbf{x}\| \leq C\|\mathbf{y}\|.$$

(2) The complex constant $\lambda \in \mathbb{C}$ is not an eigenvalue of the linear operator $\mathcal{A}$, if and only if the operator

$$\lambda\mathcal{I} - \mathcal{A} : \mathcal{X} \to (\lambda\mathcal{I} - \mathcal{A})\mathcal{X},$$

is one-to-one and onto. In this case, $\lambda \in \rho(\mathcal{A})$, if the dimension $\dim(\mathcal{X}) < \infty$.

Theorem 2. Let $(\mathcal{X}, \|\cdot\|)$ be a complex normed linear space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

be a polynomial of degree n, $a_n \neq 0$, and let

$$f(\mathcal{A}) = a_0\mathcal{I} + a_1\mathcal{A} + a_2\mathcal{A}^2 + \cdots + a_n\mathcal{A}^n.$$

Then

$$\sigma(f(\mathcal{A})) = f(\sigma(\mathcal{A})),$$

where

$$f(\sigma(\mathcal{A})) = \{f(\lambda) : \lambda \in \sigma(\mathcal{A})\}.$$

Proof. There are two steps in the proof.

$$(1) \quad f(\sigma(\mathcal{A})) \subset \sigma(f(\mathcal{A})).$$

$$(2) \quad \sigma(f(\mathcal{A})) \subset f(\sigma(\mathcal{A})).$$

First of all, let $\lambda \in \sigma(\mathcal{A})$. Then $f(\lambda) \in f(\sigma(\mathcal{A}))$. If $f(\lambda) \in \rho(f(\mathcal{A}))$, then there exists a bounded linear operator $\mathcal{B}$, such that

$$[f(\lambda)\mathcal{I} - f(\mathcal{A})]\mathcal{B} = \mathcal{B}[f(\lambda)\mathcal{I} - f(\mathcal{A})] = \mathcal{I}.$$

It is very easy to rewrite

$$f(\lambda)\mathcal{I} - f(\mathcal{A}) = (\lambda\mathcal{I} - \mathcal{A})g(\lambda\mathcal{I}, \mathcal{A}) = g(\lambda\mathcal{I}, \mathcal{A})(\lambda\mathcal{I} - \mathcal{A}).$$

Now we have

$$(\lambda\mathcal{I} - \mathcal{A})g(\lambda\mathcal{I}, \mathcal{A})\mathcal{B} = \mathcal{B}g(\lambda\mathcal{I}, \mathcal{A})(\lambda\mathcal{I} - \mathcal{A}) = \mathcal{I}.$$

This means that

$$\lambda \in \rho(\mathcal{A}).$$

This is a contradiction to the fact that

$$\lambda \in \sigma(\mathcal{A}).$$

Therefore, we find

$$f(\lambda) \in \sigma(f(\mathcal{A})).$$

Overall, we have

$$f(\sigma(\mathcal{A})) \subset \sigma(f(\mathcal{A})).$$

Let

$$\nu \notin f(\sigma(\mathcal{A})).$$

Then

$$\nu \neq f(\lambda), \qquad \nu - f(\lambda) \neq 0,$$

for all $\lambda \in \sigma(\mathcal{A})$.

We may write

$$\nu - f(\lambda) = a_n(\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)\cdots(\lambda_n - \lambda) \neq 0,$$

for complex constants λ_k, where $k = 1, 2, 3, \cdots, n$. This means that $\lambda_k \in \rho(\mathcal{A})$, for all k. Hence the inverse operator

$$(\lambda_k \mathcal{I} - \mathcal{A})^{-1},$$

exists and is bounded. Therefore,

$$\begin{aligned} & [\nu\mathcal{I} - f(\mathcal{A})]^{-1} \\ = \ & [a_n(\lambda_1\mathcal{I} - \mathcal{A})(\lambda_2\mathcal{I} - \mathcal{A})(\lambda_3\mathcal{I} - \mathcal{A})\cdots(\lambda_n\mathcal{I} - \mathcal{A})]^{-1} \\ = \ & \frac{1}{a_n}(\lambda_n\mathcal{I} - \mathcal{A})^{-1}\cdots(\lambda_3\mathcal{I} - \mathcal{A})^{-1}(\lambda_2\mathcal{I} - \mathcal{A})^{-1}(\lambda_1\mathcal{I} - \mathcal{A})^{-1} \end{aligned}$$

exists and it is bounded. Thus,

$$\nu \in \rho(f(\mathcal{A})).$$

Overall,

$$\sigma(f(\mathcal{A})) \subset f(\sigma(\mathcal{A})).$$

The proof of the theorem is finished now. □

Theorem 3. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

Suppose that

$$r \overset{\text{def}}{=} \lim_{n\to\infty} \left\{ \|\mathcal{A}^n\|^{\frac{1}{n}} \right\} < 1.$$

Then

$$\begin{aligned} &(1) \quad \lambda = 1 \in \rho(\mathcal{A}). \\ &(2) \quad (\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \mathcal{A}^n. \\ &(3) \quad \|(\mathcal{I} - \mathcal{A})^{-1}\| \le \frac{1}{1 - \|\mathcal{A}\|}, \text{ if } \|\mathcal{A}\| < 1. \end{aligned}$$

Theorem 4. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

Suppose that

$$r \overset{\text{def}}{=} \lim_{n\to\infty} \left\{ \|\mathcal{A}^n\|^{\frac{1}{n}} \right\}.$$

Then

$$\begin{aligned} &(1) \quad \lambda \in \rho(\mathcal{A}), \text{ if } |\lambda| > r. \\ &(2) \quad (\lambda\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \mathcal{A}^n, \text{ if } |\lambda| > r. \\ &(3) \quad \|(\lambda\mathcal{I} - \mathcal{A})^{-1}\| \le \frac{1}{|\lambda| - \|\mathcal{A}\|}, \text{ if } |\lambda| > \|\mathcal{A}\|. \end{aligned}$$

Theorem 5. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

Then the set $\rho(\mathcal{A})$ is open. Let $\rho(\mathcal{A}) \neq \emptyset$. For any $\lambda_0 \in \rho(\mathcal{A})$, let

$$r_0 \overset{\text{def}}{=} \lim_{n\to\infty} \left\{ \|(\lambda_0\mathcal{I} - \mathcal{A})^{-n}\|^{\frac{1}{n}} \right\}.$$

Then

$$\lambda \in \rho(\mathcal{A}), (\lambda\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda_0\mathcal{I} - \mathcal{A})^{-(n+1)} (\lambda - \lambda_0)^n,$$

if

$$|\lambda - \lambda_0| < \frac{1}{r_0}.$$

Theorem 6. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

Then the spectrum $\sigma = \sigma(\mathcal{A})$ is a closed set. Moreover, there holds the following estimate

$$\sup_{\lambda\in\sigma(\mathcal{A})} |\lambda| \leq \lim_{n\to\infty} \left\{ \|\mathcal{A}^n\|^{\frac{1}{n}} \right\}.$$

Remark. The spectrum $\sigma(\mathcal{A})$ is a closed set, even if $\mathcal{A}$ is linear unbounded.

Definition 3. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

Define the spectrum radius $r(\mathcal{A})$ by

$$r(\mathcal{A}) \overset{\text{def}}{=} \max_{\lambda\in\sigma(\mathcal{A})} |\lambda|.$$

Theorem 7. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

(1) $\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X})$,
(2) $f \in \mathcal{X}^*$.

Then

$$f((\lambda\mathcal{I}-\mathcal{A})^{-1})$$

is an analytic function of λ in the set $\rho(\mathcal{A})$.
Proof. Let $\lambda_0 \in \rho(\mathcal{A})$. Then

$$f((\lambda\mathcal{I}-\mathcal{A})^{-1}) = \sum_{n=0}^{\infty}(-1)^n f((\lambda_0\mathcal{I}-\mathcal{A})^{-n-1})(\lambda-\lambda_0)^n,$$

for all $\lambda \in \mathbb{C}$, $|\lambda-\lambda_0| < \frac{1}{r_0}$, where

$$r_0 = \lim_{n\to\infty}\left\{\|(\lambda_0\mathcal{I}-\mathcal{A})^{-n}\|^{\frac{1}{n}}\right\}.$$

Theorem 8. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

Then

$$r(\mathcal{A}) = \max_{\lambda\in\sigma(\mathcal{A})}|\lambda| = \lim_{n\to\infty}\left\{\|\mathcal{A}^n\|^{\frac{1}{n}}\right\}.$$

Proof. For all $\lambda \in \mathbb{C}$ with $|\lambda| > \|\mathcal{A}\|$, there holds

$$(\lambda\mathcal{I}-\mathcal{A})^{-1} = \sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}\mathcal{A}^n.$$

Now for any continuous linear functional $f \in \mathcal{X}^*$, we have

$$f((\lambda\mathcal{I}-\mathcal{A})^{-1}) = \sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}f(\mathcal{A}^n),$$

is a complex analytic function of λ, $|\lambda| > \|\mathcal{A}\|$. Note that this is also true for all $|\lambda| > r(\mathcal{A})$. Now for any $\varepsilon > 0$, we have

$$\sum_{n=0}^{\infty}\frac{1}{[r(\mathcal{A})+\varepsilon]^{n+1}}|f(\mathcal{A}^n)| < \infty.$$

From this result, we claim that

$$\sup_{n\geq 1}\left|f\left(\frac{1}{[r(\mathcal{A})+\varepsilon]^n}\mathcal{A}^n\right)\right|<\infty.$$

This means that the operator sequence

$$\left\{\frac{1}{[r(\mathcal{A})+\varepsilon]^n}\mathcal{A}^n\right\}$$

is weakly bounded. By using the uniform boundedness theorem in Section 4, we know that the sequence of operators is strongly bounded. There exists a positive constant $C>0$, such that

$$\frac{1}{[r(\mathcal{A})+\varepsilon]^n}\|\mathcal{A}^n\|\leq C, \qquad n\geq 1.$$

That is

$$\|\mathcal{A}^n\|^{\frac{1}{n}}\leq C^{\frac{1}{n}}[r(\mathcal{A})+\varepsilon], \qquad n\geq 1.$$

Therefore, we obtain the estimate

$$\lim_{n\to\infty}\left\{\|\mathcal{A}^n\|^{\frac{1}{n}}\right\}\leq r(\mathcal{A})+\varepsilon.$$

Letting $\varepsilon\to 0$ yields the estimate

$$\lim_{n\to\infty}\left\{\|\mathcal{A}^n\|^{\frac{1}{n}}\right\}\leq r(\mathcal{A}).$$

In Theorem 6, we have proved the estimate

$$r(\mathcal{A})\leq\lim_{n\to\infty}\left\{\|\mathcal{A}^n\|^{\frac{1}{n}}\right\}.$$

The proof of Theorem 8 is finished. □

Theorem 9. Let $(\mathcal{X},\|\cdot\|)$ be a complex nontrivial Banach space, let

$$\mathcal{I}\in\mathbb{B}(\mathcal{X}\to\mathcal{X}),$$
$$\mathcal{A}\in\mathbb{B}(\mathcal{X}\to\mathcal{X}).$$

Then

$$\sigma(\mathcal{A}) \neq \emptyset.$$

Proof. Based on the extension theorem for functionals, there exists a continuous linear functional

$$f \in [\mathbb{B}(\mathcal{X} \to \mathcal{X})]^*,$$

such that $\|f\| = 1$ and $f(\mathcal{I}) = \|\mathcal{I}\| > 0$.

For any $\lambda_0 \in \rho(\mathcal{A})$, there exists the constant $r_0 \geq 0$:

$$r_0 = \lim_{n\to\infty} \left\{ \|(\lambda_0\mathcal{I} - \mathcal{A})^{-n}\|^{\frac{1}{n}} \right\},$$

such that if $|\lambda - \lambda_0| < \frac{1}{r_0}$, then

$$\begin{aligned} (\lambda\mathcal{I} - \mathcal{A})^{-1} &= \sum_{n=0}^{\infty} (-1)^n (\lambda_0\mathcal{I} - \mathcal{A})^{-n-1} (\lambda - \lambda_0)^n, \\ f((\lambda\mathcal{I} - \mathcal{A})^{-1}) &= \sum_{n=0}^{\infty} (-1)^n f((\lambda_0\mathcal{I} - \mathcal{A})^{-n-1}) (\lambda - \lambda_0)^n. \end{aligned}$$

Suppose that

$$\sigma(\mathcal{A}) = \emptyset.$$

Hence

$$f((\lambda\mathcal{I} - \mathcal{A})^{-1})$$

is a complex analytic function of λ in the whole complex plane $\mathbb{C}$.

On the other hand, if $|\lambda| > \|\mathcal{A}\|$, then

$$\begin{aligned} (\lambda\mathcal{I} - \mathcal{A})^{-1} &= \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \mathcal{A}^n, \\ f((\lambda\mathcal{I} - \mathcal{A})^{-1}) &= \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} f(\mathcal{A}^n). \end{aligned}$$

Moreover, if $|\lambda| > 1 + \|\mathcal{A}\|$, then

$$|f((\lambda\mathcal{I} - \mathcal{A})^{-1})| \le \sum_{n=0}^{\infty} \frac{1}{|\lambda|^{n+1}} |f(\mathcal{A}^n)| \le \frac{\|f\|}{|\lambda| - \|\mathcal{A}\|}.$$

This means that $f((\lambda\mathcal{I} - \mathcal{A})^{-1})$ is bounded. By Liouville's theorem, $f((\lambda\mathcal{I} - \mathcal{A})^{-1})$ must be a constant. However, the coefficient of $\frac{1}{\lambda}$ is $f(\mathcal{I}) \neq 0$. This is a contradiction. Therefore, $\sigma(\mathcal{A}) \neq \emptyset$.

Definition 4. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

If

$$\lim_{n\to\infty} \left\{ \|\mathcal{A}^n\|^{\frac{1}{n}} \right\} = 0,$$

then we call the operator $\mathcal{A}$ general nilpotent operator. In this case

$$\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} : \lambda \neq 0\}, \qquad \sigma(\mathcal{A}) = \{0\}.$$

Moreover

$$(\lambda\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \mathcal{A}^n,$$

for all $\lambda \neq 0$.

Example 3. Consider the bounded linear operator

$$\mathcal{A} : C[a,b] \to C[a,b],$$
$$(\mathcal{A}\phi)(x) = \int_a^x \phi(t)\mathrm{d}t.$$

There holds the following estimates

$$\|\mathcal{A}^n\phi\| \le \frac{1}{n!}(b-a)^n \|\phi\|,$$

for all $\phi \in C[a,b]$.

Clearly, $\mathcal{A}$ is a general nilpotent operator. Obviously,

$$\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} : \lambda \neq 0\}, \qquad \sigma(\mathcal{A}) = \{0\}.$$

Moreover, the only spectrum

$$\lambda = 0$$

is not an eigenvalue of the linear operator $\mathcal{A}$.

Definition 5. Let $(\mathcal{X}, \|\cdot\|)$ be a complex normed linear space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

Let $\lambda \in \mathbb{C}$ be a constant. If there exists a sequence of unit vectors $\{\mathbf{x}_n\} \subset \mathcal{X}$, $\|\mathbf{x}_n\| = 1$, $n = 1, 2, 3, \cdots$, such that

$$\lim_{n\to\infty} \|(\lambda\mathcal{I} - \mathcal{A})\mathbf{x}_n\| = 0,$$

then we call λ an approximation spectrum point. Define

$$\begin{aligned}\sigma_a(\mathcal{A}) \;=\; \{\lambda \in \mathbb{C} :\; & \text{there exists a sequence of unit vectors} \\ & \{\mathbf{x}_n\} \subset \mathcal{X}, \|\mathbf{x}_n\| = 1, \text{ such that} \\ & \lim_{n\to\infty} \|(\lambda\mathcal{I} - \mathcal{A})\mathbf{x}_n\| = 0\}.\end{aligned}$$

Define the residual spectrum $\sigma_r(\mathcal{A})$ by

$$\sigma_r(\mathcal{A}) = \sigma(\mathcal{A}) - \sigma_a(\mathcal{A}).$$

Remark. Let

$$\lambda \in \sigma_a(\mathcal{A}).$$

Then, either

$$\lambda \in \sigma_p(\mathcal{A}),$$

or

$$\lambda \in \sigma_c(\mathcal{A}).$$

In the second case, the inverse operator

$$(\lambda\mathcal{I} - \mathcal{A})^{-1}$$

exists, but $(\lambda\mathcal{I} - \mathcal{A})^{-1}$ is unbounded.

Theorem 10. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

The following statements are true.

$$\begin{array}{ll}
(1) & \sigma_p(\mathcal{A}) \subset \sigma_a(\mathcal{A}), \qquad \sigma_r(\mathcal{A}) \subset \sigma_c(\mathcal{A}), \\
(2) & \sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \cup \sigma_c(\mathcal{A}), \qquad \sigma_p(\mathcal{A}) \cap \sigma_c(\mathcal{A}) = \emptyset, \\
 & \sigma(\mathcal{A}) = \sigma_a(\mathcal{A}) \cup \sigma_r(\mathcal{A}), \qquad \sigma_a(\mathcal{A}) \cap \sigma_r(\mathcal{A}) = \emptyset \\
(3) & \sigma_r(\mathcal{A}) \text{ is an open set} \\
(4) & \text{The boundary } \partial\sigma(\mathcal{A}) \subset \sigma_a(\mathcal{A}), \qquad \partial\sigma(\mathcal{A}) \neq \emptyset. \\
(5) & \sigma_a(\mathcal{A}) \text{ is a closed set and } \sigma_a(\mathcal{A}) \neq \emptyset.
\end{array}$$

In particular, if

$$\sigma_a(\mathcal{A}) = \sigma_p(\mathcal{A}),$$

then

$$\sigma_c(\mathcal{A}) = \sigma_r(\mathcal{A}).$$

Proof. Obviously, $\sigma_p(\mathcal{A}) \subset \sigma_a(\mathcal{A})$.

Let $\lambda \in \mathbb{C}$. If $\lambda \notin \sigma_c(\mathcal{A})$, then $\lambda\sigma_p(\mathcal{A})$. Clearly $\lambda \in \sigma_a(\mathcal{A}I)$, hence $\lambda \notin \sigma_r(\mathcal{A})$. In another word, if $\lambda \in \sigma_r(\mathcal{A})$, then $\lambda \in \sigma_c(\mathcal{A})$. For any $\lambda_0 \in \sigma_r(\mathcal{A})$, we know that $\lambda_0 \notin \sigma_a(\mathcal{A})$, hence there exists a positive constant $C_0 = C_0(\lambda_0) > 0$, such that

$$\|(\lambda_0\mathcal{I} - \mathcal{A})\mathbf{x}\| \geq C_0\|\mathbf{x}\|,$$

for all $\mathbf{x} \in \mathcal{X}$. Now let us show that if $\lambda_1 \in \mathbb{C}$ and

$$|\lambda_1 - \lambda_0| < \frac{1}{2} C_0,$$

then

$$\lambda_1 \in \sigma_r(\mathcal{A}).$$

First of all, there hold the following estimates

$$\begin{aligned} & \|(\lambda_1 \mathcal{I} - \mathcal{A})\mathbf{x}\| \\ \geq \ & \|(\lambda_0 \mathcal{I} - \mathcal{A})\mathbf{x}\| - |\lambda_1 - \lambda_0| \|\mathbf{x}\| \\ \geq \ & \frac{1}{2} C_0 \|\mathbf{x}\|, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{X}$. This implies that $\lambda \notin \sigma_a(\mathcal{A})$.

Secondly, if $\lambda_1 \in \rho(\mathcal{A})$, then

$$\|(\lambda_1 \mathcal{I} - \mathcal{A})^{-1}\| \leq \frac{2}{C_0}.$$

Now we have

$$\begin{aligned} r_1 &= \lim_{n \to \infty} \left\{ \|(\lambda_1 \mathcal{I} - \mathcal{A})^{-n}\|^{\frac{1}{n}} \right\} \\ &= \inf_{n \geq 1} \left\{ \|(\lambda_1 \mathcal{I} - \mathcal{A})^{-n}\|^{\frac{1}{n}} \right\} \\ &\leq \|(\lambda_1 \mathcal{I} - \mathcal{A})^{-1}\| \leq \frac{2}{C_0}. \end{aligned}$$

However, based on Theorem 5, since

$$|\lambda_0 - \lambda_1| < \frac{1}{r_1},$$

then $\lambda_0 \in \rho(\mathcal{A})$. But $\lambda_0 \in \sigma_r(\mathcal{A})$. Therefore, we have $\lambda_1 \in \sigma_r(\mathcal{A})$.

Obviously, we have the boundary

$$\partial \sigma(\mathcal{A}) \subset \sigma(\mathcal{A}).$$

Let $\lambda \in \mathbb{C}$. If $\lambda \in \partial\sigma(\mathcal{A})$, then

$$\lambda \notin \sigma_r(\mathcal{A}).$$

Since $\sigma(\mathcal{A}) \neq \emptyset$, we know that

$$\partial\sigma(\mathcal{A}) \neq \emptyset, \qquad \sigma_a(\mathcal{A}) \neq \emptyset.$$

Remark. If there is only one point in the spectrum, say $\lambda = 0$, then $\sigma(\mathcal{A}) = \{0\}$. Certainly $\lambda = 0 \in \sigma_a(\mathcal{A})$ and $\sigma_r(\mathcal{A}) = \emptyset$. However, it is possible that $\lambda = 0 \in \sigma_p(\mathcal{A})$ or $\lambda = 0 \in \sigma_c(\mathcal{A})$.

Definition 6. Let $\mathcal{X}$ be a linear space, let $\mathcal{E}$ be a subspace of $\mathcal{X}$, let

$$\mathcal{A} : \mathcal{X} \to \mathcal{X}$$

be a linear operator. If

$$\mathcal{A}\mathcal{E} \subset \mathcal{E},$$

then we call $\mathcal{E}$ an invariant subspace of $\mathcal{A}$.

Section 6: The Spectrum of Compact Operators in Complex Banach Spaces

Definition 1. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}).$$

The operator $\mathcal{A}$ is called compact, if it maps any bounded set to a compact set.

Define

$$\mathbb{C}(\mathcal{X} \to \mathcal{Y}) = \{\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}) : \mathcal{A} \text{ is a compact operator}\}.$$

Theorem 1. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces. Then $\mathbb{C}(\mathcal{X} \to \mathcal{Y})$ is a subspace of $\mathbb{B}(\mathcal{X} \to \mathcal{Y})$. Moreover, $\mathbb{C}(\mathcal{X} \to \mathcal{Y})$ is a closed subspace of $\mathbb{B}(\mathcal{X} \to \mathcal{Y})$, if $(\mathcal{X}, \|\cdot\|)$ is a Banach space.

Proof. Let $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$ be constants. Let

$$\begin{gathered}\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{Y}),\\ \mathcal{B} \in \mathbb{C}(\mathcal{X} \to \mathcal{Y}).\end{gathered}$$

Then

$$\alpha\mathcal{A} + \beta\mathcal{B} \in \mathbb{C}(\mathcal{X} \to \mathcal{Y}).$$

Now let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space. Let

$$\begin{gathered}\mathcal{A}_n \in \mathbb{C}(\mathcal{X} \to \mathcal{Y}),\\ \mathcal{A}_0 \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),\end{gathered}$$

such that

$$\lim_{n\to\infty} \|\mathcal{A}_n - \mathcal{A}_0\| = 0.$$

Then

$$\mathcal{A}_0 \in \mathbb{C}(\mathcal{X} \to \mathcal{Y}).$$

Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space and let $(\mathcal{Y}, \|\cdot\|)$ be a Banach space. Let

$$\begin{gathered}\mathcal{A}_n \in \mathbb{C}(\mathcal{X} \to \mathcal{Y}),\\ \mathcal{A}_0 \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}),\end{gathered}$$

such that

$$\lim_{n\to\infty} \|\mathcal{A}_n - \mathcal{A}_0\| = 0.$$

For any positive constant $\varepsilon > 0$, there exists a positive integer $N > 1$, such that if $n > N$, then

$$\|\mathcal{A}_n - \mathcal{A}_0\| < \frac{1}{3}\varepsilon.$$

Let

$$\mathcal{S} \subset \mathcal{X},$$

be a bounded set. Then $\mathcal{A}_{2N}\mathcal{S}$ is a compact set. For any positive constant $\varepsilon > 0$, there exist finitely many points $\mathbf{x}_k \in \mathcal{S}$, where $k = 1, 2, 3, \cdots, m$, such that

$$\mathcal{A}_{2N}\mathcal{S} \subset \bigcup_{k=1}^{m} B(\mathcal{A}_{2N}\mathbf{x}_k, \frac{1}{3}\varepsilon).$$

Now we have the following estimates

$$\begin{aligned} & \|\mathcal{A}_0\mathbf{x} - \mathcal{A}_0\mathbf{x}_k\| \\ \leq\ & \|\mathcal{A}_0\mathbf{x} - \mathcal{A}_{2N}\mathbf{x}\| + \|\mathcal{A}_{2N}\mathbf{x} - \mathcal{A}_{2N}\mathbf{x}_k\| + \|\mathcal{A}_{2N}\mathbf{x}_k - \mathcal{A}_0\mathbf{x}_k\| < \varepsilon, \end{aligned}$$

where $\|\mathcal{A}_{2N}\mathbf{x} - \mathcal{A}_{2N}\mathbf{x}_k\| < \frac{1}{3}\varepsilon$. Therefore, we see that

$$\mathcal{A}_0\mathcal{S} \subset \bigcup_{k=1}^{m} B(\mathcal{A}_0\mathbf{x}_k, \varepsilon).$$

In another word, $\mathcal{A}_0\mathcal{S}$ is compact. Recall that $(\mathcal{Y}, \|\cdot\|)$ is a Banach space. Finally, we have proved that

$$\mathcal{A}_0 \in \mathbb{C}(\mathcal{X} \to \mathcal{Y}).$$

Theorem 2. Let $(\mathcal{X}, \|\cdot\|)$, $(\mathcal{Y}, \|\cdot\|)$, $(\mathcal{Z}, \|\cdot\|)$ and $(\mathcal{E}, \|\cdot\|)$ be normed linear spaces, let

$$\begin{aligned} \mathcal{A} &\in \mathbb{C}(\mathcal{X} \to \mathcal{Y}), \\ \mathcal{B} &\in \mathbb{B}(\mathcal{Y} \to \mathcal{Z}), \\ \mathcal{C} &\in \mathbb{B}(\mathcal{E} \to \mathcal{X}). \end{aligned}$$

Then

$$\begin{array}{ll} (1) & \mathcal{A}^* \in \mathbb{C}(\mathcal{Y}^* \to \mathcal{X}^*). \\ (2) & \mathcal{B}\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{Z}). \\ (3) & \mathcal{A}\mathcal{C} \in \mathbb{C}(\mathcal{E} \to \mathcal{Y}). \\ (4) & \mathcal{A}\mathcal{X} \text{ is a separable set in } \mathcal{Y}. \end{array}$$

Let

$$\mathcal{S} = \{\mathbf{x} \in \mathcal{X} : \|\mathbf{x}\| = 1\}.$$

Then $\mathcal{A}\mathcal{S}$ is compact. For any positive constant $\varepsilon > 0$, there exist finitely many points $\mathbf{x}_k \in \mathcal{S}$, where $k = 1, 2, 3, \cdots, m$, such that

$$\mathcal{A}\mathcal{S} \subset \bigcup_{k=1}^{m} B(\mathcal{A}\mathbf{x}_k, \frac{\varepsilon}{3(C+1)}).$$

Define

$$\mathcal{F} = \overline{\mathcal{A}\mathcal{X}}.$$

Then $\mathcal{F}$ is closed and separable. There exists a sequence of points

$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \cdots, \mathbf{y}_n, \cdots \in \mathcal{F}.$$

$\{\mathbf{y}_n\}$ is dense in $\mathcal{F}$.

Let

$$f_1, f_2, f_3, \cdots, f_n, \cdots \in \mathcal{Y}^*,$$

be a bounded sequence of continuous functionals. There exists a positive constant $C > 0$, such that

$$\|f_n\| \leq C, \text{ for all } n \geq 1.$$

Consider the sequence

$$f_1(\mathbf{y}_1), f_2(\mathbf{y}_1), f_3(\mathbf{y}_1), \cdots, f_n(\mathbf{y}_1), \cdots$$

This is a bounded sequence in $\mathbb{R}$. There exists a convergent subsequence

$$f_{1n_1}(\mathbf{y}_1), f_{1n_2}(\mathbf{y}_1), f_{1n_3}(\mathbf{y}_1), \cdots, f_{1n_k}(\mathbf{y}_1), \cdots$$

Consider the new sequence

$$f_{1n_1}(\mathbf{y}_2), f_{1n_2}(\mathbf{y}_2), f_{1n_3}(\mathbf{y}_2), \cdots, f_{1n_k}(\mathbf{y}_2), \cdots$$

This is a bounded sequence in $\mathbb{R}$. There exists a convergent subsequence

$$f_{2n_1}(\mathbf{y}_2), f_{2n_2}(\mathbf{y}_2), f_{2n_3}(\mathbf{y}_2), \cdots, f_{2n_k}(\mathbf{y}_2), \cdots$$

Consider the new sequence

$$f_{2n_1}(\mathbf{y}_3), f_{2n_2}(\mathbf{y}_3), f_{2n_3}(\mathbf{y}_3), \cdots, f_{2n_k}(\mathbf{y}_3), \cdots$$

This is a bounded sequence in $\mathbb{R}$. There exists a convergent subsequence

$$f_{3n_1}(\mathbf{y}_3), f_{3n_2}(\mathbf{y}_3), f_{3n_3}(\mathbf{y}_3), \cdots, f_{3n_k}(\mathbf{y}_3), \cdots$$

If we repeat this process, then we obtain the convergent subsequences

$$f_{mn_1}(\mathbf{y}_m), f_{mn_2}(\mathbf{y}_m), f_{mn_3}(\mathbf{y}_m), \cdots, f_{mn_k}(\mathbf{y}_m), \cdots$$

for all positive integers $m = 1, 2, 3, \cdots$.

Let us consider the subsequence

$$f_{1n_1}(\mathbf{y}), f_{2n_2}(\mathbf{y}), f_{3n_3}(\mathbf{y}), \cdots, f_{mn_m}(\mathbf{y}), \cdots$$

and

$$f_{1n_1}(\mathbf{y}_1), f_{2n_2}(\mathbf{y}_2), f_{3n_3}(\mathbf{y}_3), \cdots, f_{mn_m}(\mathbf{y}_m), \cdots$$

Based on Section 3, Theorem, there exists a continuous linear functional $f_0 \in \mathcal{X}^*$, such that

$$\lim_{m\to\infty} |f_{mn_m}(\mathbf{y}) - f_0(\mathbf{y})| = 0,$$

for all $\mathbf{y} \in \mathcal{F}$. For any positive constant $\varepsilon > 0$, there exists a positive integer $N > 1$, such that if $m > N$, then

$$|f_{mn_m}(\mathbf{y}) - f_0(\mathbf{y})| < \frac{1}{3}\varepsilon.$$

Now we have the following estimate

$$\begin{aligned}
& \|\mathcal{A}^* f_{mn_m} - \mathcal{A}^* f_0\| \\
= & \sup_{\|\mathbf{x}\|=1} |f_{mn_m}(\mathcal{A}\mathbf{x}) - f_0(\mathcal{A}\mathbf{x})| \\
\leq & \sup_{\|\mathbf{x}\|=1} |f_{mn_m}(\mathcal{A}\mathbf{x} - \mathcal{A}\mathbf{x}_k) - f_0(\mathcal{A}\mathbf{x} - \mathcal{A}\mathbf{x}_k)| \\
+ & |f_{mn_m}(\mathcal{A}\mathbf{x}_k) - f_0(\mathcal{A}\mathbf{x}_k)| < \varepsilon,
\end{aligned}$$

$$\lim_{m\to\infty} \|\mathcal{A}^* f_{mn_m} - \mathcal{A}^* f_0\| = 0.$$

Theorem 3. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{X}).$$

Let $\lambda \in \mathbb{C}$ be a complex constant and $\lambda \neq 0$. Suppose that

$$(\lambda\mathcal{I} - \mathcal{A})\mathcal{X} = \mathcal{X}.$$

Then $\lambda \in \rho(\mathcal{A})$.

Proof. The main idea is to use the *onto* to prove the *one-to-one*. Define the closed subspaces

$$\mathcal{E}_n = \{\mathbf{x} \in \mathcal{X} : (\lambda\mathcal{I} - \mathcal{A})^n \mathbf{x} = \mathbf{0}\},$$

for all $n = 1, 2, 3, \cdots\cdots$.

Note that

$$(\lambda\mathcal{I}-\mathcal{A})\mathcal{X}=\mathcal{X}.$$

If there exists a vector $\mathbf{x}_1 \in \mathcal{E}_1$, $\mathbf{x}_1 \neq \mathbf{0}$, then there exists a sequence of vectors $\{\mathbf{x}_n\}$, such that

$$\begin{array}{rcl} (\lambda\mathcal{I}-\mathcal{A})\mathbf{x}_1 & = & \mathbf{0}, \\ (\lambda\mathcal{I}-\mathcal{A})\mathbf{x}_2 & = & \mathbf{x}_1, \\ (\lambda\mathcal{I}-\mathcal{A})\mathbf{x}_3 & = & \mathbf{x}_2, \\ \cdots\cdots & & \cdots\cdots \\ (\lambda\mathcal{I}-\mathcal{A})\mathbf{x}_n & = & \mathbf{x}_{n-1}, \\ (\lambda\mathcal{I}-\mathcal{A})\mathbf{x}_{n+1} & = & \mathbf{x}_n, \\ \cdots\cdots & & \cdots\cdots \end{array}$$

Overall, we have

$$\begin{array}{rcl} (\lambda\mathcal{I}-\mathcal{A})^n\mathbf{x}_n & = & \mathbf{0}, \\ (\lambda\mathcal{I}-\mathcal{A})^n\mathbf{x}_{n+1} & = & \mathbf{x}_1 \neq \mathbf{0}. \end{array}$$

There exists a sequence of unit vectors $\{\mathbf{y}_n\}$, $\|\mathbf{y}_n\| = 1$, such that

$$\mathbf{y}_{n+1} \notin \mathcal{E}_n, \qquad \mathbf{y}_{n+1} \in \mathcal{E}_{n+1}, \qquad \rho(\mathbf{y}_{n+1}, \mathcal{E}_n) > \frac{1}{2},$$

for all $n = 1, 2, 3, \cdots\cdots$. Then

$$\|\mathcal{A}\mathbf{y}_m - \mathcal{A}\mathbf{y}_n\| > \frac{1}{2}|\lambda|,$$

for all $m > n$.

Theorem 4. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{X}).$$

Let $\lambda \in \mathbb{C}$ be a constant and $\lambda \neq 0$. Then

$$(\lambda\mathcal{I}-\mathcal{A})\mathcal{X}$$

is a closed subspace of $\mathcal{X}$.

Proof. For any $\mathbf{y} \in (\lambda\mathcal{I} - \mathcal{A})\mathcal{X}$, there exists $\mathbf{x} \in \mathcal{X}$, such that

$$\mathbf{y} = (\lambda\mathcal{I} - \mathcal{A})\mathbf{x}.$$

There exists a vector $\mathbf{x}_0 \in \mathcal{X}$, such that

$$\|\mathbf{x}_0\| = \inf_{(\lambda\mathcal{I} - \mathcal{A})\mathbf{x} = \mathbf{y}} \|\mathbf{x}\|.$$

There exists a positive constant $C > 0$, such that

$$\|\mathbf{x}_0\| \leq C\|\mathbf{y}\|.$$

If not, for any positive integer $n \geq 1$, there are vectors $\mathbf{x}_n \in \mathcal{X}$ and $\mathbf{y}_n \in (\lambda\mathcal{I} - \mathcal{A})\mathcal{X}$, such that

$$(\lambda\mathcal{I} - \mathcal{A})\mathbf{x}_n = \mathbf{y}_n,$$
$$\|\mathbf{x}_n\| = \inf_{(\lambda\mathcal{I} - \mathcal{A})\mathbf{x} = \mathbf{y}_n} \|\mathbf{x}\|,$$

such that

$$\|\mathbf{x}_n\| \geq 2^n\|\mathbf{y}_n\|,$$

for all $n \geq 1$.

Without loss of generality, let $\|\mathbf{x}_n\| = 1$. Then $\|\mathbf{y}_n\| \leq \frac{1}{2^n}$. There exists a convergent subsequence $\{\mathbf{x}_{n_k}\} \subset \{\mathbf{x}_n\}$, such that

$$\lim_{k\to\infty} \|\mathbf{x}_{n_k} - \mathbf{x}_0\| = 0.$$

Hence

$$(\lambda\mathcal{I} - \mathcal{A})\mathbf{x}_0 = \mathbf{0}.$$

However, we also have

$$(\lambda\mathcal{I} - \mathcal{A})(\mathbf{x}_n - \mathbf{x}_0) = \mathbf{y}_n.$$

This implies that

$$1 = \|\mathbf{x}_{n_k}\| \leq \|\mathbf{x}_{n_k} - \mathbf{x}_0\|.$$

Letting $k \to \infty$ leads to a contradiction.

Now let $\{\mathbf{y}_n\} \subset (\lambda\mathcal{I} - \mathcal{A})\mathcal{X}$ be a Cauchy sequence. We will show that the limit is in the same space $(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}$. There exists a sequence $\{\mathbf{x}_n\} \subset \mathcal{X}$, such that

$$\begin{aligned} (\lambda\mathcal{I} - \mathcal{A})\mathbf{x}_n &= \mathbf{y}_n, \\ \|\mathbf{x}_n\| &\leq C\|\mathbf{y}_n\| \leq M, \end{aligned}$$

for all positive integers $n = 1, 2, 3, \cdots$.

There exists a convergent subsequence $\{\mathbf{x}_{n_k}\} \subset \{\mathbf{x}_n\}$,

$$(\lambda\mathcal{I} - \mathcal{A})\mathbf{x}_{n_k} = \mathbf{y}_{n_k},$$
$$\lim_{n\to\infty} \|\mathbf{x}_{n_k} - \mathbf{x}_0\| = 0, \ \lim_{n\to\infty} \|\mathbf{y}_{n_k} - \mathbf{y}_0\| = 0.$$

Therefore

$$(\lambda\mathcal{I} - \mathcal{A})\mathbf{x}_0 = \mathbf{y}_0.$$

Theorem 5. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{X}).$$

Let $\lambda \in \mathbb{C}$ be a constant and $\lambda \neq 0$. Then $\lambda \in \rho(\mathcal{A}^*)$, if λ is not an eigenvalue of $\mathcal{A}$.

Proof. First of all, the linear operator

$$(\lambda\mathcal{I} - \mathcal{A}) : \mathcal{X} \to (\lambda\mathcal{I} - \mathcal{A})\mathcal{X},$$

is one-to-one, onto. Note that

$$(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}$$

is a Banach space. Therefore, the inverse operator

$$(\lambda\mathcal{I} - \mathcal{A})^{-1} : (\lambda\mathcal{I} - \mathcal{A})\mathcal{X} \to \mathcal{X},$$

is a bounded linear operator.

Now for any $f \in \mathcal{X}^*$, define

$$g(\mathbf{x}) = f((\lambda\mathcal{I} - \mathcal{A})^{-1}\mathbf{x}),$$

for all $\mathbf{x} \in \mathcal{X}$. By using the extension theorem for functionals, we may extend it so that $g \in \mathcal{X}^*$. Obviously

$$(\lambda\mathcal{I}^* - \mathcal{A}^*)\mathcal{X}^* = \mathcal{X}^*.$$

Finally, we find that

$$\lambda \in \rho(\mathcal{A}^*).$$

Definition 2. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces, let

$$\begin{aligned} &\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}), \\ &\mathbf{x} \in \mathcal{X}, \qquad f \in \mathcal{Y}^*. \end{aligned}$$

If

$$(\mathcal{A}^* f)(\mathbf{x}) = f(\mathcal{A}\mathbf{x}) = 0,$$

then we say $\mathcal{A}\mathbf{x}$ and f are perpendicular and we write $\mathcal{A}\mathbf{x} \perp f$. In this case, we may also say $\mathcal{A}^* f$ and $\mathbf{x}$ are perpendicular and we write $\mathcal{A}^* f \perp \mathbf{x}$.

Let $(\mathcal{X}, \|\cdot\|)$ be normed linear space, let $\mathcal{X}^*$ be the dual space. Let

$$\mathbf{x} \in \mathcal{X}, \qquad f \in \mathcal{X}^*.$$

If

$$f(\mathbf{x}) = 0,$$

then we say $\mathbf{x}$ and f are perpendicular. We write $\mathbf{x} \perp f$.

Theorem 6. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces, let

$$\begin{aligned} &\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}), \\ &\mathcal{A}^* \in \mathbb{B}(\mathcal{Y}^* \to \mathcal{X}^*). \end{aligned}$$

Then

$$\begin{aligned} &(\mathcal{A}\mathcal{X})^{\perp} = \mathcal{N}(\mathcal{A}^*), \\ &(\mathcal{A}^*\mathcal{Y}^*)^{\perp} = \mathcal{N}(\mathcal{A}). \end{aligned}$$

Let

$$\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{X}).$$

Then

$$\begin{aligned} (\lambda\mathcal{I} - \mathcal{A})\mathcal{X} &= [\mathcal{N}((\lambda\mathcal{I} - \mathcal{A})^*)]^{\perp}, \\ (\lambda\mathcal{I} - \mathcal{A})^*\mathcal{X}^* &= [\mathcal{N}(\lambda\mathcal{I} - \mathcal{A})]^{\perp}. \end{aligned}$$

To establish the first result, we will prove that

$$\begin{aligned} &(1) \quad (\lambda\mathcal{I} - \mathcal{A})\mathcal{X} \subset [\mathcal{N}((\lambda\mathcal{I} - \mathcal{A})^*)]^{\perp}. \\ &(2) \quad [\mathcal{N}((\lambda\mathcal{I} - \mathcal{A})^*)]^{\perp} \subset (\lambda\mathcal{I} - \mathcal{A})\mathcal{X}. \end{aligned}$$

For any point

$$(\lambda\mathcal{I} - \mathcal{A})\mathbf{x} \in (\lambda\mathcal{I} - \mathcal{A})\mathcal{X},$$

for any

$$f \in \mathcal{N}((\lambda\mathcal{I} - \mathcal{A})^*),$$

we have

$$f((\lambda\mathcal{I}-\mathcal{A})\mathbf{x}) = ((\lambda\mathcal{I}-\mathcal{A})^* f)(\mathbf{x}) = 0.$$

Thus

$$(\lambda\mathcal{I}-\mathcal{A})\mathbf{x} \in [\mathcal{N}((\lambda\mathcal{I}-\mathcal{A})^*)]^\perp.$$

Therefore

$$(\lambda\mathcal{I}-\mathcal{A})\mathcal{X} \subset [\mathcal{N}((\lambda\mathcal{I}-\mathcal{A})^*)]^\perp.$$

Let

$$y \in [\mathcal{N}((\lambda\mathcal{I}-\mathcal{A})^*)]^\perp.$$

That is

$$f(\mathbf{y}) = 0,$$

for all f:

$$(\lambda\mathcal{I}-\mathcal{A})^* f = 0.$$

If

$$y \notin (\lambda\mathcal{I}-\mathcal{A})\mathcal{X},$$

then there exists a continuous linear functional f, such that

$$\begin{aligned} &(1) \quad \|f\| = 1\| \\ &(2) \quad f(\mathbf{y}) = \rho(\mathbf{y}, (\lambda\mathcal{I}-\mathcal{A})\mathcal{X}) > 0, \\ &(3) \quad f((\lambda\mathcal{I}-\mathcal{A})\mathbf{x}) = 0, \end{aligned}$$

for all $\mathbf{x} \in \mathcal{X}$.

On the other hand, we have

$$((\lambda\mathcal{I}-\mathcal{A})^* f)(\mathbf{x}) = f((\lambda\mathcal{I}-\mathcal{A})\mathbf{x}) = 0.$$

Hence

$$f \in \mathcal{N}((\lambda\mathcal{I} - \mathcal{A})^*).$$

Thus

$$\mathbf{y} \in (\lambda\mathcal{I} - \mathcal{A})\mathcal{X}.$$

Therefore

$$[\mathcal{N}((\lambda\mathcal{I} - \mathcal{A})^*)]^\perp \subset (\lambda\mathcal{I} - \mathcal{A})\mathcal{X}.$$

$\sum_{i=1}^n \sum_{j=1}^n \sum_{i=1}^n \sum_{j=1}^n \sup_{(1-\varepsilon)t\leq\tau\leq t} t^{2m+1+\frac{n}{2}} t^{2m+1+\frac{n}{2}} \{\} |\eta|^2\eta|\eta|^2\eta$
$\int_{\mathbb{R}^n} \mathrm{d}\mathbf{x} \int_{\mathbb{R}^n} \mathrm{d}\mathbf{x} \int_{\mathbb{R}^n} \mathrm{d}\mathbf{x} \int_{\mathbb{R}^n} \mathrm{d} \int_{\mathbb{R}^n} \mathrm{d}\mathbf{x} \int_{\mathbb{R}^n} \mathrm{d}$
$\rho()\rho()\rho() \perp\in \sigma()$

To establish the second result, we will prove that

$$\begin{aligned} &(1) \quad (\lambda\mathcal{I} - \mathcal{A})^*\mathcal{X}^* \subset [\mathcal{N}(\lambda\mathcal{I} - \mathcal{A})]^\perp. \\ &(2) \quad [\mathcal{N}(\lambda\mathcal{I} - \mathcal{A})]^\perp \subset (\lambda\mathcal{I} - \mathcal{A})^*\mathcal{X}^*. \end{aligned}$$

Let

$$f \in \mathcal{X}^*.$$

Then

$$(\lambda\mathcal{I} - \mathcal{A})^* f \in (\lambda\mathcal{I} - \mathcal{A})^*\mathcal{X}^*.$$

Simple computation shows that

$$[(\lambda\mathcal{I} - \mathcal{A})^* f](\mathbf{x}) = f((\lambda\mathcal{I} - \mathcal{A})\mathbf{x}) = 0,$$

for all $\mathbf{x}$,

$$\mathbf{x} \in \mathcal{N}(\lambda\mathcal{I} - \mathcal{A}).$$

Therefore, we have

$$(\lambda\mathcal{I} - \mathcal{A})^*\mathcal{X}^* \subset [\mathcal{N}(\lambda\mathcal{I} - \mathcal{A})]^\perp.$$

Let

$$f \in [\mathcal{N}(\lambda\mathcal{I} - \mathcal{A})]^{\perp}.$$

This is to say that

$$f(\mathbf{x}) = 0,$$

for all $\mathbf{x} \in \mathcal{N}(\lambda\mathcal{I} - \mathcal{A})$.

Let us define the functional

$$g : (\lambda\mathcal{I} - \mathcal{A})\mathcal{X} \to \mathbb{C},$$

by

$$g((\lambda\mathcal{I} - \mathcal{A})\mathbf{x}) = f(\mathbf{x}_0),$$

where $\mathbf{x}_0 \in \mathcal{X}$ is a vector such that

$$\|\mathbf{x}_0\| = \inf_{(\lambda\mathcal{I}-\mathcal{A})\mathbf{z}=(\lambda\mathcal{I}-\mathcal{A})\mathbf{x}} \|\mathbf{z}\|.$$

There exists a positive constant $C > 0$, such that

$$\|\mathbf{x}_0\| \leq C\|(\lambda\mathcal{I} - \mathcal{A})\mathbf{x}\|,$$

for all $\mathbf{x} \in \mathcal{X}$.

The functional g is well defined, because if there exist two vectors $\mathbf{x}_1 \in \mathcal{X}$ and $\mathbf{x}_2 \in \mathcal{X}$, such that

$$(\lambda\mathcal{I} - \mathcal{A})\mathbf{x}_1 = (\lambda\mathcal{I} - \mathcal{A})\mathbf{x}_2,$$

then

$$(\lambda\mathcal{I} - \mathcal{A})(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}.$$

This means that

$$f(\mathbf{x}_1 - \mathbf{x}_2) = 0, \qquad f(\mathbf{x}_1) = f(\mathbf{x}_2).$$

Moreover, we have the following estimates

$$\begin{aligned}|g((\lambda\mathcal{I}-\mathcal{A})\mathbf{x})| &= |f(\mathbf{x}_0)| \\ &\leq \ \|f\|\|\mathbf{x}_0\| \leq \|f\|\|(\lambda\mathcal{I}-\mathcal{A})\mathbf{x}\|,\end{aligned}$$

for all $\mathbf{x} \in \mathcal{X}$. Moreover it is easy to see that g is linear. Hence, g is a continuous linear functional on $(\lambda\mathcal{I}-\mathcal{A})\mathcal{X}$. We may extend it to a continuous linear functional on $\mathcal{X}$. Now

$$[(\lambda\mathcal{I}-\mathcal{A})^*g](\mathbf{x}) = g((\lambda\mathcal{I}-\mathcal{A})\mathbf{x}) = f(\mathbf{x}),$$

for all $\mathbf{x} \in \mathcal{X}$. That is

$$(\lambda\mathcal{I}-\mathcal{A})^*g = f.$$

Therefore, we have

$$[\mathcal{N}(\lambda\mathcal{I}-\mathcal{A})]^\perp \subset (\lambda\mathcal{I}-\mathcal{A})^*\mathcal{X}^*.$$

Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let

$$\begin{aligned}&\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}), \\ &f \in \mathcal{X}^*.\end{aligned}$$

Suppose that

$$f(\mathcal{A}\mathbf{x}) = 0,$$

for all $\mathbf{x} \in \mathcal{X}$, then $\mathcal{A}^*f = 0$.

In particular, if $\mathcal{A}\mathcal{X}$ is not dense in $\mathcal{X}$, then $\lambda = 0$ is an eigenvalue of $\mathcal{A}^*$.

Proof. To establish the first result, we will prove that

$$\begin{aligned}&(1) \quad (\mathcal{A}\mathcal{X})^\perp \subset \mathcal{N}(\mathcal{A}^*). \\ &(2) \quad \mathcal{N}(\mathcal{A}^*) \subset (\mathcal{A}\mathcal{X})^\perp.\end{aligned}$$

Let $f \in \mathcal{Y}^*$. If

$$f \perp \mathcal{A}\mathcal{X}, \qquad f \in (\mathcal{A}\mathcal{X})^\perp,$$

then

$$(\mathcal{A}^* f)(\mathbf{x}) = f(\mathcal{A}\mathbf{x}) = 0, \qquad \mathcal{A}^* f = 0,$$

for all $\mathbf{x} \in \mathcal{X}$. This means $f \in \mathcal{N}(\mathcal{A}^*)$.

Let $f \in \mathcal{Y}^*$. If $f \in \mathcal{N}(\mathcal{A}^*)$, then

$$f(\mathcal{A}\mathbf{x}) = (\mathcal{A}^* f)(\mathbf{x}) = 0,$$

for all $\mathbf{x} \in \mathcal{X}$. Hence

$$f \perp \mathcal{A}\mathcal{X}, \qquad f \in (\mathcal{A}\mathcal{X})^\perp.$$

To establish the second result, we will prove that

$$\begin{array}{ll} (1) & (\mathcal{A}^*\mathcal{Y}^*)^\perp \subset \mathcal{N}(\mathcal{A}). \\ (2) & \mathcal{N}(\mathcal{A}) \subset (\mathcal{A}^*\mathcal{Y}^*)^\perp. \end{array}$$

Let $\mathbf{x} \in \mathcal{X}$. If

$$\mathbf{x} \in (\mathcal{A}^*\mathcal{Y}^*)^\perp, \qquad \mathbf{x} \perp \mathcal{A}^*\mathcal{Y}^*,$$

then

$$f(\mathcal{A}\mathbf{x}) = (\mathcal{A}^* f)(\mathbf{x}) = 0,$$

for all $f \in \mathcal{Y}^*$. Hence

$$\mathcal{A}\mathbf{x} = \mathbf{0}, \qquad \mathbf{x} \in \mathcal{N}(\mathcal{A}).$$

Now let $\mathbf{x} \in \mathcal{X}$. If

$$\mathbf{x} \in \mathcal{N}(\mathcal{A}), \qquad \mathcal{A}\mathbf{x} = \mathbf{0},$$

then

$$(\mathcal{A}^*f)(\mathbf{x}) = f(\mathcal{A}\mathbf{x}) = 0,$$

for all $f \in \mathcal{Y}^*$. Hence

$$\mathbf{x} \in (\mathcal{A}^*\mathcal{Y}^*)^\perp.$$

Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

The key point is that

$$(\mathcal{A}^*g)(\mathbf{x}) = g(\mathcal{A}\mathbf{x}),$$

for all $\mathbf{x} \in \mathcal{X}$ and for all $g \in \mathcal{X}^*$.

Remark. Let $\lambda \in \mathbb{C}$ be a complex constant and $\lambda \neq 0$. Let

$$\begin{gathered} \mathcal{I} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}), \\ \mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}). \end{gathered}$$

The above results are also true if we replace

$$\begin{array}{lcl} \mathcal{A} & \text{with} & \lambda\mathcal{I} - \mathcal{A}, \\ \mathcal{A}^* & \text{with} & (\lambda\mathcal{I} - \mathcal{A})^*. \end{array}$$

Therefore, we have

$$\begin{array}{ll} (1) & [(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}]^\perp = \mathcal{N}((\lambda\mathcal{I} - \mathcal{A})^*). \\ (2) & [(\lambda\mathcal{I} - \mathcal{A})^*\mathcal{X}^*]^\perp = \mathcal{N}(\lambda\mathcal{I} - \mathcal{A}). \end{array}$$

Theorem 7. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{X}).$$

Then

(1) If $\dim(\mathcal{X}) = \infty$, then $\lambda = 0 \in \sigma(\mathcal{A})$.

(2) If $\lambda \in \sigma(\mathcal{A})$ and $\lambda \neq 0$, then $\lambda \in \sigma_p(\mathcal{A})$.

(3) If $\lambda \in \sigma(\mathcal{A})$ and $\lambda \neq 0$, then $\dim \mathcal{E}_\lambda < \infty$.

(4) Let λ_1, λ_2, λ_3, $\cdots$, λ_n be distinct eigenvalues of $\mathcal{A}$, let ξ_1, ξ_2, ξ_3, $\cdots$, ξ_n be the corresponding eigenvectors, that is

$$\mathcal{A}\xi_k = \lambda\xi_k, \qquad k = 1, 2, 3, \cdots, n,$$

then these eigenvectors are linearly independent.

(5) The only possible limit in $\sigma(\mathcal{A})$ is $\lambda = 0$.

Proof. Let $\lambda \in \mathbb{C}$ be a constant and $\lambda \neq 0$. Then

$$(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}$$

is a closed subspace of $\mathcal{X}$. Let $\lambda \in \sigma(\mathcal{A})$. Then

$$(\lambda\mathcal{I} - \mathcal{A})\mathcal{X} \neq \mathcal{X}.$$

Now based on the extension theorem for continuous linear functionals, there exists a $f \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f\| = 1, \\ &(2) \quad f((\lambda\mathcal{I} - \mathcal{A})\mathcal{X}) = \{\mathbf{0}\}. \end{aligned}$$

Now we see that

$$[(\lambda\mathcal{I} - \mathcal{A})^* f](\mathbf{x}) = f((\lambda\mathcal{I} - \mathcal{A})\mathbf{x}) = 0,$$

for all $\mathbf{x} \in \mathcal{X}$. That is

$$(\lambda\mathcal{I} - \mathcal{A})^* f = 0.$$

Thus, λ is an eigenvalue of $\mathcal{A}^*$. Based on Theorem 5, $\lambda \in \sigma_p(\mathcal{A})$.

Without loss of generality, let $\{\lambda_n\}$ be a convergent sequence of nonzero eigenvalues. Suppose that

$$\lim_{n\to\infty} \lambda_n = \lambda_0 \neq 0.$$

Let $\mathbf{x}_n$ be the corresponding eigenvectors, that is

$$\mathcal{A}\mathbf{x}_n = \lambda_n \mathbf{x}_n.$$

Let $\lambda_m \neq \lambda_n$ for $m \neq n$. It is easy to see that the eigenvectors $\{\mathbf{x}_n\}$ are linearly independent.

Define the following closed subspaces

$$\begin{aligned}
\mathcal{E}_1 &= \text{span }\{\mathbf{x}_1\},\\
\mathcal{E}_2 &= \text{span }\{\mathbf{x}_1, \mathbf{x}_2\},\\
\mathcal{E}_3 &= \text{span }\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\},\\
&\cdots\cdots\\
\mathcal{E}_n &= \text{span }\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n\},\\
&\cdots\cdots
\end{aligned}$$

There exist unit vectors $\mathbf{y}_n$, $\|\mathbf{y}_n\| = 1$:

$$\mathbf{y}_n \in \mathcal{E}_n, \qquad \mathbf{y}_n \notin \mathcal{E}_{n-1},$$

such that

$$\rho(\mathbf{y}_n, \mathcal{E}_{n-1}) > \frac{1}{2},$$

for all $n = 2, 3, 4, \cdots\cdots$.

Let

$$\mathbf{y}_n = \alpha_{n1}\mathbf{x}_1 + \alpha_{n2}\mathbf{x}_2 + \alpha_{n3}\mathbf{x}_3 + \cdots + \alpha_{nn}\mathbf{x}_n,$$

Obviously, we have

$$\begin{aligned}
& (\lambda_n \mathcal{I} - \mathcal{A})\mathbf{y}_n \\
= \ & \alpha_{n1}(\lambda_n - \lambda_1)\mathbf{x}_1 + \alpha_{n2}(\lambda_n - \lambda_2)\mathbf{x}_2 + \alpha_{n3}(\lambda_n - \lambda_3)\mathbf{x}_3 \\
+ \ & \cdots + \alpha_{n,k-1}(\lambda_n - \lambda_{n-1})\mathbf{x}_{n-1} \in \mathcal{E}_{n-1},
\end{aligned}$$

for all $n = 2, 3, 4, \cdots\cdots$. Namely,

$$\mathbf{y}_n - \mathcal{A}\left(\frac{\mathbf{y}_n}{\lambda_n}\right) \in \mathcal{E}_{n-1},$$

for all $n = 2, 3, 4, \cdots$.

Hence

$$\mathbf{y}_m - \mathcal{A}\left(\frac{\mathbf{y}_m}{\lambda_m}\right) \in \mathcal{E}_{m-1}, \qquad \mathbf{y}_n - \mathcal{A}\left(\frac{\mathbf{y}_n}{\lambda_n}\right) \in \mathcal{E}_{m-1},$$

where $m > n$. Now we have the following estimates

$$\begin{aligned} & \left\|\mathcal{A}\left(\frac{\mathbf{y}_m}{\lambda_m}\right) - \mathcal{A}\left(\frac{\mathbf{y}_n}{\lambda_n}\right)\right\| \\ = & \left\|\mathbf{y}_m - \left[\mathbf{y}_m - \mathcal{A}\left(\frac{\mathbf{y}_m}{\lambda_m}\right)\right] + \left[\mathbf{y}_n - \mathcal{A}\left(\frac{\mathbf{y}_n}{\lambda_n}\right)\right] - \mathbf{y}_n\right\| \\ \geq & \ \rho(\mathbf{y}_m, \mathcal{E}_{m-1}) > \frac{1}{2}. \end{aligned}$$

However, there must be a convergent subsequence

$$\left\{\mathcal{A}\left(\frac{\mathbf{y}_{n_k}}{\lambda_{n_k}}\right)\right\} \subset \left\{\mathcal{A}\left(\frac{\mathbf{y}_n}{\lambda_n}\right)\right\},$$

since

$$\left\|\frac{\mathbf{y}_n}{\lambda_n}\right\| \leq C,$$

for all $n = 1, 2, 3, \cdots$. This is a contradiction. Therefore, there exists at most one limit: $\lambda = 0$.

Theorem 8. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{X}).$$

The following statements are true.

(1) $\sigma(\mathcal{A}) = \sigma(\mathcal{A}^*)$.

(2) Let $\lambda \in \sigma(\mathcal{A}) = \sigma(\mathcal{A}^*)$ and $\lambda \neq 0$. Then

$$\dim[\mathcal{N}(\lambda\mathcal{I} - \mathcal{A})] = \dim[\mathcal{N}(\lambda\mathcal{I}^* - \mathcal{A}^*)].$$

(3) Let

$$\mathcal{A}\xi = \lambda\xi, \qquad \mathcal{A}^* f = \mu f.$$

If $\lambda \neq \mu$, then $f(\xi) = 0$.

(4) Let $\lambda \in \sigma(\mathcal{A})$ and $\lambda \neq 0$. Then there exists a solution to the equation

$$(\lambda\mathcal{I} - \mathcal{A})\mathbf{x} = \mathbf{y},$$

if and only if

$$y \perp \mathcal{N}(\lambda\mathcal{I}^* - \mathcal{A}^*).$$

(5) Let $\lambda \in \sigma(\mathcal{A})$ and $\lambda \neq 0$. Then there exists a solution to the equation

$$(\lambda\mathcal{I}^* - \mathcal{A}^*)g = f,$$

if and only if

$$f \perp \mathcal{N}(\lambda\mathcal{I} - \mathcal{A}).$$

There are two cases to consider. Let the dimension

$$\dim \mathcal{X} < \infty.$$

Then the operator $\mathcal{A}$ is just a constant square matrix.

$$\det(\lambda\mathcal{I} - A) = \det(\lambda\mathcal{I} - A^T).$$
$$\det(\lambda\mathcal{I} - A) = 0, \iff \lambda \in \sigma(\mathcal{A}).$$
$$\det(\lambda\mathcal{I} - A^T) = 0, \iff \lambda \in \sigma(\mathcal{A}^*).$$

Hence

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}^*).$$

Let the dimension

$$\dim \mathcal{X} = \infty.$$

Then

$$\lambda = 0 \in \sigma(\mathcal{A}) \cap \sigma(\mathcal{A}^*).$$

Let $\lambda \in \mathbb{C}$ be a constant and $\lambda \neq 0$. Then

$$\begin{aligned} \lambda \in \rho(\mathcal{A}) &\ \rightarrow\ \lambda \in \rho(\mathcal{A}^*). \\ \lambda \in \sigma(\mathcal{A}) &\ \rightarrow\ \lambda \in \sigma(\mathcal{A}^*). \end{aligned}$$

Overall, we have

$$\sigma(\mathcal{A}) = \sigma(\mathcal{A}^*).$$

Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be a normed linear spaces. Let

$$\begin{aligned} \mathcal{A} &\in \mathbb{B}(\mathcal{X} \to \mathcal{Y}), \\ \mathcal{B} &\in \mathbb{B}(\mathcal{X} \to \mathcal{Y}). \end{aligned}$$

Without loss of generality, let $\mathcal{A} \neq \mathbf{0}$ and $\mathcal{B} \neq \mathbf{0}$. Let $n \geq 1$ be any positive integer. Then

$$\begin{aligned} (\mathcal{A}\mathcal{B})^{n+1} &= \mathcal{A}(\mathcal{B}\mathcal{A})^n\mathcal{B}, \\ (\mathcal{B}\mathcal{A})^{n+1} &= \mathcal{B}(\mathcal{A}\mathcal{B})^n\mathcal{A}. \end{aligned}$$

Hence

$$\begin{aligned} \|(\mathcal{A}\mathcal{B})^{n+1}\| &\leq \|\mathcal{A}\|\|(\mathcal{B}\mathcal{A})^n\|\|\mathcal{B}\|, \\ \|(\mathcal{B}\mathcal{A})^{n+1}\| &\leq \|\mathcal{B}\|\|(\mathcal{A}\mathcal{B})^n\|\|\mathcal{A}\|. \end{aligned}$$

Therefore, we obtain the limits

$$\begin{aligned}\lim_{n\to\infty}\left\{\|(\mathcal{A}\mathcal{B})^{n+1}\|^{\frac{1}{n+1}}\right\} &\leq \lim_{n\to\infty}\left\{\|(\mathcal{B}\mathcal{A})^{n}\|^{\frac{1}{n}}\right\},\\ \lim_{n\to\infty}\left\{\|(\mathcal{B}\mathcal{A})^{n+1}\|^{\frac{1}{n+1}}\right\} &\leq \lim_{n\to\infty}\left\{\|(\mathcal{A}\mathcal{B})^{n}\|^{\frac{1}{n}}\right\}.\end{aligned}$$

Finally, we obtain

$$\lim_{n\to\infty}\left\{\|(\mathcal{A}\mathcal{B})^{n}\|^{\frac{1}{n}}\right\} = \lim_{n\to\infty}\left\{\|(\mathcal{B}\mathcal{A})^{n}\|^{\frac{1}{n}}\right\}.$$

Recall that

$$\begin{aligned}r(\mathcal{A}) &= \lim_{n\to\infty}\left\{\|\mathcal{A}^{n}\|^{\frac{1}{n}}\right\},\\ r(\mathcal{B}) &= \lim_{n\to\infty}\left\{\|\mathcal{B}^{n}\|^{\frac{1}{n}}\right\}.\end{aligned}$$

For any positive constant $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon) > 1$, such that

$$\begin{aligned}\|\mathcal{A}^{n}\| &\leq [r(\mathcal{A}) + \varepsilon]^{n},\\ \|\mathcal{B}^{n}\| &\leq [r(\mathcal{B}) + \varepsilon]^{n},\end{aligned}$$

for all positive integers $n > N$.

Let $n \geq N + 1$ be any positive integer. Since $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$, we find that

$$(\mathcal{A} + \mathcal{B})^{n} = \sum_{k=0}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\mathcal{A}^{n-k}\mathcal{B}^{k},$$

where $0! = 1$ and

$$\mathcal{A}^{0} = \mathcal{I}.$$

There hold the following estimates

$$\begin{aligned}
&\|(\mathcal{A}+\mathcal{B})^n\| \\
\leq\ & \sum_{k=0}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\|\mathcal{A}^{n-k}\|\|\mathcal{B}^k\| \\
=\ & \sum_{k=0}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\left\{\|\mathcal{A}^{n-k}\|^{\frac{1}{n-k}}\right\}^{n-k}\left\{\|\mathcal{B}^k\|^{\frac{1}{k}}\right\}^k \\
=\ & \sum_{k=N+1}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\left\{\|\mathcal{A}^{n-k}\|^{\frac{1}{n-k}}\right\}^{n-k}\left\{\|\mathcal{B}^k\|^{\frac{1}{k}}\right\}^k \\
+\ & \sum_{k=0}^{N} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\left\{\|\mathcal{A}^{n-k}\|^{\frac{1}{n-k}}\right\}^{n-k}\left\{\|\mathcal{B}^k\|^{\frac{1}{k}}\right\}^k \\
\leq\ & \sum_{k=N+1}^{n} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\left\{\|\mathcal{A}^{n-k}\|^{\frac{1}{n-k}}\right\}^{n-k}\{r(\mathcal{B})+\varepsilon\}^k \\
+\ & \sum_{k=0}^{N} \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\{r(\mathcal{A})+\varepsilon\}^{n-k}\left\{\|\mathcal{B}^k\|^{\frac{1}{k}}\right\}^k .
\end{aligned}$$

There exists a positive constant $C > 0$, such that

$$\|\mathcal{A}^{n-k}\|^{\frac{1}{n-k}} \leq C, \qquad \|\mathcal{B}^k\|^{\frac{1}{k}} \leq C,$$

for all positive integers k and n.

Finally, we obtain the desired estimate

$$\begin{aligned}
\lim_{n\to\infty}\left\{\|(\mathcal{A}+\mathcal{B})^n\|^{\frac{1}{n}}\right\} &\leq \lim_{n\to\infty}\left\{\|\mathcal{A}^n\|^{\frac{1}{n}}\right\} + \lim_{n\to\infty}\left\{\|\mathcal{B}^n\|^{\frac{1}{n}}\right\}, \\
r(\mathcal{A}+\mathcal{B}) &\leq r(\mathcal{A}) + r(\mathcal{B}).
\end{aligned}$$

Remark. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} : \mathcal{X} \to \mathcal{X},$$

be an unbounded linear operator, such that the inverse operator

$$(\lambda\mathcal{I} - \mathcal{A})^{-1} \in \mathbb{C}(\mathcal{X} \to \mathcal{X}),$$

is a compact operator, for some complex constant $\lambda \in \mathbb{C}$.

For example, let

$$\mathcal{X} = L^2(\mathbb{R}^n),$$

and

$$\mathcal{A} = \triangle = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

$$\triangle : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is an unbounded linear operator. What is the influence on the spectrum of unbounded linear operators?

Chapter 3: Hilbert Spaces and Bounded Linear Operators

Section 1: Hilbert Spaces and Elementary Pproperties

Definition 1. Let $\mathcal{H}$ be a complex or real vector space, let $(\cdot,\cdot) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ be a function. Then the space $\mathcal{H}$ is called an inner product space, if

$$\begin{aligned}
&(1) \quad (\mathbf{x},\mathbf{x}) > 0 \text{ for all } \mathbf{x} \neq \mathbf{0}, \qquad (\mathbf{x},\mathbf{x}) = 0 \text{ for } \mathbf{x} = \mathbf{0},\\
&(2) \quad (\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x},\mathbf{z}) + \beta(\mathbf{y},\mathbf{z}),\\
&(3) \quad \overline{(\mathbf{x},\mathbf{y})} = (\mathbf{y},\mathbf{x}),
\end{aligned}$$

for all $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathcal{H}$ and for all constants $\alpha, \beta \in \mathbb{C}$.

Theorem 1. Let $\mathcal{H}$ be an inner product space. There holds the Schwarz's inequality

$$|(\mathbf{x},\mathbf{y})|^2 \leq (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y}),$$

for all $\mathbf{x},\mathbf{y} \in \mathcal{H}$.

Theorem 2. Let $\mathcal{H}$ be an inner product space with the inner product $(\cdot,\cdot)$. Define $\|\cdot\|$ by

$$\|\mathbf{x}\|^2 = (\mathbf{x},\mathbf{x}),$$

for all $\mathbf{x} \in \mathcal{H}$. Then $(\mathcal{H}, \|\cdot\|)$ is a normed linear space.

Theorem 3. Let $\mathcal{H}$ be an inner product space, let

$$\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}_0\| = 0, \qquad \lim_{n\to\infty} \|\mathbf{y}_n - \mathbf{y}_0\| = 0.$$

Then

$$\lim_{n\to\infty} (\mathbf{x}_n, \mathbf{y}_n) = (\mathbf{x}_0, \mathbf{y}_0).$$

In another word, the inner product is a continuous function of the two variables.

Theorem 4. Let $\mathcal{H}$ be an inner product space. Then

$$\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2=2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2,$$

for all $\mathbf{x},\mathbf{y}\in\mathcal{H}$.

Let $(\mathcal{H},\|\cdot\|)$ be a normed linear space, such that

$$\|\mathbf{x}+\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2=2\|\mathbf{x}\|^2+2\|\mathbf{y}\|^2,$$

for all $\mathbf{x},\mathbf{y}\in\mathcal{H}$. Then $(\mathcal{H},\|\cdot\|)$ is an inner product space.

If $\mathcal{H}$ is a real vector space, then

$$(\mathbf{x},\mathbf{y})=\frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2-\|\mathbf{x}-\mathbf{y}\|^2\right),$$

for all $\mathbf{x},\mathbf{y}\in\mathcal{H}$.

If $\mathcal{H}$ is a complex space, then

$$(\mathbf{x},\mathbf{y})=\frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2-\|\mathbf{x}-\mathbf{y}\|^2+\mathrm{i}\|\mathbf{x}+\mathrm{i}\mathbf{y}\|^2-\mathrm{i}\|\mathbf{x}-\mathrm{i}\mathbf{y}\|^2\right),$$

for all $\mathbf{x},\mathbf{y}\in\mathcal{H}$.

Definition 2. Let $\mathcal{H}$ be an inner product space. $\mathcal{H}$ is called a Hilbert space, if $\mathcal{H}$ is complete.

Examples of Hilbert spaces:

(1) $L^2(\mathbb{R})$

(2) $L^2(\mathbb{R}^n)$

(3) $\mathcal{H}^{2m}(\mathbb{R}^n)=\left\{\phi\in L^2(\mathbb{R}^n):\int_{\mathbb{R}^n}(1+|\xi|^2)^m|\widehat{\phi}(\xi)|^2\mathrm{d}\xi<\infty\right\}.$

Remark. The normed linear space $L^p(\mathbb{R}^n)$ is NOT a Hilbert space, where $1<p<\infty$ and $p\neq 2$. The normed linear space $C(\mathbb{R})$ is NOT a Hilbert space.

Section 2: The Projection Theorem

Definition 1. Let $\mathcal{H}$ be an inner product space, let $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. We say $\mathbf{x}$ and $\mathbf{y}$ are orthogonal and we write $\mathbf{x} \perp \mathbf{y}$, if

$$(\mathbf{x}, \mathbf{y}) = 0.$$

Let $\mathcal{E}$ and $\mathcal{F}$ be two sets in $\mathcal{H}$. We say $\mathcal{E}$ and $\mathcal{F}$ are orthogonal and we write $\mathcal{E} \perp \mathcal{F}$, if

$$(\mathbf{x}, \mathbf{y}) = 0,$$

for all $\mathbf{x} \in \mathcal{E}$ and for all $\mathbf{y} \in \mathcal{F}$.

Theorem 1. Let $\mathcal{H}$ be an inner product space, let $\mathbf{x}, \mathbf{y} \in \mathcal{H}$, let $\mathcal{E}$ and $\mathcal{F}$ be subspaces of $\mathcal{H}$. There hold the following elementary properties:

(1) $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$, if $(\mathbf{x}, \mathbf{y}) = 0$.

(2) $\mathbf{x} \perp \mathcal{H}$, if and only if $\mathbf{x} = \mathbf{0}$.

(3) $\mathcal{E}^\perp$ is a closed subspace of $\mathcal{H}$.

(4) $\mathcal{E} \cap \mathcal{E}^\perp = \{\mathbf{0}\}$.

(5) If $\mathcal{E} \subset \mathcal{F}$, then $\mathcal{E}^\perp \supset \mathcal{F}^\perp$.

(6) If $\mathcal{E} \perp \mathcal{F}$, then $\mathcal{E} \cap \mathcal{F} = \{\mathbf{0}\}$.

Definition 2. Let $\mathcal{H}$ be an inner product space, let $\mathcal{E} \subset \mathcal{H}$ and $\mathcal{F} \subset \mathcal{H}$ be two subspaces. If $\mathcal{E} \perp \mathcal{F}$, then we define the direct sum

$$\mathcal{E} \oplus \mathcal{F} = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in \mathcal{E}, \mathbf{y} \in \mathcal{F}\}.$$

Theorem 2. Let $\mathcal{H}$ be an inner product space, let $\mathcal{E} \subset \mathcal{H}$ and $\mathcal{F} \subset \mathcal{H}$ be subspaces. Suppose that

$$\mathcal{H} = \mathcal{E} + \mathcal{F}.$$

Then it is a direct sum:

$$\mathcal{H} = \mathcal{E} \oplus \mathcal{F},$$

if and only if

$$\mathcal{E}^{\perp} = \mathcal{F}, \qquad \mathcal{F}^{\perp} = \mathcal{E}.$$

Theorem 3. Let $\mathcal{H}$ be an inner product space, let $\mathcal{E} \subset \mathcal{H}$ be a subspace and let $\mathbf{x}_1 \in \mathcal{H}$. If $\mathbf{x}_0$ is the projection of $\mathbf{x}_1$ onto $\mathcal{E}$, then

$$\|\mathbf{x}_1 - \mathbf{x}_0\| = \inf_{\mathbf{z} \in \mathcal{E}} \|\mathbf{x}_1 - \mathbf{z}\|.$$

Theorem 4. Let $\mathcal{H}$ be an inner product space, let $\mathcal{E} \subset \mathcal{H}$ be a complete convex set and let $\mathbf{x}_1 \in \mathcal{H}$. Define

$$\rho = \rho(\mathbf{x}_1, \mathcal{E}) = \inf_{\mathbf{z} \in \mathcal{E}} \|\mathbf{x}_1 - \mathbf{z}\|.$$

Then there exists a unique $\mathbf{x}_0 \in \mathcal{H}$, such that

$$\|\mathbf{x}_1 - \mathbf{x}_0\| = \rho(\mathbf{x}_1, \mathcal{E}).$$

Theorem 5. Let $\mathcal{H}$ be an inner product space, let $\mathcal{E} \subset \mathcal{H}$ be a subspace, let $\mathbf{x}_1 \in \mathcal{H}$ and $\mathbf{x}_0 \in \mathcal{E}$. If

$$\|\mathbf{x}_1 - \mathbf{x}_0\| = \inf_{\mathbf{z} \in \mathcal{E}} \|\mathbf{x}_1 - \mathbf{z}\|,$$

then $\mathbf{x}_1 - \mathbf{x}_0 \perp \mathcal{E}$.

Theorem 6. The Projection Theorem Let $\mathcal{H}$ be an inner space, let $\mathcal{E}$ be a complete subspace. For any vector $\mathbf{x}_1 \in \mathcal{H}$, there exists a unique projection $\mathbf{x}_0 \in \mathcal{E}$, such that

$$\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{x}', \qquad \mathbf{x}_0 \in \mathcal{E}, \mathbf{x}' \perp \mathcal{E}.$$

Theorem 7. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{E} \subset \mathcal{H}$ be a subspace. Then

$$\begin{aligned} &\mathcal{E}^{\perp} \neq \emptyset, \text{ if } \mathcal{E} \neq \mathcal{H} \\ &\overline{\mathcal{E}} = (\mathcal{E}^{\perp})^{\perp}. \end{aligned}$$

In particular, if $\mathcal{E}^{\perp} = \{\mathbf{0}\}$, then $\mathcal{E}$ is dense in $\mathcal{H}$.

Section 3: Complete Orthonormal Sets in Hilbert Spaces

Definition 1. Let $\mathcal{H}$ be an inner product space, let $\mathcal{S}$ be a subset of $\mathcal{H}$. If

$$(\mathbf{e}, \mathbf{f}) = 0, \text{ for all } \mathbf{e}, \mathbf{f} \in \mathcal{S}, \mathbf{e} \neq \mathbf{f},$$

then we say $\mathcal{S}$ is an orthogonal set.

If $\mathcal{S}$ is an orthogonal set and

$$\|\mathbf{e}\| = 1, \text{ for all } \mathbf{e} \in \mathcal{S},$$

then we say $\mathcal{S}$ is an orthonormal set. In this case, we call the set

$$\{(\mathbf{x}, \mathbf{e}) : \mathbf{e} \in \mathcal{S}\}$$

the set of Fourier coefficients of the vector $\mathbf{x} \in \mathcal{H}$ relative to the orthonormal set $\mathcal{S}$.

Theorem 1. Let $\mathcal{H}$ be an inner product space, let

$$\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \cdots, \mathbf{e}_n\},$$

be an orthonormal set, let

$$\mathcal{E} = \text{ span } \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \cdots, \mathbf{e}_n\}.$$

For any $\mathbf{x} \in \mathcal{H}$, the projection of $\mathbf{x}$ onto $\mathcal{E}$ is given by

$$\mathbf{x}_0 = (\mathbf{x}, \mathbf{e}_1)\mathbf{e}_1 + (\mathbf{x}, \mathbf{e}_2)\mathbf{e}_2 + (\mathbf{x}, \mathbf{e}_3)\mathbf{e}_3 + \cdots + (\mathbf{x}, \mathbf{e}_n)\mathbf{e}_n.$$

Moreover, we have

$$\begin{aligned}
&(1) \quad \|\mathbf{x}_0\|^2 = \sum_{k=1}^{n} |(\mathbf{x}, \mathbf{e}_k)|^2 \\
&(2) \quad \|\mathbf{x}\|^2 = \|\mathbf{x}_0\|^2 + \|\mathbf{x} - \mathbf{x}_0\|^2 \\
&(3) \quad \sum_{k=1}^{n} |(\mathbf{x}, \mathbf{e}_k)|^2 \leq \|\mathbf{x}\|^2 \\
&(4) \quad \left\| \mathbf{x} - \sum_{k=1}^{n} (\mathbf{x}, \mathbf{e}_k)\mathbf{e}_k \right\| \leq \|\mathbf{x} - \sum_{k=1}^{n} \alpha_k \mathbf{e}_k\|,
\end{aligned}$$

for any constants $\alpha_1 \in \mathbb{R}$, $\alpha_2 \in \mathbb{R}$, $\alpha_3 \in \mathbb{R}$, $\cdots$, $\alpha_n \in \mathbb{R}$. The equal sign $=$ holds only if $\alpha_k = (\mathbf{x}, \mathbf{e}_k)$.

Theorem 2. Let $\mathcal{H}$ be an inner product space, let

$$\mathcal{S} = \{\mathbf{e}_\lambda : \lambda \in \Lambda\}$$

be an orthonormal set in $\mathcal{H}$. Then for any $\mathbf{x} \in \mathcal{H}$, there are at most countably many nonzero Fourier coefficients in the set $\{(\mathbf{x}, \mathbf{e}_\lambda) : \lambda \in \Lambda\}$. Moreover, there holds the estimate

$$\sum_{\lambda \in \Lambda} |(\mathbf{x}, \mathbf{e}_\lambda)|^2 \leq \|\mathbf{x}\|^2.$$

$$\lim_{n \to \infty} (\mathbf{x}, \mathbf{e}_n) = 0.$$

Definition 2. Let $\mathcal{H}$ be an inner product space, let

$$\mathcal{S} = \{\mathbf{e}_\lambda : \lambda \in \Lambda\}$$

be an orthonormal set in $\mathcal{H}$. Then $\mathcal{S}$ is called a complete orthonormal set, if there holds the Parseval's identity

$$\|\mathbf{x}\|^2 = \sum_{\lambda \in \Lambda} |(\mathbf{x}, \mathbf{e}_\lambda)|^2,$$

for all $\mathbf{x} \in \mathcal{H}$.

Definition 3. Let $\mathcal{H}$ be an inner product space, let

$$\mathcal{S} = \{\mathbf{e}_\lambda : \lambda \in \Lambda\}$$

be an orthonormal set in $\mathcal{H}$. Let $\mathbf{x} \in \mathcal{H}$. We call the formal series

$$\sum_{\lambda \in \Lambda} (\mathbf{x}, \mathbf{e}_\lambda)\mathbf{e}_\lambda,$$

the Fourier series of $\mathbf{x}$ relative to the orthonormal set.

Example 1. Let

$$\mathcal{H} = L^2[0, 2\pi], \qquad (f, g) = \frac{1}{\pi}\int_0^{2\pi} f(x)g(x)\mathrm{d}x,$$

for all $f, g \in L^2[0, 2\pi]$.

The set

$$\mathcal{S} = \left\{\frac{1}{\sqrt{2}}, \cos x, \sin x, \cos(2x), \sin(2x), \cos(3x), \sin(3x), \cdots\cdots\right\}$$

is a complete, orthonormal set.

Example 2. Let

$$\mathcal{H} = L^2(\mathbb{R}), \qquad (f, g) = \int_{\mathbb{R}} f(x)g(x)\mathrm{d}x,$$

for all $f, g \in L^2(\mathbb{R})$.

The set

$$\mathcal{S} = \left\{\frac{1}{(m!2^m\sqrt{\pi})^{1/2}} \exp\left(\frac{1}{2}x^2\right) \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\exp(-x^2)\right] : m = 0, 1, 2, 3, \cdots\cdots\right\}$$

is a complete orthonormal set in $L^2(\mathbb{R})$.

Theorem 3. Let $\mathcal{H}$ be an inner product space, $\mathbf{x} \in \mathcal{H}$, let $\{\mathbf{e}_\lambda : \lambda \in \Lambda\}$ be an orthonormal set and let

$$\mathcal{E} = \text{ span } \{\mathbf{e}_\lambda : \lambda \in \Lambda\}.$$

The following statements are equivalent to each other:

$$\begin{aligned} &(1) \quad \mathbf{x} \in \mathcal{E} \\ &(2) \quad \|\mathbf{x}\|^2 = \sum_{\lambda \in \Lambda} |(\mathbf{x}, \mathbf{e}_\lambda)|^2 \\ &(3) \quad \mathbf{x} = \sum_{\lambda \in \Lambda} (\mathbf{x}, \mathbf{e}_\lambda)\mathbf{e}_\lambda \end{aligned}$$

Theorem 4. Let $\mathcal{H}$ be an inner product space, let $\mathbf{x} \in \mathcal{H}$. Let

$$\mathcal{S} = \{\mathbf{e}_\lambda : \lambda \in \Lambda\}$$

be an orthonormal set in $\mathcal{H}$. Let

$$\mathcal{E} = \overline{\text{span } \{\mathbf{e}_\lambda : \lambda \in \Lambda\}}.$$

If there exists a projection $\mathbf{x}_0$ of $\mathbf{x}$ onto $\mathcal{E}$, then

$$\mathbf{x}_0 = \sum_{\lambda \in \Lambda} (\mathbf{x}, \mathbf{e}_\lambda)\mathbf{e}_\lambda.$$

Let $\mathcal{H}$ be a Hilbert space. Then for any $\mathbf{x} \in \mathcal{H}$, the projection of $\mathbf{x}$ is given by

$$\mathbf{x}_0 = \sum_{\lambda \in \Lambda} (\mathbf{x}, \mathbf{e}_\lambda)\mathbf{e}_\lambda.$$

Theorem 5. Let $\mathcal{H}$ be a Hilbert space, let

$$\mathcal{S} = \{\mathbf{e}_\lambda : \lambda \in \Lambda\}$$

be an orthonormal set in $\mathcal{H}$. Let

$$\mathcal{E} = \overline{\text{span } \{\mathbf{e}_\lambda : \lambda \in \Lambda\}}.$$

Let

$$\{C_\lambda : \lambda \in \Lambda\}$$

be a set of numbers, such that

$$\sum_{\lambda\in\Lambda} |C_\lambda|^2 < \infty.$$

Then there exists a unique point $\mathbf{x}_0 \in \mathcal{E}$, such that

$$\mathbf{x}_0 = \sum_{\lambda\in\Lambda} C_\lambda \mathbf{e}_\lambda,$$
$$C_\lambda = (\mathbf{x}_0, \mathbf{e}_\lambda), \text{ for all } \lambda \in \Lambda.$$

Definition 4. Let $\mathcal{H}$ be an inner product space, let

$$\mathcal{S} = \{\mathbf{e}_\lambda : \lambda \in \Lambda\}$$

be an orthonormal set in $\mathcal{H}$. The set $\mathcal{S}$ is called seamless if $\mathcal{S}^\perp = \{\mathbf{0}\}$.

Theorem 6. Let $\mathcal{H}$ be an inner product space, let

$$\{\mathbf{e}_\lambda : \lambda \in \Lambda\}$$

be an orthonormal set in $\mathcal{H}$. If it is complete, then it is seamless.

Let $\mathcal{H}$ be a Hilbert space. Then any seamless orthonormal set is complete.

Definition 5. Let $\mathcal{H}$ and $\mathcal{K}$ be inner product spaces. We say $\mathcal{H}$ and $\mathcal{K}$ are product-preserving isomorphic, if there exists a linear, one-to-one, onto operator

$$\mathcal{A} : \mathcal{H} \to \mathcal{K},$$

such that

$$\begin{aligned} &(1) \quad (\mathcal{A}\mathbf{x}, \mathcal{A}\mathbf{y}) = (\mathbf{x}, \mathbf{y}), \\ &(2) \quad \mathcal{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathcal{A}\mathbf{x} + \beta\mathcal{A}\mathbf{y}, \end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$ and for all $\alpha, \beta \in \mathbb{C}$.

Theorem 7. Any finite-dimensional inner product space must be isomorphic to some Euclidean space $\mathbb{R}^n$.

Any separable Hilbert space must be isomorphic to some Euclidean space $\mathbb{R}^n$ or isomorphic to l^2.

Section 4: Dual Space and Adjoint Operator

Theorem 1. Let $\mathcal{H}$ be a Hilbert space, let $f \in \mathcal{H}^*$. Then there exists a unique point $\mathbf{y} \in \mathcal{H}$, such that

$$\begin{aligned} &(1) \quad \|f\| = \|\mathbf{y}\|, \\ &(2) \quad f(\mathbf{x}) = (\mathbf{x}, \mathbf{y}), \end{aligned}$$

for all $\mathbf{x} \in \mathcal{H}$.

Theorem 2. Let $\mathcal{H}$ be a Hilbert space. Define the mapping

$$\begin{aligned} &\mathcal{A} : \mathcal{H} \to \mathcal{H}^*, \\ &\mathcal{A}\mathbf{y} = f = (\cdot, \mathbf{y}), \qquad \mathbf{y} \in \mathcal{H}, \end{aligned}$$

where $f(\mathbf{x}) = (\mathbf{x}, \mathbf{y})$, for all $\mathbf{x} \in \mathcal{H}$. Then $\mathcal{A}$ is one-to-one, onto, and

$$\begin{aligned} &(1) \quad \|\mathcal{A}\mathbf{x}\| = \|\mathbf{x}\|, \\ &(2) \quad \mathcal{A}(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \overline{\alpha}\mathcal{A}\mathbf{x}_1 + \overline{\beta}\mathcal{A}\mathbf{x}_2, \end{aligned}$$

for all $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{H}$.

Theorem 3. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{K}$ be an inner product space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{K}).$$

Then there exists a unique bounded linear operator

$$\mathcal{B} \in \mathbb{B}(\mathcal{K} \to \mathcal{H}),$$

such that

$$(\mathcal{A}\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathcal{B}\mathbf{y}),$$

for all $\mathbf{x} \in \mathcal{H}$ and for all $\mathbf{y} \in \mathcal{K}$.

Definition 1. Let $\mathcal{H}$ and $\mathcal{K}$ be inner product spaces, let

$$\begin{aligned} &(1) \quad \mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{K}), \\ &(2) \quad \mathcal{A}^* \in \mathbb{B}(\mathcal{K} \to \mathcal{H}). \end{aligned}$$

The mapping $\mathcal{A}^*$ is called the adjoint operator of $\mathcal{A}$, if

$$(\mathcal{A}\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathcal{A}^*\mathbf{y}),$$

for all $\mathbf{x} \in \mathcal{H}$ and for all $\mathbf{y} \in \mathcal{K}$.

Theorem 4. Let $\mathcal{X}$ and $\mathcal{H}$ be Hilbert spaces, let $\mathcal{K}$ be an inner product space. Let $\alpha, \beta \in \mathbb{C}$ be constants, let

$$\begin{aligned} &(1) \quad \mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{K}), \\ &(2) \quad \mathcal{B} \in \mathbb{B}(\mathcal{H} \to \mathcal{K}), \\ &(3) \quad \mathcal{C} \in \mathbb{B}(\mathcal{X} \to \mathcal{H}). \end{aligned}$$

Then

$$\begin{aligned} &(1) \quad (\mathcal{A}^*)^* = \mathcal{A}. \\ &(2) \quad \|\mathcal{A}^*\mathcal{A}\| = \|\mathcal{A}^*\|^2 = \|\mathcal{A}\|^2. \\ &(3) \quad (\alpha\mathcal{A} + \beta\mathcal{B})^* = \overline{\alpha}\mathcal{A}^* + \overline{\beta}\mathcal{B}^*. \\ &(4) \quad (\mathcal{A}\mathcal{C})^* = \mathcal{C}^*\mathcal{A}^*. \\ &(5) \quad \lambda = 0 \in \rho(\mathcal{A}), \text{ iff } \lambda = 0 \in \rho(\mathcal{A}^*). \\ &(6) \quad (\mathcal{A}^*)^{-1} = (\mathcal{A}^{-1})^*, \text{ if } \lambda = 0 \in \rho(\mathcal{A}). \end{aligned}$$

Theorem 5. Let $\mathcal{H}$ be a complex Hilbert space and let

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}).$$

Then

$$\begin{array}{ll}(1) & \rho(\mathcal{A}^*) = \overline{\rho(\mathcal{A})}, \\ (2) & \sigma(\mathcal{A}^*) = \overline{\sigma(\mathcal{A})}.\end{array}$$

Let $\xi \in \mathcal{N}(\lambda\mathcal{I} - \mathcal{A})$ and $\eta \in \mathcal{N}((\mu\mathcal{I} - \mathcal{A})^*)$. Then $\xi \perp \eta$, if $\lambda \neq \mu$.

Theorem 6. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}).$$

Then

$$\begin{array}{ll}(1) & \overline{\mathcal{A}\mathcal{H}} = [\mathcal{N}(\mathcal{A}^*)]^\perp. \\ (2) & \overline{\mathcal{A}^*\mathcal{H}^*} = [\mathcal{N}(\mathcal{A})]^\perp. \\ (3) & (\mathcal{A}\mathcal{H})^\perp = \mathcal{N}(\mathcal{A}^*). \\ (4) & (\mathcal{A}^*\mathcal{H}^*)^\perp = \mathcal{N}(\mathcal{A}).\end{array}$$

Definition 2. Let $\mathcal{H}$ be a Hilbert space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}).$$

The operator $\mathcal{A}$ is called self-adjoint, if $\mathcal{A}^* = \mathcal{A}$.

Theorem 7. Let $\mathcal{H}$ be a complex Hilbert space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}).$$

Then $\mathcal{A}$ is self-adjoint, if and only if

$$(\mathcal{A}\mathbf{x}, \mathbf{x}) \in \mathbb{R},$$

for all $\mathbf{x} \in \mathcal{H}$.

There hold the following polarized identities

$$\begin{aligned}
& (\mathcal{A}\mathbf{x},\mathbf{y}) \\
= \;& \frac{1}{4}\left\{[(\mathcal{A}(\mathbf{x}+\mathbf{y}),\mathbf{x}+\mathbf{y}) - (\mathcal{A}(\mathbf{x}-\mathbf{y}),\mathbf{x}-\mathbf{y})]\right. \\
+ \;& \left.\mathrm{i}[(\mathcal{A}(\mathbf{x}+\mathrm{i}\mathbf{y}),\mathbf{x}+\mathrm{i}\mathbf{y})(\mathcal{A}(\mathbf{x}-\mathrm{i}\mathbf{y}),\mathbf{x}-\mathrm{i}\mathbf{y})]\right\}, \\
& (\mathcal{A}\mathbf{y},\mathbf{x}) \\
= \;& \frac{1}{4}\left\{[(\mathcal{A}(\mathbf{x}+\mathbf{y}),\mathbf{x}+\mathbf{y}) - (\mathcal{A}(\mathbf{x}-\mathbf{y}),\mathbf{x}-\mathbf{y})]\right. \\
- \;& \left.\mathrm{i}[(\mathcal{A}(\mathbf{x}+\mathrm{i}\mathbf{y}),\mathbf{x}+\mathrm{i}\mathbf{y})(\mathcal{A}(\mathbf{x}-\mathrm{i}\mathbf{y}),\mathbf{x}-\mathrm{i}\mathbf{y})]\right\},
\end{aligned}$$

for all $\mathbf{x},\mathbf{y}\in\mathcal{H}$.

Therefore, we see that

$$(\mathcal{A}\mathbf{x},\mathbf{y}) = \overline{(\mathcal{A}\mathbf{y},\mathbf{x})} = (\mathbf{x},\mathcal{A}\mathbf{y}),$$

for all $\mathbf{x},\mathbf{y}\in\mathcal{H}$.

Theorem 8. Let $\mathcal{H}$ be a Hilbert space, let

$$\begin{aligned}
&(1) \quad \mathcal{A}\in\mathbb{B}(\mathcal{H}\to\mathcal{H}). \\
&(2) \quad \mathcal{B}\in\mathbb{B}(\mathcal{H}\to\mathcal{H}). \\
&(3) \quad \mathcal{A}_n\in\mathbb{B}(\mathcal{H}\to\mathcal{H}),
\end{aligned}$$

be self-adjoint operators, where $n=1,2,3,\cdots$. Then

$$\alpha\mathcal{A}+\beta\mathcal{B}$$

is self-adjoint, for any $\alpha,\beta\in\mathbb{R}$.

If

$$\lim_{n\to\infty}\mathcal{A}_n \overset{\text{stronglyorweakly}}{=\!=\!=\!=\!=} \mathcal{A}_0,$$

then $\mathcal{A}_0$ is self-adjoint.

Section 5: Projection Operators and Their Operations

Definition 1. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{E}$ be a closed subspace of $\mathcal{H}$. Define the projection operator

$$\mathcal{P} : \mathcal{H} \to \mathcal{E},$$
$$\mathbf{x} = \mathcal{P}\mathbf{x} + \mathbf{x}',$$

for all $\mathbf{x} \in \mathcal{H}$, where $\mathcal{P}\mathbf{x}$ is the projection of $\mathbf{x}$ onto $\mathcal{E}$, $\mathbf{x}' \perp \mathcal{E}$.

Theorem 1. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{E}$ be a closed subspace, let

$$\mathcal{P} : \mathcal{H} \to \mathcal{E},$$

be a projection operator. Then

$$\begin{aligned} &(1) \quad \mathcal{P} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}). \\ &(2) \quad \mathcal{P}\mathcal{H} = \mathcal{E} = \{\mathbf{x} \in \mathcal{H} : \mathcal{P}\mathbf{x} = \mathbf{x}\}. \\ &(3) \quad \|\mathcal{P}\| = 1, \text{ if } \mathcal{E} \neq \{\mathbf{0}\}. \\ &(4) \quad \|\mathcal{P}\| = 0, \text{ if } \mathcal{E} = \{\mathbf{0}\}. \end{aligned}$$

Theorem 2. Let $\mathcal{H}$ be a Hilbert space, let

$$\mathcal{P} : \mathcal{H} \to \mathcal{H}$$

be a linear operator. Then $\mathcal{P}$ is a projection operator, if and only if

$$\mathcal{P}^2 = \mathcal{P}, \qquad \mathcal{P}^* = \mathcal{P}.$$

Theorem 3. Let $\mathcal{H}$ be a complex Hilbert space, let

$$\mathcal{P} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}).$$

Then $\mathcal{P}$ is a projection operator, if and only if

$$\|\mathcal{P}\mathbf{x}\|^2 = (\mathcal{P}\mathbf{x}, \mathbf{x}),$$

for all $\mathbf{x} \in \mathcal{H}$.

Theorem 4. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{E}$ and $\mathcal{F}$ be closed subspaces, let

$$\begin{aligned} &(1) \quad \mathcal{P} : \mathcal{H} \to \mathcal{E}, \\ &(2) \quad \mathcal{Q} : \mathcal{H} \to \mathcal{F}, \end{aligned}$$

be projection operators. Then

(1) $\mathcal{E} \perp \mathcal{F}$, if and only if $\mathcal{P}\mathcal{Q} = \mathbf{0}$.

(2) $\mathcal{P} + \mathcal{Q}$ is a projection operator, if and only if $\mathcal{P}\mathcal{Q} = \mathbf{0}$. If $\mathcal{P} + \mathcal{Q}$ is a projection operator, then $(\mathcal{P} + \mathcal{Q})\mathcal{H} = \mathcal{E} \oplus \mathcal{F}$.

(3) $\mathcal{P}\mathcal{Q}$ is a projection operator, if and only if $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P}$. If $\mathcal{P}\mathcal{Q}$ is a projection operator, then $\mathcal{P}\mathcal{Q}\mathcal{H} = \mathcal{E} \cap \mathcal{F}$.

(4) $\mathcal{P} - \mathcal{Q}$ is a projection operator, if and only if $\mathcal{E} \supset \mathcal{F}$. If $\mathcal{P} - \mathcal{Q}$ is a projection operator, then $(\mathcal{P} - \mathcal{Q})\mathcal{H} = \mathcal{E} \ominus \mathcal{F}$.

(5) $\mathcal{I} - \mathcal{P}$ is a projection operator and $(\mathcal{I} - \mathcal{P})\mathcal{H} = \mathcal{E}^{\perp}$.

(6) $\mathcal{P}\mathcal{Q}$ is a projection operator, if and only if $[\mathcal{E} \ominus (\mathcal{E} \cap \mathcal{F})] \perp [\mathcal{F} \ominus (\mathcal{E} \cap \mathcal{F})]$.

(7) Let $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P}$. Then $\mathcal{P} - \mathcal{P}\mathcal{Q} + \mathcal{Q}$ is a projection operator onto $[\mathcal{F} - (\mathcal{E} \cap \mathcal{F})] \oplus \mathcal{E}$.

Theorem 5. Let $\mathcal{H}$ be a Hilbert space, let

$$\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \cdots, \mathcal{E}_n, \cdots$$

be a sequence of mutually perpendicular, closed subspaces of $\mathcal{H}$, let

$$\begin{aligned}
&\mathcal{P}_1 : \mathcal{H} \to \mathcal{E}_1, \\
&\mathcal{P}_2 : \mathcal{H} \to \mathcal{E}_2, \\
&\mathcal{P}_3 : \mathcal{H} \to \mathcal{E}_3, \\
&\cdots\cdots \\
&\mathcal{P}_n : \mathcal{H} \to \mathcal{E}_n, \\
&\cdots\cdots
\end{aligned}$$

be a sequence of mutually perpendicular projection operators. Then there exists a projection operator $\mathcal{P}$:

$$\mathcal{P}\mathbf{x} = \mathcal{P}_1\mathbf{x} + \mathcal{P}_2\mathbf{x} + \mathcal{P}_3\mathbf{x} + \cdots + \mathcal{P}_n\mathbf{x} + \cdots ,$$

for all $\mathbf{x} \in \mathcal{H}$.

Define

$$\begin{aligned}
\mathcal{E} &= \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3 \oplus \cdots \oplus \mathcal{E}_n \oplus \cdots \\
&= \left\{ \sum_{n=1}^{\infty} \mathbf{x}_n : \sum_{n=1}^{\infty} \|\mathbf{x}_n\|^2 < \infty, \mathbf{x}_n \in \mathcal{E}_n, n = 1, 2, 3, \cdots \right\}.
\end{aligned}$$

Then

$$\mathcal{P}\mathcal{H} = \mathcal{E}.$$

Definition 2. Let $\mathcal{H}$ be a Hilbert space, let

$$\begin{aligned}
&(1) \quad \mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}), \\
&(2) \quad \mathcal{B} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}),
\end{aligned}$$

be bounded linear, self-adjoint operators. We say $\mathcal{A} \leq \mathcal{B}$, if

$$(\mathcal{A}\mathbf{x}, \mathbf{x}) \leq (\mathcal{B}\mathbf{x}, \mathbf{x}),$$

for all $\mathbf{x} \in \mathcal{H}$.

Theorem 6. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{E}$ and $\mathcal{F}$ be closed subspaces of $\mathcal{H}$, let

$$
\begin{aligned}
&(1) \quad \mathcal{P} : \mathcal{H} \to \mathcal{E}, \\
&(2) \quad \mathcal{Q} : \mathcal{H} \to \mathcal{F},
\end{aligned}
$$

be projection operators. Then the following statements are equivalent:

(1) $\mathcal{P} \geq \mathcal{Q}$.

(2) $\|\mathcal{P}\mathbf{x}\| \geq \|\mathcal{Q}\mathbf{x}\|$, for all $\mathbf{x} \in \mathcal{H}$.

(3) $\mathcal{E} \supset \mathcal{F}$.

(4) $\mathcal{P}\mathcal{Q} = \mathcal{Q}$.

(5) $\mathcal{Q}\mathcal{P} = \mathcal{Q}$.

Theorem 7. Let $\mathcal{H}$ be a Hilbert space, let

$$
\begin{aligned}
&\mathcal{P}_1 \leq \mathcal{P}_2 \leq \mathcal{P}_3 \leq \cdots \leq \mathcal{P}_n \leq \cdots \text{or} \\
&\mathcal{P}_1 \geq \mathcal{P}_2 \geq \mathcal{P}_3 \geq \cdots \geq \mathcal{P}_n \geq \cdots
\end{aligned}
$$

Then there exists a projection operator $\mathcal{P}$, such that

$$\lim_{n\to\infty} \|\mathcal{P}_n\mathbf{x} - \mathcal{P}\mathbf{x}\| = 0,$$

for all $\mathbf{x} \in \mathcal{H}$.

Theorem 8. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{E}$ be a closed subspace, let

$$\mathcal{P} : \mathcal{H} \to \mathcal{E}$$

be a projection operator. Let

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}).$$

Then $\mathcal{E}$ is an invariant subspace of $\mathcal{A}$, if and only if

$$\mathcal{PAP} = \mathcal{AP}.$$

Theorem 9. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{E}$ be a closed subspace, let

$$\mathcal{P} : \mathcal{H} \to \mathcal{E}$$

be a projection operator. Let

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}).$$

Then both $\mathcal{E}$ and $\mathcal{E}^{\perp}$ are invariant subspaces of $\mathcal{A}$, if and only if

$$\mathcal{AP} = \mathcal{PA};$$

if and only if $\mathcal{E}$ is a subspace of both $\mathcal{A}$ and $\mathcal{A}^{*}$.

Remark. Let $\mathcal{A}$ be a self-adjoint operator and let $\mathcal{E}$ be an invariant subspace of $\mathcal{A}$. Then $\mathcal{E}^{\perp}$ is also an invariant subspace of $\mathcal{A}$.

Section 6: Bilinear Hermite Functionals and Self-Adjoint Operators

Definition 1. Let $\mathcal{H}$ be a vector space, let $f : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ or $\mathbb{C}$ be a functional. f is called a bilinear functional, if

$$\begin{aligned} &(1) \quad f(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \alpha f(\mathbf{x}, \mathbf{z}) + \beta f(\mathbf{y}, \mathbf{z}), \\ &(2) \quad f(\mathbf{x}, \alpha\mathbf{y} + \beta\mathbf{z}) = \overline{\alpha} f(\mathbf{x}, \mathbf{y}) + \overline{\beta} f(\mathbf{x}, \mathbf{z}), \end{aligned}$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$.

Definition 2. Let $\mathcal{H}$ be a vector space, let $f : \mathcal{H} \times \mathcal{H} \to \mathcal{C}$ be a functional. f is called Hermite functional, if

$$f(\mathbf{x}, \mathbf{y}) = \overline{f(\mathbf{y}, \mathbf{x})},$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$.

Definition 3. Let $\mathcal{H}$ be an inner product space, let

$$\mathcal{A} : \mathcal{H} \to \mathcal{H}$$

be a linear operator. Then

$$(\mathcal{A}\mathbf{x}, \mathbf{y})$$

is called the functional induced by $\mathcal{A}$. If $\mathcal{A}^* = \mathcal{A}$, then

$$(\mathcal{A}\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathcal{A}\mathbf{y}),$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. We call this the Hermite functional induced by $\mathcal{A}$.

Theorem 1. Let $\mathcal{H}$ be a complex vector space, let

$$f : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

be a bilinear functional. Then f is Hermite, if and only if

$$f(\mathbf{x}, \mathbf{x}) \in \mathbb{R},$$

for all $\mathbf{x} \in \mathcal{H}$.

There hold the following polarized identities

$$\begin{aligned} & f(\mathbf{x}, \mathbf{y}) \\ = & \frac{1}{4}\{[f(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) - f(\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y})] \\ + & \mathrm{i}[f(\mathbf{x}+\mathrm{i}\mathbf{y}, \mathbf{x}+\mathrm{i}\mathbf{y}) - f(\mathbf{x}-\mathrm{i}\mathbf{y}, \mathbf{x}-\mathrm{i}\mathbf{y})]\}, \\ & f(\mathbf{y}, \mathbf{x}) \\ = & \frac{1}{4}\{[f(\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}) - f(\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y})] \\ - & \mathrm{i}[f(\mathbf{x}+\mathrm{i}\mathbf{y}, \mathbf{x}+\mathrm{i}\mathbf{y}) - f(\mathbf{x}-\mathrm{i}\mathbf{y}, \mathbf{x}-\mathrm{i}\mathbf{y})]\}, \end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$.

Therefore, we see that

$$\overline{f(\mathbf{y},\mathbf{x})} = f(\mathbf{x},\mathbf{y}),$$

for all $\mathbf{x},\mathbf{y} \in \mathcal{H}$.

Definition 4. Let $\mathcal{H}$ be an inner product space, let

$$f : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

be a bilinear functional. f is called a bounded bilinear functional, if there exists a positive constant $C > 0$, such that

$$|f(\mathbf{x},\mathbf{y})| \leq C\|\mathbf{x}\|\|\mathbf{y}\|,$$

for all $\mathbf{x},\mathbf{y} \in \mathcal{H}$.

Theorem 2. Let $\mathcal{H}$ be an inner product space, let

$$f : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

be a bilinear Hermite functional. Suppose that there exists a positive constant $C > 0$, such that

$$|f(\mathbf{x},\mathbf{x})| \leq C\|\mathbf{x}\|^2,$$

for all $\mathbf{x} \in \mathcal{H}$. Then f is bounded and $\|f\| \leq C$.

Theorem 3. Let $\mathcal{H}$ be a Hilbert space, let

$$f : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

be a bounded, bilinear functional. Then there exists a bounded linear operator

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}),$$

such that

$$f(\mathbf{x},\mathbf{y}) = (\mathcal{A}\mathbf{x},\mathbf{y}),$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Moreover, $\mathcal{A}$ is self-adjoint, if f is Hermite.

Definition 5. Let $\mathcal{H}$ be an inner product space, let

$$f : \mathcal{H} \to \mathbb{C}$$

be a functional. f is called a functional, if

$$\begin{aligned} &(1) \quad f(\alpha\mathbf{x}) = |\alpha|^2 f(\mathbf{x}), \\ &(2) \quad f(\mathbf{x}+\mathbf{y}) + f(\mathbf{x}-\mathbf{y}) = 2f(\mathbf{x}) + 2f(\mathbf{y}), \end{aligned}$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Moreover, if

$$\sup_{\|\mathbf{x}\|=1} |f(\mathbf{x})| < \infty,$$

then

Theorem 5. Let $\mathcal{H}$ be a Hilbert space, let

$$f : \mathcal{H} \to \mathbb{R}$$

be a real bounded functional. Then there exists a unique, self-adjoint bounded linear operator $\mathcal{A}$,

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H})$$

such that

$$f(\mathbf{x}, \mathbf{y}) = (\mathcal{A}\mathbf{x}, \mathbf{y}),$$

for all $\mathbf{x} \in \mathcal{H}$. $\mathcal{A}\mathbb{B}\mathbb{B}$.

Let $0 < \alpha < 1$, $\lambda > 0$, $1 \leq p < \infty$ be positive constants. Let $m \geq 1$ and $n \geq 1$ be positive integers.

$$\mathcal{M} \overset{\text{def}}{=} \mathbb{R}^n$$

with the standard norm

$$\|\mathbf{u}\|^2 = |u_1|^2 + |u_2|^2 + |u_3|^2 + \cdots + |u_n|^2, \qquad \mathbf{u} \in \mathbb{R}^n$$

Define the following spaces

$$\mathbf{C}[a,b],$$

$$\mathbf{C}\left(\prod_{k=1}^{n}[a_k, b_k]\right),$$

with the standard norm

$$\|\phi\| = \max_{[a,b]} |\phi(x)|.$$

$$\mathbf{BC}^{m,\alpha}(\mathbb{R}) \overset{\text{def}}{=} \left\{\phi \in \mathbf{BC}(\mathbb{R}) : \frac{\partial^k}{\partial x^k}\phi \in \mathbf{BC}(\mathbb{R}), k = 1, 2, 3, \cdots, m \right.$$
$$\left. \sup_{x \neq y}\left[\frac{1}{|x-y|^\alpha}\left|\frac{\partial^m}{\partial x^m}\phi(x) - \frac{\partial^m}{\partial x^m}\phi(y)\right|\right] < \infty\right\}.$$

with the standard norm

$$\|\phi\|_{\mathbf{C}^{m,\alpha}(\mathbb{R})} \overset{\text{def}}{=} \sup_{x \in \mathbb{R}}\left\{\sum_{k=0}^{m}\left|\frac{\partial^k}{\partial x^k}\phi(x)\right|\right\}$$
$$+ \sup_{x,y \in \mathbb{R}, x \neq y}\left\{\frac{1}{|x-y|^\alpha}\left|\frac{\partial^m}{\partial x^m}\phi(x) - \frac{\partial^m}{\partial x^m}\phi(y)\right|\right\}.$$

$$\mathbf{H}^\lambda(\mathbb{R}^n) \overset{\text{def}}{=} \left\{\phi \in L^2(\mathbb{R}^n) : \qquad \int_{\mathbb{R}^n}(1 + |\xi|^2)^{\frac{\lambda}{2}}|\widehat{\phi}(\xi)|^2 \mathrm{d}\xi < \infty\right\}$$

with the norm defined by

$$\|\phi\|^2_{\mathbf{H}^\lambda(\mathbb{R}^n)} \overset{\text{def}}{=} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{\frac{\lambda}{2}} |\widehat{\phi}(\xi)|^2 \mathrm{d}\xi.$$

$$\mathbf{W}^{m,p}(\mathbb{R}^n) \overset{\text{def}}{=} \Bigg\{ \phi \in L^p(\mathbb{R}^n) : \sum_{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n \le m}$$

$$\int_{\mathbb{R}^n} \left| \frac{\partial \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n}{\partial x_1^{\alpha 1} \partial x_2^{\alpha 2} \partial x_3^{\alpha 3} \cdots \partial x_n^{\alpha_n}} \phi(\mathbf{x}) \right|^p \mathrm{d}\mathbf{x} < \infty \Bigg\}$$

with the norm defined by

$$\|\phi\|^p_{\mathbf{W}^{m,p}(\mathbb{R}^n)} \overset{\text{def}}{=} \sum_{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n} \int_{\mathbb{R}^n} \left| \frac{\partial^{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}}{\partial x_1^{\alpha 1} \partial x_2^{\alpha 2} \partial x_3^{\alpha 3} \cdots \partial x_n^{\alpha_n}} \phi(\mathbf{x}) \right|^p \mathrm{d}\mathbf{x}.$$

1. Let $(\mathcal{M}_1, \rho_1)$ and $(\mathcal{M}_2, \rho_2)$ be complete metric spaces. Let

$$\mathcal{A} : (\mathcal{M}_1, \rho_1) \to (\mathcal{M}_2, \rho_2)$$

be a continuous isomorphism. Let

$$\mathcal{S}_1 \subset \mathcal{M}_1, \qquad \mathcal{S}_2 \subset \mathcal{M}_2,$$

be subsets. Define

$$\begin{aligned} \mathcal{A}\mathcal{S}_1 &= \{\mathcal{A}(\mathbf{u}) : \mathbf{u} \in \mathcal{S}_1\}, \\ \mathcal{A}^{-1}\mathcal{S}_2 &= \{\mathbf{u} \in \mathcal{M}_1 : \mathcal{A}\mathbf{u} \in \mathcal{S}_2\}. \end{aligned}$$

(A) Prove the following results.

1 If $\mathcal{S}_1$ is bounded, then $\mathcal{A}\mathcal{S}_1$ is bounded.

2 If $\mathcal{S}_1$ is open, then $\mathcal{A}\mathcal{S}_1$ is open.

3 If $\mathcal{S}_1$ is closed, then $\mathcal{A}\mathcal{S}_1$ is closed.

4 If $\mathcal{S}_1$ is convex, then $\mathcal{A}\mathcal{S}_1$ is convex.

5 If $\mathcal{S}_1$ is dense in $\mathcal{M}_1$, then $\mathcal{A}\mathcal{S}_1$ is dense in $\mathcal{M}_2$.

6 If $\mathcal{S}_1$ is compact, then $\mathcal{A}\mathcal{S}_1$ is compact.

7 If $\mathcal{S}_1$ is finite-dimensional, then $\mathcal{A}\mathcal{S}_1$ is finite-dimensional.

(B) Prove the following results.

1 If $\mathcal{S}_2$ is bounded, then $\mathcal{A}^{-1}\mathcal{S}_2$ is bounded.

2 If $\mathcal{S}_2$ is open, then $\mathcal{A}^{-1}\mathcal{S}_2$ is open.

3 If $\mathcal{S}_2$ is closed, then $\mathcal{A}^{-1}\mathcal{S}_2$ is closed.

4 If $\mathcal{S}_2$ is convex, then $\mathcal{A}^{-1}\mathcal{S}_2$ is convex.

5 If $\mathcal{S}_2$ is dense in $\mathcal{M}_2$, then $\mathcal{A}^{-1}\mathcal{S}_2$ is dense in $\mathcal{M}_1$.

6 If $\mathcal{S}_2$ is compact, then $\mathcal{A}^{-1}\mathcal{S}_2$ is compact.

7 If $\mathcal{S}_2$ is finite-dimensional, then $\mathcal{A}^{-1}\mathcal{S}_2$ is finite-dimensional.

1. Let $\mathcal{S} \subset \mathbb{R}^n$ be a bounded closed set. Let the mapping $\mathcal{A} : \mathcal{S} \to \mathcal{S}$ satisfy the following conditions

$$\rho(\mathcal{A}(\mathbf{u}), \mathcal{A}(\mathbf{v})) < \rho(\mathbf{u}, \mathbf{v}),$$

for all $\mathbf{u} \in \mathcal{S}$ and $\mathbf{v} \in \mathcal{S}$, $\mathbf{u} \neq \mathbf{v}$. Prove that there exists a unique point $\mathbf{u}_0 \in \mathcal{S}$, such that

$$\mathcal{A}(\mathbf{u}_0) = \mathbf{u}_0.$$

2. Let $(\mathcal{M}, \rho)$ be a complete metric space, let

$$\mathcal{A} : (\mathcal{M}, \rho) \to (\mathcal{M}, \rho),$$

be a mapping. Define

$$\alpha_n = \sup_{\mathbf{u} \neq \mathbf{v}} \frac{\rho(\mathcal{A}^n \mathbf{u}, \mathcal{A}^n \mathbf{v})}{\rho(\mathbf{u}, \mathbf{v})}.$$

(1) If the series

$$\sum_{n=1}^{\infty} \alpha_n < \infty,$$

is convergent, then prove that for any point $\mathbf{u}_0 \in \mathcal{M}$, the sequence $\{\mathcal{A}^n \mathbf{u}_0\}$ converges to the unique fixed point of $\mathcal{A}$.
(2) If

$$\inf_n \alpha_n < 1,$$

then prove that there exists a unique fixed point to $\mathcal{A}$.

3. Let $A = (a_{ij})$ be a real $n \times n$ matrix, let $\mathbf{b} \in \mathbb{R}^n$ be a real vector. Consider the system of equations

$$A\mathbf{x} = \mathbf{b}.$$

Suppose that there hold the following conditions

$$\begin{aligned}
&(a_{11}-1)^2+a_{12}^2+a_{13}^2+\cdots+a_{1n}^2<1,\\
&a_{21}^2+(a_{22}-1)^2+a_{23}^2+\cdots+a_{2n}^2<1,\\
&a_{31}^2+a_{32}^2+(a_{33}-1)^2+\cdots+a_{3n}^2<1,\\
&\cdots\cdots\\
&a_{n1}^2+a_{n2}^2+a_{n3}^2+\cdots+(a_{nn}-1)^2<1.
\end{aligned}$$

Prove that for any vector $\mathbf{b}$, there exists a unique solution to $A\mathbf{x}=\mathbf{b}$.

4. Make and prove a theorem on the existence and uniqueness of the solution to the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}=\mathbf{f}(\mathbf{u},t), \qquad \mathbf{u}(t_0)=\mathbf{u}_0.$$

5. Let $f=f(x)$ be a continuous function defined on $(0,\infty)$. Prove that there exists a unique solution to the integral equation

$$y(t)=f(t)+\lambda\int_0^t e^{t-\tau}y(\tau)\mathrm{d}\tau.$$

Find the solution explicitly.

6. Let $(\mathcal{M},\rho)$ be a complete metric space. Let

$$\mathcal{A}:(\mathcal{M},\rho)\to(\mathcal{M},\rho)$$

be a mapping. Suppose that there exist positive constants $0<\alpha<1$, $\gamma>0$ and $\delta>0$, such that

$$\begin{aligned}
&\rho(\mathcal{A}(\mathbf{u}),\mathcal{A}(\mathbf{v}))<\alpha\rho(\mathbf{u},\mathbf{v}),\\
&\rho(\mathbf{u}_0,\mathcal{A}\mathbf{u}_0)\le\gamma\delta,
\end{aligned}$$

for all $\mathbf{u}\in B(\mathbf{u}_0,\delta)$ and $\mathbf{v}\in B(\mathbf{u}_0,\delta)$, $\mathbf{u}\neq\mathbf{v}$.

(1) Prove that there exists a unique fixed point in $B(\mathbf{u}_0,\delta)$ to $\mathcal{A}$,

if $\alpha+\gamma<1$.
(2) Prove that there exists a unique fixed point in $S(\mathbf{u}_0,\delta)$ to $\mathcal{A}$, if $\alpha+\gamma=1$ and $\mathcal{A}$ is continuous on $S(\mathbf{u}_0,\delta)$.

Make problems (with applications of ideas, concepts, definitions, theorems, results)

Presentations 2

Students: Pick up three problems for your presentation.
Dates of presentation: Thursday, September 22 and 29.

1. Show that the linear operator $\mathcal{A}: L[a,b]\to C[a,b]$:

$$(\mathcal{A}\phi)(x)=\int_a^x \phi(t)\mathrm{d}t,$$

is bounded, and prove that

$$\|\mathcal{A}\|=1.$$

2. Show that the linear operator $\mathcal{A}: L[a,b]\to L[a,b]$:

$$(\mathcal{A}\phi)(x)=\int_a^x \phi(t)\mathrm{d}t,$$

is bounded, and prove that

$$\|\mathcal{A}\|=b-a.$$

3. Let $K=K(x,y)$ be a continuous function defined on $[a,b]\times[a,b]$. Let $\mathcal{A}: C[a,b]\to C[a,b]$ be a bounded linear operator, defined by

$$(\mathcal{A}\phi)(x)=\int_a^b K(x,y)\phi(y)\mathrm{d}y, \qquad \phi\in C[a,b].$$

Prove that

$$\|\mathcal{A}\|=\max_{a\le x\le b}\int_a^b |K(x,y)|\mathrm{d}y.$$

4. Let $K = K(x, y)$ be a continuous function defined on $[a, b] \times [a, b]$. Let $\mathcal{A} : C[a, b] \to C[a, b]$ be a bounded linear operator, defined by

$$(\mathcal{A}\phi)(x) = \int_a^x K(x, y)\phi(y)\mathrm{d}y.$$

Compute the limit

$$\lim_{n\to\infty} (\|\mathcal{A}^n\|)^{1/n}.$$

$$\mathcal{A}$$

5. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces. Let $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ be a linear operator.
(1) Is $\mathcal{A}$ bounded, if the null space $\mathcal{N}(\mathcal{A})$ is closed?
(2) Is the null space $\mathcal{N}(\mathcal{A})$ closed, if $\mathcal{A}$ is bounded?

6. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let

$$\mathcal{A} \in \mathcal{B}(\mathcal{X} \to \mathcal{X}), \qquad \mathcal{C} \in \mathcal{B}(\mathcal{X} \to \mathcal{X}).$$

(1) Prove that there exists a positive constant $\lambda_0 > 0$, such that

$$\|(\lambda_0 I - \mathcal{A})^m \mathbf{x}\| > 0,$$

for all $\mathbf{x} \in \mathcal{X}$, $\mathbf{x} \neq \mathbf{0}$, and for all $m = 1, 2, 3, \cdots$.
(2) Prove that there exist no bounded linear operators $\mathcal{A}$ and $\mathcal{C}$, such that

$$\mathcal{A}\mathcal{C} - \mathcal{C}\mathcal{A} = \mathcal{I}.$$

Hint: Without loss of generality, let $\mathcal{C}^m \neq 0$. Then show that

$$\mathcal{A}\mathcal{C}^{m+1} - \mathcal{C}^{m+1}\mathcal{A} = (m+1)\mathcal{C}^m,$$

for all positive integers $m \geq 1$.

7. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let $\mathcal{E} \subset \mathcal{X}$ be a closed linear subspace, and $\mathbf{x}_n \in \mathcal{E}$, for all $n = 1, 2, 3, \cdots$. Prove that if

$$\lim_{n\to\infty} \mathbf{x}_n \overset{\text{weakly}}{=} \mathbf{x}_0,$$

then $\mathbf{x}_0 \in \mathcal{E}$.

8. Let $(\mathcal{X}, \|\cdot\|)$, $(\mathcal{Y}, \|\cdot\|)$ and $(\mathcal{Z}, \|\cdot\|)$ be normed linear spaces. Let

$$\mathcal{A} \in \mathcal{B}(\mathcal{X} \to \mathcal{Y}),$$

and

$$\mathcal{C} \in \mathcal{B}(\mathcal{Y} \to \mathcal{Z}).$$

Prove that

$$(\mathcal{C}\mathcal{A})^* = \mathcal{A}^*\mathcal{C}^*$$
$$(\mathcal{C}\mathcal{A})^{**} = \mathcal{C}^{**}\mathcal{A}^{**}.$$

9. Let the positive integer $n > 1$. Let the constant $1 < p < \infty$. Let the constants a and b satisfy $a < b$. Let $\Omega \subset \mathbb{R}^n$ be the unit closed ball. Prove that the following normed linear spaces are self-reflexive.

(1) $L^p[a, b]$

(2) $L^p(\Omega)$

(3) $L^p(\mathbb{R})$

(4) $L^p(\mathbb{R}^n)$

10. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces. Let

$$\mathcal{A}_0 \in \mathcal{B}(\mathcal{X} \to \mathcal{Y}),$$
$$\mathcal{A}_n \in \mathcal{B}(\mathcal{X} \to \mathcal{Y}), n \geq 1.$$

(1) Prove that

$$\begin{aligned}&\lim_{n\to\infty}\|\mathcal{A}_n^* - \mathcal{A}_0^*\| = 0,\\ &\lim_{n\to\infty}\|\mathcal{A}_n^{**} - \mathcal{A}_0^{**}\| = 0, \text{ if}\\ &\lim_{n\to\infty}\|\mathcal{A}_n - \mathcal{A}_0\| = 0.\end{aligned}$$

(2) Is it true that

$$\lim_{n\to\infty}\|\mathcal{A}_n^*(g) - \mathcal{A}_0^*(g)\| = 0,$$

for all $g \in \mathcal{Y}^*$, if

$$\lim_{n\to\infty}\|\mathcal{A}_n(\mathbf{x}) - \mathcal{A}_0(\mathbf{x})\| = 0,$$

for all $\mathbf{x} \in \mathcal{X}$?

(3) Is it true that

$$\lim_{n\to\infty}|(\mathcal{A}_n^* g)(\mathbf{x}) - (\mathcal{A}_0^* g)(\mathbf{x})| = 0,$$

for all $g \in \mathcal{Y}^*$ and for all $\mathbf{x} \in \mathcal{X}$, if

$$\lim_{n\to\infty}\|\mathcal{A}_n\mathbf{x} - \mathcal{A}_0\mathbf{x}\| = 0,$$

for all $\mathbf{x} \in \mathcal{X}$.

11. Let $K \in L^2(\mathcal{R})$, where

$$\mathcal{R} = [0,1] \times [0,1].$$

Define the operator $\mathcal{A}$ by

$$(\mathcal{A}\phi)(x) = \int_0^1 K(x,y)\phi(y)\mathrm{d}y, \phi \in L^2[0,1].$$

Find the explicit representation of the adjoint operator $\mathcal{A}^*$.

12. Let $(\mathcal{X}, \|\cdot\|)$ be a self-reflexive Banach space. Prove that there exists a dense countable set in $\mathcal{X}$, if and only if there exists a dense countable set in $\mathcal{X}^*$.

13. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space. Prove that $\mathcal{X}$ is self-reflexive, if and only if any closed subspace $\mathcal{E} \subset \mathcal{X}$ is self-reflexive.

14. Let $(\mathcal{X}, \|\cdot\|)$ be a self-reflexive normed linear space. Prove that any bounded set in $\mathcal{X}$ is weakly compact.

15. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. Prove that $\mathcal{X}$ is self-reflexive, if and only if $\mathcal{X}^*$ is self-reflexive.

Bonus Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and let $(\mathcal{Y}, \|\cdot\|)$ be a normed linear space. Prove that the normed linear space

$$\mathcal{B}(\mathcal{X} \to \mathcal{Y})$$

is weakly complete, if and only if $\mathcal{Y}$ is weakly complete. (By weakly complete, we mean that any weak Cauchy sequence is weakly convergent.)

Homework Assignment 1

1. Let $(\mathcal{M}, \rho)$ be a metric space. Let

$$\{\mathcal{O}_\lambda : \lambda \in \Lambda\}$$

be a family of open sets, let

$$\{\mathcal{C}_\mu : \mu \in \Gamma\}$$

be a family of closed sets. Prove that

$$\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda$$

is open and

$$\bigcap_{\mu \in \Gamma} \mathcal{C}_\mu$$

is closed.

2. Let

$$-\infty < a_k < b_k < \infty, \qquad k = 1, 2, 3, \cdots, n.$$

Let

$$\Omega = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \times [a_n, b_n].$$

Let $\mathcal{S} \subset C(\Omega)$ be a bounded set. Prove that $\mathcal{A}$ is totally bounded, if and only if for every positive constant $\varepsilon > 0$, there exists another positive constant $\delta = \delta(\varepsilon) > 0$, such that

$$|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon,$$

for all $f \in \mathcal{A}$, for all $\mathbf{x} \in \Omega$ and $\mathbf{y} \in \Omega$, such that $|\mathbf{x} - \mathbf{y}| < \delta$.

3. Let $(\mathbf{X}, \|\cdot\|)$ be a n-dimensional normed linear space, let

$$\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n\}$$

be a basis of $\mathbf{X}$. Prove that there exist two positive constants $C_2 > C_1 > 0$, such that there holds the following estimates

$$C_1 \sum_{k=1}^{n} \alpha_k^2 \leq \|\mathbf{u}\|^2 \leq C_2 \sum_{k=1}^{n} \alpha_k^2$$

for all

$$\mathbf{u} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n \in \mathbf{X}.$$

4. Let $(\mathbf{X}, \|\cdot\|)$ be a normed linear space and let $\mathbf{Y} \subset \mathbf{X}$ be a closed subspace, $\mathbf{Y} \neq \mathbf{X}$. Prove that for every positive constant $0 < \varepsilon < 1$, there exists a unit vector $\mathbf{u}_0 \in \mathbf{X}$, $\|\mathbf{u}_0\| = 1$, such that

$$\rho(\mathbf{u}_0, \mathbf{Y}) > \varepsilon.$$

5. Let $(\mathbf{X}, \|\cdot\|)$ be an infinite-dimensional normed linear space. Prove that the unit ball

$$\{\mathbf{u} \in \mathbf{X} : \|\mathbf{u}\| = 1\}$$

is not compact.

6. Let $(\mathbf{X}, \|\cdot\|)$ be a Banach space and let $\mathcal{S} \subset \mathbf{X}$ be a compact subset. Prove that the convex closure

$$h(\mathcal{S}) \overset{\text{def}}{=} \left\{ \sum_{k=1}^{m} \alpha_k \mathbf{u}_k : \alpha_k > 0, \mathbf{u}_k \in \mathcal{S}, \sum_{k=1}^{m} \alpha_k = 1, m \geq 1 \right\}$$

is compact as well. **This is a Bonus Problem.**

7. Let $(\mathbf{X}, \|\cdot\|)$ be a Banach space and let $\mathcal{S} \subset \mathbf{X}$ be a convex, closed subset. Let

$$\mathcal{A} : \mathcal{S} \to \mathcal{S}$$

be a continuous mapping, such that $\mathcal{A}(\mathcal{S})$ is compact. Prove that there exists a fixed point $\mathbf{u}_0 \in \mathcal{S}$, such that

$$\mathcal{A}(\mathbf{u}_0) = \mathbf{u}_0.$$

Make basic problems about: definitions, theorems, results, necessary and sufficient conditions of results or theorems.

Homework 2

Due October 2 at 11:59 PM

Definition 1. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let

$$\mathcal{A} \in \mathcal{B}(\mathcal{X} \to \mathcal{X}).$$

The mapping is called a projection operator, if

$$\mathcal{A}^2 = \mathcal{A}.$$

Definition 2. Let $\mathcal{X}$ be a vector space, let $\mathcal{Y}$ and $\mathcal{Z}$ be subspaces of $\mathcal{X}$. If every vector $\mathbf{x} \in \mathcal{X}$ can be written uniquely as

$$\mathbf{x} = \mathbf{y} + \mathbf{z},$$

where $\mathbf{y} \in \mathcal{Y}$ and $\mathbf{z} \in \mathcal{Z}$, then we say $\mathcal{X}$ is the direct sum of $\mathcal{Y}$ and $\mathcal{Z}$. We write

$$\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}.$$

1. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and let $(\mathcal{Y}, \|\cdot\|)$ be a normed linear space, let

$$\mathcal{A}_\lambda \in \mathcal{B}(\mathcal{X} \to \mathcal{Y}),$$

for all $\lambda \in \Lambda$. Prove that there exists a positive constant $C > 0$, such that

$$\|\mathcal{A}_\lambda\| \leq C,$$

for all $\lambda \in \Lambda$, if for every $g \in \mathcal{Y}^*$ and for every $\mathbf{x} \in \mathcal{X}$, there exists a positive constant $C_0 = C_0(g, \mathbf{x}) > 0$, such that

$$|g(\mathcal{A}_\lambda \mathbf{x})| \leq C(g, \mathbf{x}),$$

for all $\lambda \in \Lambda$.

2. Let $\mathcal{X}$ be a vector space. Let

$$\mathcal{A} : \mathcal{X} \to \mathcal{X}$$

be a projection operator. Prove that

$$\mathcal{I} - \mathcal{A}$$

is also a projection operator.

Define

$$\begin{aligned} \mathcal{Y} &\overset{\text{def}}{=} \{\mathbf{x} \in \mathcal{X} : \mathcal{A}\mathbf{x} = \mathbf{x}\}, \\ \mathcal{Z} &\overset{\text{def}}{=} \{\mathbf{x} \in \mathcal{X} : (\mathcal{I} - \mathcal{A})\mathbf{x} = \mathbf{x}\}. \end{aligned}$$

Prove that

$$\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}.$$

On the other hand, if

$$\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z},$$

then define the operator $\mathcal{A}$:

$$\mathcal{A}\mathbf{x} = \mathbf{y},$$

where

$$\mathbf{x} = \mathbf{y} + \mathbf{z}.$$

Prove that

$$\mathcal{A} : \mathcal{X} \to \mathcal{X}$$

is a projection operator and

$$\mathcal{Y} = \{\mathbf{x} \in \mathcal{X} : \mathcal{A}\mathbf{x} = \mathbf{x}\}.$$

3. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let

$$\mathcal{A} \in \mathcal{B}(\mathcal{X} \to \mathcal{X})$$

be a projection operator, that is $\mathcal{A}^2 = \mathcal{A}$. Prove that

$$\begin{aligned} \mathcal{Y} &= \{\mathbf{x} \in \mathcal{X} : \mathcal{A}\mathbf{x} = \mathbf{x}\}, \\ \mathcal{Z} &= \{\mathbf{x} \in \mathcal{X} : (\mathcal{I} - \mathcal{A})\mathbf{x} = \mathbf{x}\}, \end{aligned}$$

are closed subspaces of $\mathcal{X}$.

4. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space, let $\mathcal{Y}$ and $\mathcal{Z}$ be closed subspaces of $\mathcal{X}$, such that

$$\mathcal{X} = \mathcal{Y} \oplus \mathcal{Z}.$$

Prove that the operators $\mathcal{A}$ and $\mathcal{P}$ defined by

$$\begin{aligned} \mathcal{A}\mathbf{x} &= \mathbf{y}, \\ \mathcal{P}\mathbf{x} &= \mathbf{z}, \end{aligned}$$

are bounded linear operators and are projection operators, where

$$\mathbf{x} = \mathbf{y} + \mathbf{z},$$

$$\mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}, \mathbf{z} \in \mathcal{Z}.$$

Moreover, show that

$$\mathcal{A} + \mathcal{P} = \mathcal{I}.$$

5. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be Banach spaces, let

$$\mathcal{A} : \mathcal{X} \to \mathcal{Y}$$

be a linear operator. Prove that

$$\mathcal{A} \in \mathcal{B}(\mathcal{X} \to \mathcal{Y}),$$

if for every $g \in \mathcal{Y}^*$, the functional on $\mathcal{X}$ define by

$$g(\mathcal{A}\mathbf{x})$$

is a continuous linear functional.

6. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be Banach spaces, let

$$\mathcal{A} \in \mathcal{B}(\mathcal{X} \to \mathcal{Y}),$$

such that $\mathcal{A}\mathcal{X} = \mathcal{Y}$. Let $\mathbf{y}_0 \in \mathcal{Y}$. Prove that there exists a positive constant $C > 0$, such that for any convergent sequence

$$\{\mathbf{y}_n\} \subset \mathcal{Y}, \qquad \lim_{n\to\infty} \|\mathbf{y}_n - \mathbf{y}_0\| = 0,$$

there exists a convergent sequence

$$\{\mathbf{x}_n\} \subset \mathcal{X}, \qquad \lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}_0\| = 0,$$

such that

$$\|\mathbf{x}_n\| \le C\|\mathbf{y}_n\|, \qquad \mathbf{y}_n = \mathcal{A}\mathbf{x}_n, \qquad \lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}_0\| = 0.$$

7. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space, let

$$\mathcal{B} = \{\mathbf{x}_n\} \subset \mathcal{X}$$

be a sequence of points. $\mathcal{B}$ is called a basis of $\mathcal{X}$, if for each point $\mathbf{x} \in \mathcal{X}$, there exists a sequence $\{\alpha_n(\mathbf{x})\}$ of real numbers, such that

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{n} \alpha_k(\mathbf{x})\mathbf{x}_k - \mathbf{x} \right\| = 0.$$

Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space with the basis $\mathcal{B}$. Prove that $\alpha_k \in \mathcal{X}^*$, for all $k \geq 1$, where α_k appears in the representation

$$\mathbf{x} = \sum_{k=1}^{\infty} \alpha_k(\mathbf{x})\mathbf{x}_k,$$

for all $\mathbf{x} \in \mathcal{X}$.

Presentation 3

1. Let

$$A = \begin{pmatrix} \alpha & -\beta & -\beta & -\beta \\ -\beta & \alpha & -\beta & -\beta \\ -\beta & -\beta & \alpha & -\beta \\ -\beta & -\beta & -\beta & \alpha \end{pmatrix}, \qquad B = \begin{pmatrix} \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \\ \alpha & -\beta & \alpha & -\beta \\ -\beta & \alpha & -\beta & \alpha \end{pmatrix},$$

where α and β are real constants.

Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n,$$

be a real polynomial of degree n, where $a_n \neq 0$. Define

$$\begin{aligned} f(A) &= a_0 I + a_1 A + a_2 A^2 + a_3 A^3 + \cdots + a_n A^n, \\ f(B) &= a_0 I + a_1 B + a_2 B^2 + a_3 B^3 + \cdots + a_n B^n. \end{aligned}$$

(1) Find all eigenvalues and all eigenspaces of $f(A)$.
(2) Find all eigenvalues and all eigenspaces of $f(B)$.

2. Let the Fourier transformation operator $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be defined by

$$(\mathcal{A}\phi)(\xi) = \int_{\mathbb{R}} \exp(-\mathrm{i}x\xi)\phi(x)\mathrm{d}x, \qquad \phi \in L^2(\mathbb{R}).$$

Find all eigenvalues and all eigenspaces of $\mathcal{A}$.

3. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A}_n \in \mathbb{B}(\mathcal{X} \to \mathcal{X}),$$
$$\mathcal{A}_0 \in \mathbb{B}(\mathcal{X} \to \mathcal{X}),$$

where $n = 1, 2, 3, \cdots$, such that

$$\lim_{n\to\infty} \|\mathcal{A}_n - \mathcal{A}_0\| = 0.$$

Let the complex constant $\lambda_0 \in \rho(\mathcal{A}_0)$. Show that $\lambda_0 \in \rho(\mathcal{A}_n)$, for all sufficiently large $n \gg 1$. Morevoer, show that

$$\lim_{n\to\infty} \|(\lambda_0\mathcal{I} - \mathcal{A}_n)^{-1} - (\lambda_0\mathcal{I} - \mathcal{A}_0)^{-1}\| = 0.$$

4. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

Show that $\sigma_r(\mathcal{A}) \subset \sigma_p(\mathcal{A}^*)$.

5. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}),$$
$$\mathcal{B} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

(1) Prove that

$$r(\mathcal{A}\mathcal{B}) = r(\mathcal{B}\mathcal{A})$$

(2) Prove that

$$r(\mathcal{A} + \mathcal{B}) \leq r(\mathcal{A}) + r(\mathcal{B}),$$

if $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$.

6. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces, let

$$f_1, f_2, f_3, \cdots, f_n \in \mathcal{X}^*,$$
$$\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \cdots, \mathbf{y}_n \in \mathcal{Y}.$$

Define the operator $\mathcal{A}$ by

$$\mathcal{A}\mathbf{x} = f_1(\mathbf{x})\mathbf{y}_1 + f_2(\mathbf{x})\mathbf{y}_2 + f_3(\mathbf{x})\mathbf{y}_3 + \cdots + f_n(\mathbf{x})\mathbf{y}_n,$$

for all $\mathbf{x} \in \mathcal{X}$. Show that

$$\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{Y}).$$

7. Let the bounded linear operator

$$\mathcal{A} : C[0,1] \to C[0,1],$$

be defined by

$$(\mathcal{A}\phi)(x) = x\phi(x),$$

for all $\phi \in C[0,1]$. Find

$$\rho(\mathcal{A}), \qquad \sigma(\mathcal{A}), \qquad \sigma_p(\mathcal{A}), \qquad \sigma_c(\mathcal{A}), \qquad \sigma_a(\mathcal{A}), \qquad \sigma_r(\mathcal{A}).$$

8. Let $(\mathcal{X}, \|\cdot\|)$ be an infinite-dimensional complex Banach space, let

$$\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{Y}).$$

Prove that

$$\mathcal{A} \neq \lambda\mathcal{I},$$

for any complex constant $\lambda \in \mathbb{C}$ with $\lambda \neq 0$.

9. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{Y}).$$

Let $\lambda \in \mathbb{C}$ be a complex constant and $\lambda \neq 0$. Prove that the dimension of the null space is finite

$$\dim[\mathcal{N}(\lambda\mathcal{I} - \mathcal{A})] < \infty.$$

10. Let the bounded linear operator $\mathcal{A} : C[a,b] \to C[a,b]$ be defined by

$$(\mathcal{A}\phi)(x) = \int_a^x \phi(t)\mathrm{d}t, \qquad \phi \in C[a,b].$$

Show that $\mathcal{A}$ is a compact operator. Find

$$\rho(\mathcal{A}), \qquad \sigma(\mathcal{A}), \qquad \sigma_p(\mathcal{A}), \qquad \sigma_c(\mathcal{A}), \qquad \sigma_a(\mathcal{A}), \qquad \sigma_r(\mathcal{A}).$$

11. Let T be a positive constant and let M be a constant $n \times n$ matrix. Prove that the operator

$$\mathcal{A} : C[0,T] \to C[0,T],$$
$$(\mathcal{A}\mathbf{u})(t) = \int_0^t M\mathbf{u}(\tau)\mathrm{d}\tau,$$

where $\mathbf{u} = \mathbf{u}(t) \in C[0,T]$ is a real vector valued function, is a compact operator. Find

$$\rho(\mathcal{A}), \qquad \sigma(\mathcal{A}), \qquad \sigma_p(\mathcal{A}), \qquad \sigma_c(\mathcal{A}), \qquad \sigma_a(\mathcal{A}), \qquad \sigma_r(\mathcal{A}).$$

12. Let $D > 0$ be a positive constant. Define

$\Omega_1 = B(\mathbf{0}, 1)$, this is the unit open ball in $\mathbb{R}^n$,
$\Omega_2 = [-D, D] \times [-D, D] \times [-D, D] \times \cdots \times [-D, D]$,
$\Omega_3 = \mathbb{R}^n$.

Define

$$\mathcal{A} = (I - \triangle)^{-1},$$

where

$$\triangle = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2},$$

represents the second order differential operator. Is $\mathcal{A}$ from the Banach space to itself (see below) a compact operator

(1) $C(\Omega_1) \to C(\Omega_1)$?

(2) $L^2(\Omega_2) \to L^2(\Omega_2)$?

(3) $L^3(\Omega_3) \to L^3(\Omega_3)$?

13. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X}).$$

Let $\lambda \in \mathbb{C}$ be any constant. Show that

$$\begin{aligned}[](\lambda\mathcal{I} - \mathcal{A})\mathcal{X}]^{\perp} &= \mathcal{N}(\lambda\mathcal{I}^* - \mathcal{A}^*),\\ [(\lambda\mathcal{I}^* - \mathcal{A}^*)\mathcal{X}^*]^{\perp} &= \mathcal{N}(\lambda\mathcal{I} - \mathcal{A}).\end{aligned}$$

14. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}).$$

Let $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}$ be constants and $\lambda \neq \mu$. Show that

$$\mathcal{N}(\lambda\mathcal{I} - \mathcal{A}) \perp \mathcal{N}(\mu\mathcal{I}^* - \mathcal{A}^*).$$

15. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let[1]

$$\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{Y}).$$

Let $\lambda \in \mathbb{C}$ be a constant and $\lambda \neq 0$. Show that

$$(\lambda\mathcal{I} - \mathcal{A})\mathcal{X} = \mathcal{X},$$

if and only if

$$(\lambda\mathcal{I}^* - \mathcal{A}^*)\mathcal{X}^* = \mathcal{X}^*.$$

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[1] Is the result true, if

$$\mathcal{A} \in \mathbb{B}(\mathcal{X} \to \mathcal{X})?$$

16. Let $\mathcal{X} = L^2(\mathbb{R}^n)$. Let $\alpha > 0$ and $T > 0$ be fixed positive constants. Is $\mathcal{A}$ given by

$$\begin{aligned} &\mathcal{A} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \\ &(\mathcal{A}\phi)(\mathbf{x}) = \exp(-\alpha|\xi|^2 T)\widehat{\phi}(\xi), \end{aligned}$$

a compact operator, where $\widehat{\phi}$ represents the Fourier transformation of ϕ?

Homework 3

1. Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, \|\cdot\|)$ be normed linear spaces, let

$$\{\mathcal{A}_\lambda \in \mathbb{B}(\mathcal{X} \to \mathcal{Y}) : \lambda \in \Lambda\}$$

be a family of bounded linear operators. Suppose that the set

$$\sup_{\lambda\in\Lambda}\{|f(\mathcal{A}_\lambda \mathbf{x})| : \lambda \in \Lambda\} < \infty,$$

for every point $\mathbf{x} \in \mathcal{X}$ and every functional $f \in \mathcal{Y}^*$. Prove that there exists a positive constant $C > 0$, such that

$$\sup_{\lambda\in\Lambda} \|\mathcal{A}_\lambda\| \leq C.$$

2. Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space, let

$$f : \mathcal{X} \to \mathbb{R},$$

be a functional, such that

$$\begin{aligned}
&(1) \quad f(\mathbf{x}) \geq 0,\\
&(2) \quad f(\mathbf{x}_1 + \mathbf{x}_2) \leq f(\mathbf{x}_1) + f(\mathbf{x}_2),\\
&(3) \quad f(\alpha\mathbf{x}) = |\alpha| f(\mathbf{x}), \alpha \in \mathbb{R},\\
&(4) \quad \liminf_{n\to\infty} f(\mathbf{x}_n) \geq f(\mathbf{x}_0),\\
&\qquad\ \text{if } \lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}_0\| = 0, \mathbf{x}_0 \in \mathcal{X},
\end{aligned}$$

for all vectors $\mathbf{x} \in \mathcal{X}$, $\mathbf{x}_0 \in \mathcal{X}$, $\mathbf{x}_1 \in \mathcal{X}$, $\mathbf{x}_2 \in \mathcal{X}$, and for all constants $\alpha \in \mathbb{R}$. Prove that there exists a positive constant $C > 0$, such that

$$|f(\mathbf{x})| \leq C\|\mathbf{x}\|,$$

for all $\mathbf{x} \in \mathcal{X}$.

3. Let $(\mathcal{X}, \|\cdot\|)$ be a complex Banach space, let

$$\mathcal{A} \in \mathbb{C}(\mathcal{X} \to \mathcal{X}),$$

be a compact operator. Let $\lambda \in \mathbb{C}$ be a constant and $\lambda \neq 0$. Prove that there exist at most finitely many, linearly independent, unit vectors

$$\begin{aligned}
&\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n \in \mathcal{X}, \\
&\|\mathbf{x}_1\| = 1, \|\mathbf{x}_2\| = 1, \|\mathbf{x}_3\| = 1, \cdots, \|\mathbf{x}_n\| = 1,
\end{aligned}$$

such that

$$\mathcal{X} = \overline{\text{span}\ \{(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n\}}.$$

Let

$$\begin{aligned}
&\mathcal{X}_1 = \overline{\text{span}\ \{(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}, \mathbf{x}_1\}}, \\
&\mathcal{X}_2 = \overline{\text{span}\ \{(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}, \mathbf{x}_1, \mathbf{x}_2\}}, \\
&\mathcal{X}_3 = \overline{\text{span}\ \{(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}}, \\
&\cdots\cdots \\
&\mathcal{X}_n = \overline{\text{span}\ \{(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n\}}.
\end{aligned}$$

Moreover, prove that there exist continuous linear functionals

$$f_1, f_2, f_3, \cdots, f_n \in \mathcal{X}^*,$$

which are linearly independent, such that

$$\begin{aligned}
&\|f_1\| = 1, \qquad f_1(\mathbf{x}_1) = \rho(\mathbf{x}_1, (\lambda\mathcal{I} - \mathcal{A})\mathcal{X}) > \frac{1}{2}, \qquad f_1((\lambda\mathcal{I} - \mathcal{A})\mathcal{X}) = \{\mathbf{0} \\
&\|f_2\| = 1, \qquad f_2(\mathbf{x}_2) = \rho(\mathbf{x}_2, \mathcal{X}_1) > \frac{1}{2}, \qquad f_2((\lambda\mathcal{I} - \mathcal{A})\mathcal{X}) = \{\mathbf{0}\}, \\
&\|f_3\| = 1, \qquad f_3(\mathbf{x}_3) = \rho(\mathbf{x}_3, \mathcal{X}_2) > \frac{1}{2}, \qquad f_3((\lambda\mathcal{I} - \mathcal{A})\mathcal{X}) = \{\mathbf{0}\}, \\
&\cdots\cdots \\
&\|f_n\| = 1, \qquad f_n(\mathbf{x}_n) = \rho(\mathbf{x}_n, \mathcal{X}_{n-1}) > \frac{1}{2}, \qquad f_n((\lambda\mathcal{I} - \mathcal{A})\mathcal{X}) = \{\mathbf{0}\}.
\end{aligned}$$

The continuous linear functional also satisfy the conditions

$$\begin{aligned}
&f_2(\mathbf{x}_1) = 0,\\
&f_3(\mathbf{x}_1) = 0, f_3(\mathbf{x}_2) = 0,\\
&f_4(\mathbf{x}_1) = 0, f_4(\mathbf{x}_2) = 0, f_4(\mathbf{x}_3) = 0,\\
&\cdots\cdots\\
&f_n(\mathbf{x}_1) = 0, f_n(\mathbf{x}_2) = 0, f_3(\mathbf{x}_3) = 0, \cdots, f_n(\mathbf{x}_{n-1}) = 0.
\end{aligned}$$

Proof. There is nothing to prove, if

$$(\lambda \mathcal{I} - \mathcal{A})\mathcal{X} = \mathcal{X}.$$

If $(\lambda \mathcal{I} - \mathcal{A})\mathcal{X} \neq \mathcal{X}$, then there exists a vector $\mathbf{x}_1 \in \mathcal{X}$, such that

$$\|\mathbf{x}_1\| = 1, \qquad \rho(\mathbf{x}_1, (\lambda \mathcal{I} - \mathcal{A})\mathcal{X}) > \frac{1}{2}.$$

Moreover, there exists a continuous linear functional $f_1 \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f_1\| = 1, \\ &(2) \quad f_1(\mathbf{x}_1) = \rho(\mathbf{x}_1, (\lambda \mathcal{I} - \mathcal{A})\mathcal{X}) > \frac{1}{2}, \\ &(3) \quad f_1((\lambda \mathcal{I} - \mathcal{A})\mathcal{X}) = \{\mathbf{0}\}. \end{aligned}$$

Define

$$\mathcal{X}_1 = \overline{\text{span}\ \{(\lambda \mathcal{I} - \mathcal{A})\mathcal{X}, \mathbf{x}_1\}}.$$

We have finished the proof, if

$$\overline{\text{span}\ \{(\lambda \mathcal{I} - \mathcal{A})\mathcal{X}, \mathbf{x}_1\}} = \mathcal{X}.$$

If $\mathcal{X}_1 \neq \mathcal{X}$, then there exists a vector $\mathbf{x}_2 \in \mathcal{X}$, such that

$$\|\mathbf{x}_2\| = 1, \qquad \rho(\mathbf{x}_2, \mathcal{X}_1) > \frac{1}{2}.$$

Moreover, there exists a continuous linear functional $f_2 \in \mathcal{X}^*$, such that

$$\begin{aligned} &(1) \quad \|f_2\| = 1, \\ &(2) \quad f_2(\mathbf{x}_2) = \rho(\mathbf{x}_2, \mathcal{X}_1) > \frac{1}{2}, \\ &(3) \quad f_2((\lambda \mathcal{I} - \mathcal{A})\mathcal{X}) = \{\mathbf{0}\}, f(\mathbf{x}_1) = 0. \end{aligned}$$

Define

$$\mathcal{X}_2 = \overline{\text{span}\ \{(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}, \mathbf{x}_1, \mathbf{x}_2\}}.$$

We have finished the proof, if

$$\mathcal{X}_2 = \mathcal{X}.$$

If $\mathcal{X}_2 \neq \mathcal{X}$, then there exists a vector $\mathbf{x}_3 \in \mathcal{X}$, such that

$$\|\mathbf{x}_3\| = 1, \qquad \rho(\mathbf{x}_3, \mathcal{X}_2) > \frac{1}{2}.$$

Moreover, there exists a continuous linear functional $f_3 \in \mathcal{X}^*$, such that

$$\begin{aligned}
&(1) \quad \|f_3\| = 1,\\
&(2) \quad f_3(\mathbf{x}_3) = \rho(\mathbf{x}_3, \mathcal{X}_2) > \frac{1}{2},\\
&(3) \quad f_3((\lambda\mathcal{I} - \mathcal{A})\mathcal{X}) = \{\mathbf{0}\}, f(\mathbf{x}_k) = 0,
\end{aligned}$$

for $k = 1, 2$.

Define

$$\mathcal{X}_3 = \overline{\text{span}\ \{(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}}.$$

We have finished the proof, if

$$\mathcal{X}_3 = \mathcal{X}.$$

If $\mathcal{X}_3 \neq \mathcal{X}$, then we may continue this process. $\cdots\cdots$ There exists a vector $\mathbf{x}_n \in \mathcal{X}$, such that

$$\|\mathbf{x}_n\| = 1, \qquad \rho(\mathbf{x}_n, \mathcal{X}_{n-1}) > \frac{1}{2}.$$

Moreover, there exists a continuous linear functional $f_n \in \mathcal{X}^*$, such that

$$\begin{aligned}
&(1) \quad \|f_n\| = 1,\\
&(2) \quad f_n(\mathbf{x}_n) = \rho(\mathbf{x}_n, \mathcal{X}_{n-1}) > \frac{1}{2},\\
&(3) \quad f_n((\lambda\mathcal{I} - \mathcal{A})\mathcal{X}) = \{\mathbf{0}\}, f(\mathbf{x}_k) = 0,
\end{aligned}$$

for $k = 1, 2, 3, \cdots, n-1$.

Define

$$\mathcal{X}_n = \overline{\text{span } \{(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n\}}.$$

Clearly, this process must stop after finitely many steps. Therefore, there exists a unique positive integer $n \geq 1$, such that

$$\mathcal{X} = \overline{\text{span } \{(\lambda\mathcal{I} - \mathcal{A})\mathcal{X}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_n, \cdots, \mathbf{x}_n\}}.$$

Overall, we have

$$[(\lambda\mathcal{I} - \mathcal{A})^* f_k](\mathbf{x}) = f_k((\lambda\mathcal{I} - \mathcal{A})\mathbf{x}) = 0,$$

for all $\mathbf{x} \in \mathcal{X}$. We have found eigenvalues and eigenvectors of $\mathcal{A}^*$:

$$\mathcal{A}^* f_k = \lambda f_k,$$

for all $k = 1, 2, 3, \cdots, n$. It is easy to see that the continuous linear functionals are linearly independent. Hence

$$\dim[\mathcal{N}(\lambda\mathcal{I}^* - \mathcal{A}^*)] \geq n.$$

The proof is completely finished now. $\square$

$\mathcal{A} \in \mathcal{C}(\mathcal{X} \to \mathcal{Y})$

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$\mathcal{A} \in \mathcal{C}(\mathcal{X} \to \mathcal{Y})$

1. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, let

$$\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \cdots, \mathbf{e}_n\},$$

be a basis of $\mathbb{R}^n$. All vectors in $\mathbb{R}^n$ may be represented as

$$\mathbf{x} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3 + \cdots + \alpha_n\mathbf{e}_n,$$

where $\alpha_k \in \mathbb{R}$. Prove that $(\cdot, \cdot)$ is an inner product in $\mathbb{R}^n$, if and only if there exists an $n \times n$ symmetric, positively definite matrix

A, such that

$$\left(\sum_{k=1}^{n}\alpha_k\mathbf{e}_k, \sum_{l=1}^{n}\beta_l\mathbf{e}_l\right) = \sum_{k=1}^{n}\sum_{l=1}^{n}a_{kl}\alpha_k\beta_l.$$

2. Let $\mathcal{H}$ be an inner product space, let $\mathbf{y}_0 \in \mathcal{H}$. Define the functional f_0:

$$f_0(\mathbf{x}) = (\mathbf{x}, \mathbf{y}_0), \qquad \mathbf{x} \in \mathcal{H}.$$

Show that $f_0 \in \mathcal{H}^*$ is a continuous linear functional. Additionally, show that $\|f_0\| = \|\mathbf{y}_0\|$.

3. Let $\mathcal{H}$ be an inner product space, let $\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n\}$ be a set of vectors, such that

$$\begin{aligned} &(1) \quad (\mathbf{x}_k, \mathbf{x}_k) = 1, \\ &(2) \quad (\mathbf{x}_k, \mathbf{x}_l) = 0, k \neq l. \end{aligned}$$

Prove that $\mathcal{S}$ is linearly independent.

4. Prove that any inner product space may be completed to a Hilbert space.

5. Define the complex space

$$\mathcal{H} = \left\{\phi : \phi \text{ is Lebesgue measurable on } \mathbb{R}^n, \int_{\mathbb{R}^n} |\phi(\mathbf{x})|^2 \mathrm{d}\mathbf{x} < \infty\right\}.$$

Define

$$(\phi, \psi) = \int_{\mathbb{R}^n} \phi(\mathbf{x})\overline{\psi(\mathbf{x})}\mathrm{d}\mathbf{x},$$

for all $\phi, \psi \in \mathcal{H}$. Prove that $\mathcal{H}$ is a Hilbert space.

6. Let $\mathcal{H}$ be an inner product space, let $\mathcal{E}$ be a complete subspace of $\mathcal{H}$. Prove that

$$\overline{\mathcal{E}} = (\mathcal{E}^{\perp})^{\perp}.$$

7. Let $\mathcal{H}$ be an inner product space, let $\mathcal{S}$ and $\mathcal{T}$ be subsets of $\mathcal{H}$, let

$$\mathcal{E} = \left\{ \sum_{k=1}^{m} \alpha_k \mathbf{x}_k + \sum_{l=1}^{n} \beta_l \mathbf{y}_l : \alpha_k \in \mathbb{R}, \mathbf{x}_k \in \mathcal{S}, m \geq 1, \beta_l \in \mathbb{R}, \mathbf{y}_l \in \mathcal{T}, n \geq 1 \right\}$$

Prove that

$$\mathcal{E}^{\perp} = \mathcal{S}^{\perp} \cap \mathcal{T}^{\perp}.$$

8. Let

$$\mathcal{H} = L^2[0, 2\pi],$$

and

$$\mathcal{S} = \left\{ \frac{1}{\sqrt{2}}, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \cdots, \cos(mx), \sin(\right.$$

Show that $\mathcal{S}$ is a complete orthonormal set in $\mathcal{H}$.

9. Let

$$\mathcal{H} = \left\{ \phi : \int_{\mathbb{R}} |\phi(x)|^2 \mathrm{d}x < \infty \right\},$$

and

$$\mathcal{S} = \left\{ \frac{1}{(2^m m! \sqrt{\pi})^{1/2}} \exp(\frac{1}{2}x^2) \frac{\mathrm{d}^m}{\mathrm{d}x^m} [\exp(-x^2)] : m = 0, 1, 2, 3, \cdots \right\}.$$

Show that $\mathcal{S}$ is a complete orthonormal set.

10. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{K}$ be an inner product space. Let

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{K}),$$
$$\mathcal{A}^{-1} \in \mathbb{B}(\mathcal{H}\mathcal{H}).$$

Show that $\mathcal{K}$ is a Hilbert space.

11. Let $\mathcal{H}$ be a Hilbert space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{H}\mathcal{H}).$$

Prove that $\mathcal{A}^* = -\mathcal{A}$, if and only if

$$\text{Re } (\mathcal{A}\mathbf{x}, \mathbf{x}) = 0,$$

for all $\mathbf{x} \in \mathcal{H}$.

12. Let $\mathcal{H}$ be a Hilbert space, let λ_0 be an eigenvalue of

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}).$$

Is $\overline{\lambda}$ an eigenvalue of $\mathcal{A}^*$?

13. Let $\mathcal{H}$ be a complex Hilbert space, let $\mathcal{A}$ be a bounded linear, self-adjoint operator. Suppose that there exists a positive constant $C > 0$, such that

$$(\mathcal{A}\mathbf{x}, \mathbf{x}) \geq C(\mathbf{x}, \mathbf{x}),$$

for all $\mathbf{x} \in \mathcal{H}$. Define a new inner product

$$(\mathbf{x}, \mathbf{y})_{\mathcal{A}} = (\mathcal{A}\mathbf{x}, \mathbf{y}),$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$. Prove that $\mathcal{H}$ is a Hilbert space with respect to the new inner product.

14. Let $\mathcal{H}$ be a Hilbert space, let

$$\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \cdots, \mathcal{E}_n, \cdots$$

be a sequence of mutually perpendicular closed subspaces. Prove that

$$\oplus_{n=1}^{\infty} \mathcal{E}_n = \overline{\text{span } \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \cdots, \mathcal{E}_n, \cdots\}}.$$

15. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{E}$ and $\mathcal{F}$ be closed subspaces, let

$$\mathcal{P} : \mathcal{H} \to \mathcal{E},$$
$$\mathcal{Q} : \mathcal{H} \to \mathcal{F},$$

be projection operators. Prove that $\mathcal{PQ}$ is a projection operator, if and only if

$$[\mathcal{E} \ominus (\mathcal{E} \cap \mathcal{F})] \perp [\mathcal{F} \ominus (\mathcal{E} \cap \mathcal{F})].$$

16. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{P}$ and $\mathcal{Q}$ be projection operators, let

$$\mathcal{P} + \mathcal{Q} - \mathcal{PQ}$$

be a projection operator as well. Question: Is $\mathcal{PQ} = \mathcal{QP}$ true?

17. Let $\mathcal{H}$ be a Hilbert space, let $\{\mathcal{P}_\lambda : \lambda \in \Lambda\}$ be a family of mutually perpendicular projection operators and let $\{\mathcal{E}_\lambda : \lambda \in \Lambda\}$ be the corresponding projection subspaces. Prove that there exists a projection operator $\mathcal{P}$:

$$\mathcal{P}\mathbf{x} = \sum_{\lambda \in \Lambda} \mathcal{P}_\lambda \mathbf{x},$$

Let $\mathcal{PH} = \mathcal{E}$. Prove that

$$\mathcal{E} = \oplus_{\lambda \in \Lambda} \mathcal{E}_\lambda = \left\{ \sum_{\lambda \in \Lambda} \mathbf{x}_\lambda : \sum_{\lambda \in \Lambda} \|\mathbf{x}_\lambda\|^2 < \infty, \mathbf{x}_\lambda \in \mathcal{E}_\lambda \right\}.$$

18. Let $\mathcal{H}$ be a Hilbert space, let λ_n be a sequence of bounded constants, let

$$\mathcal{P}_n : \mathcal{H} \to \mathcal{E}_n,$$

be a sequence of mutually perpendicular projection operators. Prove that there exists a bounded linear operator

$$\mathcal{A} \in \mathbb{B}(\mathcal{H}\mathcal{H}),$$

such that

$$\mathcal{A} = \lim_{n \to \infty} \sum_{k=1}^{n} \lambda_k \mathcal{P}_k \text{ strongly}.$$

Moreover, prove that $\{\lambda_n\}$ are the eigenvalues of $\mathcal{A}$ and

$$\mathcal{E}_{\lambda_n} = \mathcal{P}_n\mathcal{H}.$$

Additionally, prove that

$$\|\mathcal{A}\| = \sup_{n\geq 1} |\lambda_n|.$$

19. Let $\mathcal{H}$ be a Hilbert space, let $\mathcal{E}$ be a closed subspace and let

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}).$$

Suppose that $\mathcal{E}$ and $\mathcal{E}^\perp$ are invariant subspaces of $\mathcal{A}$. Prove that

$\mathcal{E}$ and $\mathcal{E}^\perp$ are invariant subspaces of $\mathcal{A}^*$.

$\mathcal{E}$ and $\mathcal{E}^\perp$ are invariant subspaces of $\mathcal{A}^{-1}$, if $\mathcal{A}^{-1} \in \mathbb{B}(\mathcal{H} \to \mathcal{H})$.

20. Let $\mathcal{H}$ be a Hilbert space, let

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}).$$

Suppose that

$$\mathcal{A}^*\mathcal{A} - \mathcal{A}\mathcal{A}^* \geq 0.$$

Prove that

$\mathcal{N}(\mathcal{A})$ and $[\mathcal{N}(\mathcal{A})]^\perp$ are invariant subspaces of $\mathcal{A}$.

$\mathcal{N}(\lambda\mathcal{I} - \mathcal{A})$ and $[\mathcal{N}(\lambda\mathcal{I} - \mathcal{A})]^\perp$ are invariant subspaces of $\mathcal{A}$.

21. Let $\mathcal{H}$ be an inner product space, let

$$f : \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

be a bilinear functional. Prove that f is bounded, if and only if f is continuous with respect to the two variables.

22. Let $\mathcal{H}$ be a Hilbert space, let

$$f : \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

be a bilinear functional. Prove that f is bounded, if and only if f is a continuous function of one variable when the other variable is fixed.

23. Let $\mathcal{H}$ be an inner product space, let

$$f : \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

be a bilinear functional. Suppose that

$$\sup_{\|\mathbf{x}\|=1} |f(\mathbf{x}, \mathbf{x})| < \infty.$$

Is f bounded?

24. Let $\mathcal{H}$ be a complex inner product space, let

$$f : \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

be a bilinear functional, such that

$$\mathrm{Re}\ f(\mathbf{x}, \mathbf{x}) = 0,$$

for all $\mathbf{x} \in \mathcal{H}$. Is

$$\sup_{\|\mathbf{x}\|=1, \|\mathbf{y}\|=1} |f(\mathbf{x}, \mathbf{y})| = \sup_{\|\mathbf{x}\|=1} |f(\mathbf{x}, \mathbf{x})|,$$

true?

25. Let $\mathcal{H}$ be a Hilbert space, let

$$f : \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

be a bilinear functional. f is called non-degenerate, if there exists no nonzero vector $\mathbf{y} \in \mathcal{H}$, such that

$$\begin{aligned} &(1) \quad f(\mathbf{x}, \mathbf{y}) = 0, \ \text{for all } \mathbf{y} \in \mathcal{H}, \\ &(2) \quad f(\mathbf{y}, \mathbf{x}) = 0, \ \text{for all } \mathbf{y} \in \mathcal{H}. \end{aligned}$$

Prove that f is non-degenerate, if and only if there exists a bounded linear operator

$$\mathcal{A} \in \mathbb{B}(\mathcal{H} \to \mathcal{H}),$$

such that

$$f(\mathbf{x}, \mathbf{y}) = (\mathcal{A}\mathbf{x}, \mathbf{y}),$$

and that

$$\mathcal{N}(\mathcal{A}) = \{\mathbf{0}\}, \qquad \overline{\mathcal{A}\mathcal{H}} = \mathcal{H}.$$

Definition. Let $\mathcal{H}$ be a vector space, let

$$[\cdot, \cdot] : \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

be a functional, such that

$$\begin{aligned} &(1) \quad [\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}] = \alpha[\mathbf{x}, \mathbf{z}] + \beta[\mathbf{y}, \mathbf{z}], \\ &(2) \quad \overline{[\mathbf{y}, \mathbf{x}]} = [\mathbf{x}, \mathbf{y}], \\ &(3) \quad [\mathbf{x}, \mathbf{x}] \geq 0, \end{aligned}$$

for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{H}$. We call $\mathcal{H}$ an inner product$*$ space.

26. Let $\mathcal{H}$ be an inner product$*$ space. Prove that

$$|[\mathbf{x}, \mathbf{y}]|^2 \leq [\mathbf{x}, \mathbf{x}][\mathbf{y}, \mathbf{y}],$$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{H}$.

27. Let $\mathcal{H}$ be an inner product$*$ space. Suppose that

$$\lim_{n\to\infty} [\mathbf{x}_n - \mathbf{x}_0, \mathbf{x}_n - \mathbf{x}_0] = 0, \ \lim_{n\to\infty} [\mathbf{y}_n - \mathbf{y}_0, \mathbf{y_n} - \mathbf{y}_0] = 0.$$

Prove that

$$\lim_{n\to\infty} |[\mathbf{x}_n, \mathbf{y}_n] - [\mathbf{x}_0, \mathbf{y}_0]| = 0.$$

Definition. If

$$[\mathbf{x}, \mathbf{y}] = 0,$$

then we say $\mathbf{x}$ and $\mathbf{y}$ are perpendicular and we write $\mathbf{x} \perp \mathbf{y}$.

Definition. Let $\mathcal{H}$ be an inner product$*$ space, let $\mathcal{E}$ be a closed subspace. Let $\mathbf{x}_1 \in \mathcal{H}$. If there exists $\mathbf{x}_0 \in \mathcal{E}$ and $\mathbf{x}' \perp \mathcal{E}$, such that

$$\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{x}',$$

then we say $\mathbf{x}_0$ is the projection of $\mathbf{x}_1$ in $\mathcal{E}$.

28. Let $\mathcal{H}$ be an inner product$*$ space, let $\mathcal{E}$ be a complete convex set in $\mathcal{H}$, let $\mathbf{x}_1 \in \mathcal{H} - \mathcal{E}$. Prove that there exists a point $\mathbf{x}_0 \in \mathcal{E}$, such that

$$[\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0] = \inf_{\mathbf{x} \in \mathcal{E}} [\mathbf{x}_1 - \mathbf{x}, \mathbf{x}_1 - \mathbf{x}].$$

29. Let $\mathcal{H}$ be an inner product$*$ space, let $\mathcal{E}$ be a closed subspace, let $\mathbf{x}_1 \in \mathcal{H} - \mathcal{E}$. Suppose that there exists a point $\mathbf{x}_0 \in \mathcal{E}$, such that

$$[\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_1 - \mathbf{x}_0] = \inf_{\mathbf{x} \in \mathcal{E}} [\mathbf{x}_1 - \mathbf{x}, \mathbf{x}_1 - \mathbf{x}].$$

Prove that

$$\mathbf{x}_1 - \mathbf{x}_0 \perp \mathcal{E}.$$

30. Let $\mathcal{H}$ be an inner product$*$ space, let $\mathcal{E}$ be a complete convex subspace, let $\mathbf{x}_1 \in \mathcal{H}$. Then the projection $\mathbf{x}_0$ of $\mathbf{x}_1$ onto $\mathcal{E}$ always exists.

$$\mathcal{H}\mathcal{H}\mathcal{H} \to \mathcal{H}\mathcal{H} \to \mathcal{H}$$

$$\mathcal{H} \to \mathcal{H}$$

$$\mathcal{E}\mathcal{E}\mathcal{E}\mathcal{E}$$