Chapter 8
Limit theorems in discrete stochastic geometry

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Abstract We survey two general methods for establishing limit theorems for functionals in discrete stochastic geometry. The functionals are linear statistics with the general representation $\sum_{x\in\mathcal{X}} \xi(x,\mathcal{X})$, where $\mathcal{X}$ is finite and where the interactions of x with respect to $\mathcal{X}$, given by $\xi(x,\mathcal{X})$, are spatially correlated. We focus on subadditive methods and stabilization methods as a way to obtain weak laws of large numbers, variance asymptotics, and central limit theorems for normalized and rescaled versions of $\sum_{i=1}^{n} \xi(\eta_i, \{\eta_j\}_{j=1}^{n})$, where $\eta_j, j \geq 1$, are i.i.d. random variables. The general theory is applied to deduce the limit theory for functionals arising in Euclidean combinatorial optimization, convex hulls of i.i.d. samples, random sequential packing, and dimension estimation.

8.1 Introduction

This overview surveys two general methods for establishing limit theorems, including weak laws of large numbers, variance asymptotics, and central limit theorems, for functionals of large random geometric structures. By geometric structures, we mean for example networks arising in computational geometry, graphs arising in Euclidean optimization problems, models for random sequential packing, germ-grain models, and the convex hull of high density point sets. Such diverse structures share only the common feature that they are defined in terms of random points belonging to Euclidean space $\mathbb{R}^d$. The points are often the realization of i.i.d. random variables, but they could also be the realization of Poisson point processes or even Gibbs point processes. There is scope here for generalization to functionals of point processes in more general spaces, including manifolds and general metric spaces, but for ease of exposition we shall usually restrict attention to point processes in $\mathbb{R}^d$. As such,

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this introductory overview makes few demands involving prior familiarity with the literature.

Our goals are to provide an accessible survey of asymptotic methods involving (i) subadditivity and (ii) stabilization and to illustrate the applicability of these methods to problems in discrete stochastic geometry. The treatment of subadditivity parallels that in [524].

8.1.1 Functionals of interest

Functionals of geometric structures are often formulated as *linear statistics* on finite point sets $\mathcal{X}$ of $\mathbb{R}^d$, that is to say consist of sums represented as

$$H(\mathcal{X}) := H^{\xi}(\mathcal{X}) := \sum_{x \in \mathcal{X}} \xi(x, \mathcal{X}), \tag{8.1}$$

where the function ξ, defined on all pairs $(x, \mathcal{X})$, $x \in \mathcal{X}$, represents the *interaction* of x with respect to input $\mathcal{X}$.

The focus of this chapter is to develop the large n limit theory for the normalized sums

$$n^{-1} H^{\xi}(\{\eta_i\}_{i=1}^n), \tag{8.2}$$

where $\eta_i, i \geq 1$, are i.i.d. with values in $[0,1]^d$. We seek mean and variance asymptotics for (8.2) as well as central limit theorems for $n^{-1/2}(H^{\xi}(\{\eta_i\}_{i=1}^n) - \mathbf{E}H^{\xi}(\{\eta_i\}_{i=1}^n))$, as $n \to \infty$. In nearly all problems of interest, the values of $\xi(x, \mathcal{X})$ and $\xi(y, \mathcal{X}), x \neq y$, are not unrelated but, loosely speaking, become more related as the Euclidean distance $\|x - y\|$ becomes smaller. This 'spatial dependency' is the chief source of difficulty when developing the limit theory for H^{ξ} on random point sets.

Typical questions motivating this survey, which may be framed in terms of the linear statistics (8.1), include the following:

1. Given i.i.d. points $\eta_1, ..., \eta_n$ in the unit cube $[0,1]^d$, what is the asymptotic length of the shortest tour through $\eta_1, ..., \eta_n$? To see that this question fits into the framework of (8.1), it suffices to let $\xi(x, \mathcal{X})$ be one half the sum of the lengths of edges incident to x in the shortest tour on $\mathcal{X}$. $H^{\xi}(\mathcal{X})$ is the length of the shortest tour through $\mathcal{X}$.

2. Given i.i.d. points $\eta_1, ...\eta_n$ in the unit volume d-dimensional ball, what is the asymptotic distribution of the number of k-dimensional faces, $k \in \{0, 1, ..., d-1\}$, in the random polytope given by the convex hull of $\eta_1, ..., \eta_n$? To fit this question into the framework of (8.1), we let $\xi_k(x, \mathcal{X})$ be zero if x is not a vertex in the convex hull of $\mathcal{X}$ and otherwise we let it be the product of $(k+1)^{-1}$ and the number of k-dimensional faces containing x. $H^{\xi_k}(\mathcal{X})$ is the number of k-faces in the convex hull of $\mathcal{X}$.

3. Open balls $B_1,\ldots,B_n$ of volume n^{-1} arrive sequentially and uniformly at random in $[0,1]^d$. The first ball B_1 is *packed*, and recursively for $i = 2,3,\ldots$, the i-th ball B_i is packed iff B_i does not overlap any ball in $B_1,\ldots,B_{i-1}$ which has already been packed. If not packed, the i-th ball is discarded. The process continues until no more balls can be packed. As $n \to \infty$, what is the asymptotic distribution of the number of balls which are packed in $[0,1]^d$? To fit this into the set-up of (8.1), we let $\xi(x,\mathcal{X})$ be equal to one or zero depending on whether the ball with center at $x \in \mathcal{X}$ is accepted or not. $H^\xi(\mathcal{X})$ is the total number of accepted balls.

When $\mathcal{X}$ is the realization of a growing point set of random variables, the large scale asymptotic analysis of the sums (8.1) is sometimes handled by M-dependent methods, ergodic theory, or mixing methods; see for example Chapter 10. However, these classical methods, when applicable, may not give explicit asymptotics in terms of the underlying interaction and point densities, they may not yield second order results, or they may not easily yield rates of convergence. Our goal is to provide an abridged treatment of two alternate methods suited to the asymptotic theory of the sums (8.2), namely to discuss (i) subadditivity and stabilization.

Subadditive methods lean heavily on the self-similarity of the unit cube, but to obtain distributional results, variance asymptotics, and explicit limiting constants in laws of large numbers, one needs tools going beyond subadditivity. When the spatial dependency may be localized, in a sense to be made precise, then this localization yields distributional and second order results, and it also shows that the *large scale macroscopic behaviour of* H^ξ on random point sets, for example laws of large numbers and central limit theorems, *is governed by the local interactions involving* ξ.

The subadditive approach, described in detail in the monographs [482], [524], yields a.s. laws of large numbers for problems in Euclidean combinatorial optimization, including the length of minimal spanning trees, minimal matchings, and shortest tours on random point sets. Formal definitions of these archetypical problems are given below. Subadditive methods also yield the a.s. limit theory of problems in computational geometry, including the total edge length of nearest neighbour graphs, the Voronoi and Delaunay graphs, the sphere of influence graph, as well as graphs graphs arising in minimal triangulations and the k-means problem. The approach based on stabilization, originating in Penrose and Yukich [398] and further developed in [57, 395, 396, 400, 402], is useful in proving laws of large numbers, central limit theorems, and variance asymptotics for many of these functionals; as such it provides closed form expressions for the limiting constants arising in the mean and variance asymptotics. This approach has been used to study linear statistics arising in random packing [400], convex hulls [459], ballistic deposition models [57, 400], quantization [460, 525], loss networks [460], high-dimensional spacings [56], distributed inference in random networks [12], and geometric graphs in Euclidean combinatorial optimization [398, 399].

8.1.2 Examples

Letting input $\mathcal{X} := \{x_1, ..., x_n\}$ be a finite point set in $\mathbb{R}^d$, functionals and graphs of interest include:

1. **Traveling salesman functional; TSP.** A closed tour on $\mathcal{X}$ or closed Hamiltonian tour is a closed path traversing each vertex in $\mathcal{X}$ exactly once. Let $\mathrm{TSP}(\mathcal{X})$ be the length of the shortest closed tour T on $\mathcal{X}$. Thus

$$\mathrm{TSP}(\mathcal{X}) := \min_T \sum_{e \in T} |e|, \tag{8.3}$$

where the minimum is over all tours T on $\mathcal{X}$ and where $|e|$ denotes the Euclidean edge length of the edge e. Thus,

$$\mathrm{TSP}(\mathcal{X}) := \min_\sigma \left\{ \|x_{\sigma(n)} - x_{\sigma(1)}\| + \sum_{i=1}^{n-1} \|x_{\sigma(i)} - x_{\sigma(i+1)}\| \right\},$$

where the minimum is taken over all permutations σ of the integers $1, 2, ..., n$ and where $\|\cdot\|$ denotes the Euclidean norm.

2. **Minimum spanning tree; MST.** Let $\mathrm{MST}(\mathcal{X})$ be the length of the shortest spanning tree on $\mathcal{X}$, namely

$$\mathrm{MST}(\mathcal{X}) := \min_T \sum_{e \in T} |e|, \tag{8.4}$$

where the minimum is over all spanning trees T on $\mathcal{X}$.

3. **Minimal matching; MM.** The *minimal matching* on $\mathcal{X}$ has length given by

$$\mathrm{MM}(\mathcal{X}) := \min_\sigma \sum_{i=1}^{n/2} \|x_{\sigma(2i-1)} - x_{\sigma(2i)}\|, \tag{8.5}$$

where the minimum is over all permutations of the integers $1, 2, ..., n$. If n has odd parity, then the minimal matching on $\mathcal{X}$ is the minimum of the minimal matchings on the n distinct subsets of $\mathcal{X}$ of size $n-1$.

4. ***k*-nearest neighbours graph.** Let $k \in \mathbb{N}$. The k-nearest neighbours (undirected) graph on $\mathcal{X}$, here denoted $G^N(k, \mathcal{X})$, is the graph with vertex set $\mathcal{X}$ obtained by including $\{x, y\}$ as an edge whenever y is one of the k nearest neighbours of x and/or x is one of the k nearest neighbours of y. The k-nearest neighbours (directed) graph on $\mathcal{X}$, denoted $\overrightarrow{G}^N(k, \mathcal{X})$, is the graph with vertex set $\mathcal{X}$ obtained by placing an edge between each point and its k nearest neighbours. Let $\mathrm{NN}(k, \mathcal{X})$ denote the total edge length of $G^N(k, \mathcal{X})$, i.e.,

$$\mathrm{NN}(k, \mathcal{X}) := \sum_{e \in G^N(k, \mathcal{X})} |e|, \tag{8.6}$$

with a similar definition for the total edge length of $\overrightarrow{G}^N(k,\mathcal{X})$.

5. **Steiner minimal spanning tree.** Consider the problem of finding the graph of shortest length which connects the vertices of $\mathcal{X}$. Such a graph is a tree, known as the Steiner minimal spanning tree, and it may include vertices other than those in $\mathcal{X}$. If not, the graph coincides with the minimal spanning tree graph. The total edge length of the Steiner minimal spanning tree on $\mathcal{X}$ is

$$\mathrm{ST}(\mathcal{X}) := \min_S \sum_{e\in S} |e|, \tag{8.7}$$

where the minimum ranges over all connected graphs S on $\mathcal{X}$.

6. **Minimal semi-matching.** A semi-matching on $\mathcal{X}$ is a graph in which all vertices have degree 2, with the understanding that an isolated edge between two vertices represents two copies of that edge. The graph thus contains tours with an odd number of edges as well as isolated edges. The minimal semi-matching functional on $\mathcal{X}$ is

$$\mathrm{SM}(\mathcal{X}) := \min_{\mathrm{SM}} \sum_{e\in \mathrm{SM}} |e|, \tag{8.8}$$

where the minimum ranges over all semi-matchings SM on $\mathcal{X}$.

7. ***k*-TSP functional.** Fix $k \in \mathbb{N}$. Let $\mathcal{C}$ be a collection of k sub-tours on points of $\mathcal{X}$, each sub-tour containing a distinguished shared vertex x_0 and such that each $x \in \mathcal{X}$ belongs to exactly one sub-tour. $T(k;\mathcal{C},\mathcal{X})$ is the sum of the combined lengths of the k sub-tours in $\mathcal{C}$. The k-TSP functional is the infimum

$$T(k;\mathcal{X}) := \inf_{\mathcal{C}} T(k;\mathcal{C},\mathcal{X}). \tag{8.9}$$

Power-weighted edge versions of these functionals are found in [524]. For example, $MST^{(p)}(\mathcal{X})$ is the length of the shortest spanning tree on $\mathcal{X}$ with pth power weighted edges, namely

$$\mathrm{MST}^{(p)}(\mathcal{X}) := \min_T \sum_{e\in T} |e|^p, \tag{8.10}$$

where the minimum is over all spanning trees T on $\mathcal{X}$.

To allow for power weighted edges, we henceforth let the interaction ξ depend on a parameter $p \in (0,\infty)$ and we will write $\xi(\cdot,\cdot) := \xi_p(\cdot,\cdot)$. We henceforth work in this context, but to lighten the notation we shall suppress mention of p.

8.2 Subadditivity

This section gives an introductory account of asymptotic methods based on the subadditivity of the functionals H^ξ defined at (8.1). It culminates with a general umbrella theorem providing an a.s. law of large numbers for H^ξ.

8.2.1 Subadditive functionals

Let $x_n \in \mathbb{R},\ n \geq 1$, satisfy the 'subadditive inequality'

$$x_{m+n} \leq x_m + x_n \ \text{ for all } m,\, n \in \mathbb{N}. \tag{8.11}$$

Subadditive sequences are nearly additive in the sense that they satisfy the *subadditive limit theorem*, namely $\lim_{n\to\infty} x_n/n = \alpha$ where $\alpha := \inf\{x_m/m : m \geq 1\} \in [-\infty, \infty)$. This classic result, proved in Hille [245], may be viewed as a limit result about subadditive functions indexed by intervals.

For certain choices of the interaction ξ, the functionals H^ξ defined at (8.1) satisfy geometric subadditivity over *rectangles* and, as we will see, consequently satisfy a subadditive limit theorem analogous to the classic one just mentioned.

Let $\mathcal{R} := \mathcal{R}(d)$ denote the collection of d-dimensional rectangles in $\mathbb{R}^d$. Recall that $\xi(\cdot,\cdot) := \xi_p(\cdot,\cdot)$ depends on the parameter p. Write $H^\xi(\mathcal{X}, R)$ for $H^\xi(\mathcal{X} \cap R)$, $R \in \mathcal{R}$. Say that H^ξ is *geometrically subadditive*, or simply *subadditive*, if there is a constant $c_1 := c_1(p) < \infty$ such that for all $R \in \mathcal{R}$, all partitions of R into rectangles R_1 and R_2, and all finite point sets $\mathcal{X}$ we have

$$H^\xi(\mathcal{X}, R) \leq H^\xi(\mathcal{X}, R_1) + H^\xi(\mathcal{X}, R_2) + c_1(\mathrm{diam}(R))^p. \tag{8.12}$$

Unlike scalar subadditivity (8.11), the relation (8.12) carries an error term.

Classic optimization problems as well as certain functionals of Euclidean graphs, satisfy geometric subadditivity (8.12). For example, the length of the minimal spanning tree defined at (8.4) satisfies (8.12) when p is set to 1, which may be seen as follows. Put $\mathrm{MST}(\mathcal{X}, R)$ to be the length of the minimal spanning tree on $\mathcal{X} \cap R$. Given a finite set $\mathcal{X}$ and a rectangle $R := R_1 \cup R_2$, let $\mathcal{T}_i$ denote the minimal spanning tree on $\mathcal{X} \cap R_i$, $1 \leq i \leq 2$. Tie together the two spanning trees $\mathcal{T}_1$ and $\mathcal{T}_2$ with an edge having a length bounded by the sum of the diameters of the rectangles R_1 and R_2. Performing this operation generates a feasible spanning tree on $\mathcal{X}$ at a total cost bounded by $\mathrm{MST}(\mathcal{X}, R_1)$ + $\mathrm{MST}(\mathcal{X}, R_2)$ + $\mathrm{diam}(R)$. Putting $p = 1$, (8.12) follows by minimality.

Exercise 8.1. Using edge deletion and insertion techniques, show that the TSP functional(8.3), minimal matching functional (8.5), and nearest neighbour functionals (8.6) satisfy geometric subadditivity (8.12) with $p = 1$.

8.2.2 Superadditive functionals

If geometric functionals H^{ξ} were to simultaneously satisfy a superadditive relation analogous to (8.12), then the resulting 'near additivity' of H^{ξ} would lead directly to laws of large numbers. This is too much to expect. On the other hand, many geometric functionals $H^{\xi}(\cdot,R)$ admit a 'dual' version - one which essentially treats the boundary of the rectangle R as a single point, that is to say edges on the boundary ∂R have zero length or 'zero cost'. This boundary version, introduced in [415] and used in [416] and [417] and here denoted $H_B^{\xi}(\cdot,R)$, closely approximates $H^{\xi}(\cdot,R)$ in a sense to be made precise (see (8.18) below) and is *superadditive without any error term*. More exactly, the boundary version $H_B^{\xi}(\cdot,R)$ satisfies

$$H_B^{\xi}(\mathcal{X},R) \geq H_B^{\xi}(\mathcal{X}\cap R_1,R_1)+H_B^{\xi}(\mathcal{X}\cap R_2,R_2). \tag{8.13}$$

Boundary functionals are defined on a case-by-case basis. For example, the boundary minimal spanning tree functional is defined as follows. For all rectangles $R \in \mathcal{R}$ and finite sets $\mathcal{X} \subset R$ put

$$\mathrm{MST}_B(\mathcal{X},R) := \min\left(\mathrm{MST}(\mathcal{X},R),\ \inf\sum_i \mathrm{MST}(\mathcal{X}_i \cup \{a_i\})\right),$$

where the infimum ranges over all partitions $(\mathcal{X}_i)_{i\geq 1}$ of $\mathcal{X}$ and all sequences of points $(a_i)_{i\geq 1}$ belonging to ∂R. When $\mathrm{MST}_B(\mathcal{X},R) \neq \mathrm{MST}(\mathcal{X},R)$ the graph realizing the boundary functional $\mathrm{MST}_B(\mathcal{X},R)$ may be thought of as a collection of small trees connected via the boundary ∂R into a single large tree, where the connections on ∂R incur no cost. See Figure 8.1. It is a simple matter to see that the boundary MST functional satisfies subadditivity (8.12) with $p=1$ and is also superadditive (8.13). Later we will see that the boundary MST functional closely approximates the standard MST functional.

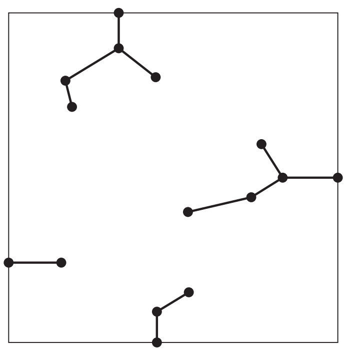

Fig. 8.1 The boundary MST graph; edges on boundary have zero cost.

Exercise 8.2. Show that the TSP (8.3), minimal matching (8.5), and nearest neighbour functionals (8.6) have boundary versions which are superadditive (8.13).

8.2.3 Subadditive and superadditive Euclidean functionals

Recall that $\xi(\cdot,\cdot) := \xi_p(\cdot,\cdot)$. The following conditions endow the functional $H^\xi(\cdot,\cdot)$ with a *Euclidean structure*:

$$H^\xi(\mathcal{X}, R) = H^\xi(\mathcal{X}+y, R+y) \tag{8.14}$$

for all $y \in \mathbb{R}^d$, $R \in \mathcal{R}$, $\mathcal{X} \subset R$ and

$$H^\xi(\alpha\mathcal{X}, \alpha R) = \alpha^p H^\xi(\mathcal{X}, R) \tag{8.15}$$

for all $\alpha > 0$, $R \in \mathcal{R}$ and $\mathcal{X} \subset R$. By αB we understand the set $\{\alpha x, x \in B\}$ and by $y+\mathcal{X}$ we mean $\{y+x : x \in \mathcal{X}\}$. Conditions (8.14) and (8.15) express the *translation invariance* and *homogeneity of order p* of H^ξ, respectively. Homogeneity (8.15) is satisfied whenever the interaction ξ is itself homogeneous of order p, that is to say whenever

$$\xi(\alpha x, \alpha\mathcal{X}) = \alpha^p \xi(x, \mathcal{X}), \quad \alpha > 0. \tag{8.16}$$

Functionals satisfying translation invariance and homogeneity of order 1 include the total edge length of graphs, including those defined at (8.3)-(8.9).

Exercise 8.3. Show that the TSP functional (8.3), MST functional (8.4), and minimal matching functional (8.5) are homogeneous of order 1 and are thus subadditive Euclidean functionals.

Definition 8.1. Let $H^\xi(\emptyset, R) = 0$ for all $R \in \mathcal{R}$ and suppose H^ξ satisfies geometric subadditivity (8.12), translation invariance (8.14), and homogeneity of order p (8.15). Then H^ξ is a *subadditive Euclidean functional*.

If a functional $H^\xi(\mathcal{X}, R)$, $(\mathcal{X}, R) \in \mathcal{N} \times \mathcal{R}$, is superadditive over rectangles and has a Euclidean structure over $\mathcal{N} \times \mathcal{R}$, where $\mathcal{N}$ is the collection of locally finite point sets in $\mathbb{R}^d$, then we say that H^ξ is a *superadditive Euclidean functional*, formally defined as follows:

Definition 8.2. Let $H^\xi(\emptyset, R) = 0$ for all $R \in \mathcal{R}$ and suppose H^ξ satisfies (8.14) and (8.15). If H^ξ satisfies

$$H^\xi(\mathcal{X}, R) \geq H^\xi(\mathcal{X} \cap R_1, R_1) + H^\xi(\mathcal{X} \cap R_2, R_2), \tag{8.17}$$

whenever $R \in \mathcal{R}$ is partitioned into rectangles R_1 and R_2 then H^ξ is a *superadditive Euclidean functional*.

It may be shown that the functionals TSP, MST and MM are subadditive Euclidean functionals and that they admit dual boundary versions which are superadditive Euclidean functionals; see Chapter 2 of [524].

Pointwise close property

To be useful in establishing asymptotics, dual boundary functionals must closely approximate the corresponding functional. The following closeness condition is sufficient for these purposes. Recall that we suppress the dependence of ξ on p, writing $\xi(\cdot,\cdot) := \xi_p(\cdot,\cdot)$.

Definition 8.3. Say that $H^\xi := H^{\xi_p}$ and the boundary version $H_B^\xi := H_B^{\xi_p}$, $p \in (0,\infty)$, are *pointwise close* if for all finite subsets $\mathcal{X} \subset [0,1]^d$ we have

$$|H^\xi(\mathcal{X},[0,1]^d) - H_B^\xi(\mathcal{X},[0,1]^d)| = o\left((\mathrm{card}(\mathcal{X}))^{(d-p)/d}\right). \tag{8.18}$$

The TSP, MST, MM and nearest neighbour functionals all admit respective boundary versions which are pointwise close in the sense of (8.18); see Lemma 3.7 of [524]. See [524] for description of other functionals having boundary versions which are pointwise close in the sense of (8.18).

Growth bounds

Iteration of geometric subadditivity (8.12) leads to growth bounds on subadditive Euclidean functionals H^ξ, namely for all $p \in (0,d)$ there is a constant $c_2 := c_2(\xi_p,d)$ such that for all rectangles $R \in \mathcal{R}$ and all $\mathcal{X} \subset R$, $\mathcal{X} \in \mathcal{N}$, we have

$$H^\xi(\mathcal{X},R) \le c_2(\mathrm{diam}(R))^p(\mathrm{card}\,\mathcal{X})^{(d-p)/d}. \tag{8.19}$$

Smooth of order p

Subadditivity (8.12) and growth bounds (8.19) by themselves do not provide enough structure to yield the limit theory for Euclidean functionals; one also needs to control the oscillations of these functionals as points are added or deleted. Some functionals, such as TSP, necessarily increase with increasing argument size, whereas others, such as MST, do not have this property. A useful continuity condition goes as follows.

Definition 8.4. A Euclidean functional $H^\xi := H^{\xi_p}$, $p \in (0,\infty)$, is *smooth of order p* if there is a finite constant $c_3 := c_3(\xi_p,d)$ such that for all finite sets $\mathcal{X}_1, \mathcal{X}_2 \subset [0,1]^d$ we have

$$|H^\xi(\mathcal{X}_1 \cup \mathcal{X}_2) - H^\xi(\mathcal{X}_1)| \le c_3(\mathrm{card}(\mathcal{X}_2))^{(d-p)/d}. \tag{8.20}$$

8.2.4 Examples of functionals satisfying smoothness (8.20)

1. Let TSP be as in (8.3). For all finite sets $\mathcal{X}_1$ and $\mathcal{X}_2 \subset [0,1]^d$ we have

$$\mathrm{TSP}(\mathcal{X}_1) \le \mathrm{TSP}(\mathcal{X}_1 \cup \mathcal{X}_2) \le \mathrm{TSP}(\mathcal{X}_1) + \mathrm{TSP}(\mathcal{X}_2) + c\mathrm{diam}([0,1]^d),$$

where the first inequality follows by monotonicity and the second by subadditivity (8.12). By (8.19) we have $\mathrm{TSP}(\mathcal{X}_2) \le c_2\sqrt{d}(\mathrm{card}\,\mathcal{X}_2)^{(d-1)/d}$ and since clearly $c\mathrm{diam}([0,1]^d) \le cd^{1/2}(\mathrm{card}(\mathcal{X}_2)^{(d-1)/d}$, it follows that the TSP is smooth of order 1.

2. Let MST be as in (8.4). Subadditivity (8.12) and the growth bounds (8.19) imply that for all finite sets $\mathcal{X}_1, \mathcal{X}_2 \subset [0,1]^d$ we have $\mathrm{MST}(\mathcal{X}_1 \cup \mathcal{X}_2) \le \mathrm{MST}(\mathcal{X}_1) + (c_1\sqrt{d} + c_2\sqrt{d}(\mathrm{card}\,\mathcal{X}_2)^{(d-1)/d} \le \mathrm{MST}(\mathcal{X}_1) + c(\mathrm{card}\,\mathcal{X}_2)^{(d-1)/d}$. It follows that the MST is smooth of order 1 once we show the reverse inequality

$$\mathrm{MST}(\mathcal{X}_1 \cup \mathcal{X}_2) \ge \mathrm{MST}(\mathcal{X}_1) - c(\mathrm{card}\,\mathcal{X}_2)^{(d-1)/d}. \tag{8.21}$$

To show (8.21) let $\mathcal{T}$ denote the graph of the minimal spanning tree on $\mathcal{X}_1 \cup \mathcal{X}_2$. Remove the edges in $\mathcal{T}$ which contain a vertex in $\mathcal{X}_2$. Since each vertex has bounded degree, say D, this generates a subgraph $\mathcal{T}_1 \subset \mathcal{T}$ which has at most $D \cdot \mathrm{card}\,\mathcal{X}_2$ components. Choose one vertex from each component and form the minimal spanning tree $\mathcal{T}_2$ on these vertices. By the growth bounds (8.19), the edge length of $\mathcal{T}_2$ is bounded by $c(D \cdot \mathrm{card}\,\mathcal{X}_2)^{(d-1)/d}$. Since the union of the trees $\mathcal{T}_1$ and $\mathcal{T}_2$ is a feasible spanning tree on $\mathcal{X}_1$, it follows that

$$\mathrm{MST}(\mathcal{X}_1) \le \sum_{e \in \mathcal{T}_1 \cup \mathcal{T}_2} |e| \le \mathrm{MST}(\mathcal{X}_1 \cup \mathcal{X}_2) + c(D \cdot \mathrm{card}\,\mathcal{X}_2)^{(d-1)/d}.$$

Thus smoothness (8.20) holds for the MST functional.

It may be shown that a modification of the Steiner functional (8.7) is smooth of order 1 (see Chapter 10 of [524]). Smoothness is a common property of geometric functionals, as indicated in the next exercise.

Exercise 8.4. Show that the minimal matching functional MM defined at (8.5) is smooth of order 1. Likewise, show that the semi-matching, nearest neighbour, and k-TSP functionals are smooth of order 1. Hints; see Chapter 3.3 of [524]), Sections 8.2, 8.3 and 8.4 of [524], respectively.

The functionals TSP, MST and MM defined at (8.3)-(8.5) are thus smooth subadditive Euclidean functionals which are pointwise close to a canonical boundary functional. The functionals (8.6)-(8.9) satisfy the same properties. Now we give some limit theorems for such functionals.

8.2.5 Laws of large numbers for superadditive Euclidean functionals

We state a basic law of large numbers for Euclidean functionals on i.i.d. uniform random variables $U_1, \ldots, U_n$ in $[0,1]^d$. Recall that a sequence of random variables

ζ_n converges completely, here denoted c.c., to a limit random variable ζ, if for all $\varepsilon > 0$, we have $\sum_{n=1}^{\infty} \mathbf{P}(|\zeta_n - \zeta| > \varepsilon) < \infty$.

Theorem 8.1. *Let $p \in [1,d)$. If $H_B^{\xi} := H_B^{\xi_p}$ is a smooth superadditive Euclidean functional of order p on $\mathbb{R}^d$, then*

$$\lim_{n\to\infty} n^{(p-d)/d} H_B^{\xi}(U_1,...,U_n) = \alpha(H_B^{\xi}, d) \ \ c.c., \tag{8.22}$$

where $\alpha(H_B^{\xi}, d)$ is a positive constant. If $H^{\xi} := H^{\xi_p}$ is a subadditive Euclidean functional which is pointwise close to $H_B^{\xi} := H_B^{\xi_p}$ as in (8.18), then

$$\lim_{n\to\infty} n^{(p-d)/d} H^{\xi}(U_1,...,U_n) = \alpha(H_B^{\xi}, d) \ \ c.c. \tag{8.23}$$

Remarks.

1. In practice, Theorem 8.1 involves taking $H_B^{\xi} := H_B^{\xi_p}$ to be a boundary version of $H^{\xi} := H^{\xi_p}$, but it is conceivable that there are functionals $H_B^{\xi_p}$ which satisfy the conditions of Theorem 8.1 and which are not boundary versions. By considering boundary functionals, Theorem 8.1 gives laws of large numbers for the functionals (8.3)-(8.9); see [524] for details.
2. Smooth subadditive Euclidean functionals which are point-wise close to smooth superadditive Euclidean functionals are ‘nearly additive’ and consequently satisfy Donsker-Varadhan-style *large deviation principles*, as shown in [463].
3. The papers [242] and [295] provide further accounts of the limit theory for subadditive Euclidean functionals.

Proof of Theorem 8.1. We only prove a mean version of (8.22), namely

$$\lim_{n\to\infty} n^{(p-d)/d} \mathbf{E} L_B^p(U_1,...,U_n) = \alpha(L_B^p, d), \tag{8.24}$$

referring the reader to [524] for a complete proof. To prove (8.24), we will follow the proof of Theorem 4.1 of [524]. Fix $1 \le p < d$ and set $\varphi(n) := \mathbf{E} L_B^p(U_1,...,U_n)$. The number of points from the sample $(U_1,...,U_n)$ belonging to a given subcube of $[0,1]^d$ of volume m^{-d} is a binomial random variable $\mathrm{Binom}(n, m^{-d})$ with parameters n and m^{-d}. Superadditivity of L_B^p, homogeneity (8.15), smoothness (8.20), and Jensen's inequality in this order yield

$$\varphi(n) \ge m^{-p} \sum_{i \le m^d} \varphi(\mathrm{Binom}(n, m^{-d}))$$

$$\ge m^{-p} \sum_{i \le m^d} \left(\varphi(nm^{-d}) - c_3 \mathbf{E}(|\mathrm{Binom}(n, m^{-d}) - nm^{-d}|^{(d-p)/d}) \right)$$

$$\ge m^{-p} \sum_{i \le m^d} \left(\varphi(nm^{-d}) - c_3 (nm^{-d})^{(d-p)/2d} \right).$$

Simplifying, we get

$$\varphi(n) \geq m^{d-p}\varphi(nm^{-d}) - c_3 m^{(d-p)/2} n^{(d-p)/2d}.$$

Dividing by $n^{(d-p)/d}$ and replacing n by nm^d yields the homogenized relation

$$\frac{\varphi(nm^d)}{(nm^d)^{(d-p)/d}} \geq \frac{\varphi(n)}{n^{(d-p)/d}} - \frac{c_3}{n^{(d-p)/2d}}. \tag{8.25}$$

Set $\alpha := \alpha(L_B^p, d) := \limsup_{n\to\infty} \varphi(n)/n^{(d-p)/d}$ and note that $\alpha \leq c_3$ by the assumed smoothness. For all $\varepsilon > 0$, choose n_o such that for all $n \geq n_o$ we have $c_3/n^{(d-p)/2d} \leq \varepsilon$ and $\varphi(n_o)/n_o^{(d-p)/d} \geq \alpha - \varepsilon$. Thus, for all $m = 1, 2, ...$ it follows that

$$\frac{\varphi(n_o m^d)}{(n_o m^d)^{(d-p)/d}} \geq \alpha - 2\varepsilon.$$

To now obtain (8.24) we use the smoothness of L and an interpolation argument. For an arbitrary integer $k \geq 1$ find the unique integer m such that

$$n_o m^d < k \leq n_o(m+1)^d.$$

Then $|n_o m^d - k| \leq C n_o m^{d-1}$ and by smoothness (8.20) we therefore obtain

$$\frac{\varphi(k)}{k^{(d-p)/d}} \geq \frac{\varphi(n_o m^d)}{(n_o(m+1)^d)^{(d-p)/d}} - \frac{(C n_o m^{d-1})^{(d-p)/d}}{(m+1)^{d-p}\, n_o^{(d-p)/d}}$$

$$\geq (\alpha - 2\varepsilon)\left(\frac{m}{m+1}\right)^{d-p} - \frac{(C n_o m^{d-1})^{(d-p)/d}}{(m+1)^{d-p}\, n_o^{(d-p)/d}}.$$

Since the last term in the above goes to zero as m goes to infinity, it follows that

$$\liminf_{k\to\infty} k^{(p-d)/d}\varphi(k) \geq \alpha - 2\varepsilon.$$

Now let ε tend to zero to see that the liminf and the limsup of the sequence $\varphi(k)/k^{(d-p)/d}$, $k \geq 1$, coincide, that is

$$\lim_{k\to\infty} k^{(p-d)/d}\varphi(k) = \alpha.$$

We have thus shown $\lim_{n\to\infty} n^{(p-d)/d}\mathbf{E}L_B^p(U_1, ..., U_n) = \alpha$ as desired. This completes the proof of (8.24). □

8.2.6 Rates of convergence of Euclidean functionals

Recall that we write $\xi(\cdot,\cdot) := \xi_p(\cdot,\cdot)$. If a subadditive Euclidean functional H^ξ is *close in mean* (cf. Definition 3.9 in [524]) to the associated superadditive Euclidean functional H_B^ξ, namely if

$$|\mathbf{E}H^{\xi}(U_1,...,U_n) - \mathbf{E}H_B^{\xi}(U_1,...,U_n)| = o(n^{(d-p)/d}), \tag{8.26}$$

where we recall that U_i are i.i.d. uniform on $[0,1]^d$, then we may upper bound $|\mathbf{E}H^{\xi}(U_1,...,U_n) - \alpha(H_B^{\xi},d)n^{(d-p)/d}|$, thus yielding rates of convergence of

$$n^{(p-d)/d}\mathbf{E}H^{\xi}(U_1,...,U_n)$$

to its mean. Since the TSP, MST, and MM functionals satisfy closeness in mean $(p \neq d-1,\ d \geq 3)$ the following theorem immediately provides rates of convergence for our prototypical examples.

Theorem 8.2. *(Rates of convergence of means) Let H^{ξ} and H_B^{ξ} be subadditive and superadditive Euclidean functionals, respectively, satisfying the close in mean approximation (8.26). If H^{ξ} is smooth of order $p \in [1,d)$ as defined at (8.20), then for $d \geq 2$ and for $\alpha(H_B^{\xi},d)$ as at (8.22), we have*

$$|\mathbf{E}H^{\xi}(U_1,...,U_n) - \alpha(H_B^{\xi},d)n^{(d-p)/d}| \leq c\left(n^{(d-p)/2d} \vee n^{(d-p-1)/d}\right). \tag{8.27}$$

For a complete proof of Theorem 8.2, we refer to [524]. Koo and Lee [309] give conditions under which Theorem 8.2 can be improved.

8.2.7 General umbrella theorem for Euclidean functionals

Here is the main result of this section. Let $\eta_1,...,\eta_n$ be i.i.d. random variables with values in $[0,1]^d,\ d \geq 2$, and put $\mathcal{X}_n := \{\eta_i\}_{i=1}^n$.

Theorem 8.3. *(Umbrella theorem for Euclidean functionals) Let H^{ξ} and H_B^{ξ} be subadditive and superadditive Euclidean functionals, respectively, both smooth of order $p \in [1,d)$. Assume that H^{ξ} and H_B^{ξ} are close in mean (8.26). Then*

$$\lim_{n\to\infty} n^{(p-d)/d}H^{\xi}(\mathcal{X}_n) = \alpha(H_B^{\xi},d)\int_{[0,1]^d}\kappa(x)^{(d-p)/d}\,dx \quad c.c., \tag{8.28}$$

where κ is the density of the absolutely continuous part of the law of η_1.

Remarks.

1. The above theorem captures the limit behavior of the total edge length of the functionals described in Section 8.1.1, hence the term ‘umbrella’. Indeed, the TSP functional satisfies the conditions of Theorem 8.3 and we thus recover as a corollary the Beardwood-Halton-Hammersley theorem [61]. See [524] for details.
2. Umbrella theorems for Euclidean functionals satisfying monotonicity and other assumptions not involving boundary functionals appear in Theorem 2 of [481]. Theorem 8.3 has its origins in [415] and [416].

3. Theorem 8.3 is used by Baltz et al. [39] to analyze asymptotics for the multiple vehicle routing problem; Costa and Hero [130] show asymptotics similar to Theorem 8.3 for the MST on suitably regular Riemannian manifolds and they apply their results to estimation of Rényi entropy and manifold dimension. Costa and Hero [131], using the theory of subadditive and superadditive Euclidean functionals, obtain asymptotics for the total edge length of k-nearest neighbour graphs on manifolds. The paper [242] provides further applications to imaging and clustering.
4. If the η_i fail to have a density then the right-hand side of (8.28) vanishes. On the other hand, Hölder's inequality shows that the right-hand side of (8.28) is largest when κ is uniform on $[0,1]^d$.
5. See Chapter 7 of [524] for extensions of Theorem 8.3 to functionals of random variables on unbounded domains.

Proof. (Sketch of proof of Theorem 8.3) The Azuma-Hoeffding concentration inequality shows that it is enough to prove convergence of means in (8.28). Smoothness then shows that it is enough to prove convergence of $n^{(p-d)/d}\mathbf{E}H^{\xi}(\mathcal{X}_n)$ for the so-called blocked distributions, i.e. those whose absolutely continuous part is a linear combination of indicators over congruent sub-cubes forming a partition of $[0,1]^d$. To establish convergence for the blocked distributions, one combines Theorem 8.1 with the subadditive and superadditive relations. We refer to [524] for complete details of these standard methods. $\square$

The limit (8.28) exhibits the asymptotic dependency of the total edge length of graphs on the underlying point density κ. Still, (8.28) is unsatisfying in that we don't have a closed form expression for the constant $\alpha(H_B^{\xi}, d)$. Stabilization methods, described below, are used to explicitly identify $\alpha(H_B^{\xi}, d)$.

8.3 Stabilization

Subadditive methods yield a.s. limit theory for the functionals H^{ξ} defined at (8.1) but they do not express the macroscopic behaviour of H^{ξ} in terms of the local interactions described by ξ. Stabilization methods overcome this limitation, they yield second order and distributional results, and they also provide limit results for the empirical measures

$$\sum_{x \in \mathcal{X}} \xi(x, \mathcal{X}) \delta_x, \tag{8.29}$$

where δ_x is the point mass at x. The empirical measure (8.29) has total mass given by H^{ξ}.

We will often assume that the interaction or 'score' function ξ, defined on pairs $(x, \mathcal{X})$, with $\mathcal{X}$ locally finite in $\mathbb{R}^d$, is translation invariant, i.e., for all $y \in \mathbb{R}^d$ we have $\xi(x+y, \mathcal{X}+y) = \xi(x, \mathcal{X})$. When $x \in \mathbb{R}^d \setminus \mathcal{X}$, we abbreviate notation and write $\xi(x, \mathcal{X})$ instead of $\xi(x, \mathcal{X} \cup \{x\})$.

When $\mathcal{X}$ is random the range of spatial dependence of ξ at $x \in \mathcal{X}$ is random and the purpose of stabilization is to quantify this range in a way useful for asymptotic analysis. There are several notions of stabilization, with the simplest being that of stabilization of ξ with respect to a rate τ homogeneous Poisson point process Π_τ on $\mathbb{R}^d$, defined as follows. Let $B_r(x)$ denote the Euclidean ball centered at x with radius r and let o denote a point at the origin of $\mathbb{R}^d$.

8.3.1 Homogeneous stabilization

We say that a translation invariant ξ is *homogeneously stabilizing* if for all τ and almost all realizations Π_τ there exists $R := R(\Pi_\tau) < \infty$ such that

$$\xi(o, (\Pi_\tau \cap B_R(o)) \cup \mathcal{A}) = \xi(o, \Pi_\tau \cap B_R(o)) \tag{8.30}$$

for all locally finite $\mathcal{A} \subset \mathbb{R}^d \setminus B_R(o)$. Thus the value of ξ at o is unaffected by changes in the configuration outside $B_R(o)$. The random range of dependency given by R depends on the realization of Π_τ. When ξ is homogeneously stabilizing we may write

$$\xi(o, \Pi_\tau) = \lim_{r \to \infty} \xi(o, \Pi_\tau \cap B_r(o)).$$

Examples of homogeneously stabilizing functionals.

1. **Nearest neighbour distances.** Recalling (8.6), consider the nearest neighbour graph $G^N(1, \mathcal{X})$ on the point set $\mathcal{X}$ and let $\xi(x, \mathcal{X})$ denote one half the sum of the lengths of edges in $G^N(1, \mathcal{X})$ which are incident to x. Thus $H^\xi(\mathcal{X})$ is the sum of edge lengths in $G^N(1, \mathcal{X})$. Partition $\mathbb{R}^2$ into six congruent cones $K_i, 1 \le i \le 6$, having apex at the origin of $\mathbb{R}^2$ and for all $1 \le i \le 6$, put R_i to be the distance between the origin and the nearest point in $\Pi_\tau \cap K_i$. We assert that $R := 2\max_{1 \le i \le 6} R_i$ is a radius of stabilization, i.e., points in $B^c_{2R}(o)$ do not change the value of $\xi(o, \Pi_\tau)$. Indeed, edges in $G^N(1, \Pi_\tau)$ incident to the origin are not changed by the addition of points in $B^c_{2R}(o)$. Such points will be closer to at least one point in $\Pi_\tau \cap B_R(o)$ than to the origin and so will not connect to the origin. Also, edges between points in $\Pi_\tau \cap B_R(o)$ and the origin will not be affected by the insertion of points in $B^c_{2R}(o)$.

2. **Voronoi graphs**. Consider the graph of the Voronoi tessellation of $\mathcal{X}$ and let $\xi(x, \mathcal{X})$ be one half the sum of the lengths of the edges in the Voronoi cell $C(x)$ around x. The Voronoi flower around x, or fundamental region, is the union of those balls having as center a vertex of $C(x)$ and exactly two points of $\mathcal{X}$ on their boundary and no points of $\mathcal{X}$ inside. Then it may be shown (see Zuyev [532]) that the geometry of $C(x)$ is completely determined by the Voronoi flower and thus the radius of a ball centered at x containing the Voronoi flower qualifies as a stabilization radius.

3. **Minimal spanning trees.** Let $\mathcal{X} \subset \mathbb{R}^d, d \geq 2$, be locally finite. Given $a > 0$, let $\mathcal{G}_a(\mathcal{X})$ be the graph with vertex set $\mathcal{X}$ and with edge set $\{\{x,y\} : |x-y| < a\}$. Let $\mathcal{G}_{\mathrm{MST}}(\mathcal{X})$ be the graph with vertex set $\mathcal{X}$ obtained by including each edge $\{x,y\}$ such that x and y lie in different components of $\mathcal{G}_{|x-y|}(\mathcal{X})$ and at least one of the components is finite. When $\mathcal{X}$ is finite, then $\mathcal{G}_{\mathrm{MST}}(\mathcal{X})$ is the minimal spanning tree graph on $\mathcal{X}$, with total edge length $\mathrm{MST}(\mathcal{X})$, as in (8.4). Let $\xi(x,\mathcal{X})$ be one half the sum of the lengths of the edges in $\mathcal{G}_{\mathrm{MST}}(\mathcal{X})$ which are incident to x. Then ξ is homogeneously stabilizing, which follows from arguments involving the uniqueness of the infinite component in continuum percolation [401].

Given $\mathcal{X} \subset \mathbb{R}^d$ and $a > 0$, recall that $a\mathcal{X} := \{ax : x \in \mathcal{X}\}$. For all $\lambda > 0$ define the λ *re-scaled version* of ξ by

$$\xi_\lambda(x,\mathcal{X}) := \xi(\lambda^{1/d}x, \lambda^{1/d}\mathcal{X}). \tag{8.31}$$

Re-scaling is natural when considering point sets in compact sets K having cardinality roughly λ; dilation by $\lambda^{1/d}$ means that unit volume subsets of $\lambda^{1/d}K$ host on the average one point.

It is useful to consider point processes on $\mathbb{R}^d$ more general than the homogeneous Poisson point processes. In what follows, let $\eta_1, \ldots, \eta_n$ be i.i.d., with a distribution which is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$, with density κ having support K. For all $\lambda > 0$, let $\Pi_{\lambda\kappa}$ denote a Poisson point process in $\mathbb{R}^d$ with intensity measure $\lambda\kappa(x)\,dx$. We shall assume throughout that κ is bounded with supremum denoted $\|\kappa\|_\infty$.

Homogeneous stabilization is an example of 'point stabilization' [457] in that ξ is required to stabilize around a given point $x \in \mathbb{R}^d$ with respect to homogeneously distributed Poisson points Π_τ. A related 'point stabilization' requires that the re-scaled $\xi_\lambda, \lambda \in [1,\infty)$, stabilize around x, but now with respect to $\Pi_{\lambda\kappa}$ uniformly in $\lambda \in [1,\infty)$. This goes as follows.

8.3.2 Stabilization with respect to the probability density κ

ξ is *stabilizing with respect to the probability density κ and the subset K of* $\mathbb{R}^d$ if for all $\lambda \in [1,\infty)$ and all $x \in K$, there exists almost surely a $R := R(x,\lambda) < \infty$ (*a radius of stabilization* for ξ_λ at x) such that for all locally finite $\mathcal{A} \subset (\mathbb{R}^d \setminus B_{\lambda^{-1/d}R}(x))$, we have

$$\xi_\lambda\left(x, [\Pi_{\lambda\kappa} \cap B_{\lambda^{-1/d}R}(x)] \cup \mathcal{A}\right) = \xi_\lambda\left(x, \Pi_{\lambda\kappa} \cap B_{\lambda^{-1/d}R}(x)\right). \tag{8.32}$$

If the tail probability $\tau(t)$ defined for $t > 0$ by $\tau(t) := \sup_{\lambda \geq 1,\, x \in K} \mathbf{P}(R(x,\lambda) > t)$ satisfies $\limsup_{t\to\infty} t^{-1}\log\tau(t) < 0$ then we say that ξ is *exponentially stabilizing with respect to κ and K*.

Roughly speaking, $R := R(x,\lambda)$ is a radius of stabilization if for all $\lambda \in [1,\infty)$, the value of $\xi_\lambda(x,\Pi_{\lambda\kappa})$ is unaffected by changes in point configurations outside

$B_{\lambda^{-1/d}R}(x)$. In most examples of interest, methods showing that functionals ξ homogeneously stabilize are easily modified to show stabilization of ξ with respect to densities κ. While it is straightforward to determine conditions under which the interaction function ξ from examples 1 and 2 stabilizes exponentially fast, it is not known whether the interaction ξ from example 3 stabilizes exponentially fast.

Exercise 8.5. Show that the interaction function ξ from examples 1 and 2 stabilizes exponentially fast when κ is bounded away from zero on its support K, assumed compact and convex.

We may weaken homogeneous stabilization by requiring that the point sets $\mathcal{A}$ in (8.30) belong to the homogeneous Poisson point process Π_τ. This weaker version of stabilization, called *localization*, is used in [111] and [459] to establish variance asymptotics and central limit theorems for functionals of convex hulls of random samples in the unit ball. Given $r > 0$, let $\xi^r(x, \mathcal{X}) := \xi(x, \mathcal{X} \cap B_r(x))$.

Say that $\hat{R} := \hat{R}(x, \Pi_\tau)$ is a *radius of localization* for ξ at x with respect to Π_τ if almost surely $\xi(x, \Pi_\tau) = \xi^{\hat{R}}(x, \Pi_\tau)$ and for all $s > \hat{R}$ we have $\xi^s(x, \Pi_\tau) = \xi^{\hat{R}}(x, \Pi_\tau)$.

8.3.3 A weak law of large numbers for stabilizing functionals

Recall that $\Pi_{\lambda\kappa}$ is the Poisson point process on $\mathbb{R}^d$ with intensity measure $\lambda\kappa(x)dx$. It is easy to show that $\lambda^{1/d}(\Pi_{\lambda\kappa} - x_0)$ converges to $\Pi_{\kappa(x_0)}$ as $\lambda \to \infty$, where convergence is in the sense of weak convergence of point processes. If $\xi(\cdot,\cdot)$ is a functional defined on $\mathbb{R}^d \times \mathcal{N}$, where we recall that $\mathcal{N}$ is the space of locally finite point sets in $\mathbb{R}^d$, one might hope that ξ is *continuous* on the pairs $(o, \lambda^{1/d}(\Pi_{\lambda\kappa} - x_0))$ in the sense that $\xi(o, \lambda^{1/d}(\Pi_{\lambda\kappa} - x_0))$ converges in distribution to $\xi(o, \Pi_{\kappa(x_0)})$ as $\lambda \to \infty$. This turns out to be the case whenever ξ is homogeneously stabilizing as in (8.30). This is the content of the next lemma; for a complete proof see Section 3 of [395]. Recall that almost every $x \in \mathbb{R}^d$ is a *Lebesgue point* of κ, that is to say for almost all $x \in \mathbb{R}^d$ we have that $\varepsilon^{-d}\int_{B_\varepsilon(x)} |\kappa(y) - \kappa(x)|\, dy$ tends to zero as ε tends to zero.

Lemma 8.1. *Let x_0 be a Lebesgue point for κ. If ξ is homogeneously stabilizing as in (8.30), then as $\lambda \to \infty$*

$$\xi_\lambda(x_0, \Pi_{\lambda\kappa}) \xrightarrow{d} \xi(o, \Pi_{\kappa(x_0)}). \tag{8.33}$$

Proof. (Sketch) We have $\xi_\lambda(x_0, \Pi_{\lambda\kappa}) = \xi(o, \lambda^{1/d}(\Pi_{\lambda\kappa} - x_0))$ by translation invariance of ξ. By the stabilization of ξ, it may be shown [394] that $(o, \Pi_{\kappa(x_0)})$ is a continuity point for ξ with respect to the product topology on $\mathbb{R}^d \times \mathcal{N}$, when the space of locally finite point sets $\mathcal{N}$ in $\mathbb{R}^d$ is equipped with the metric

$$d(\mathcal{X}_1, \mathcal{X}_2) := (\max\{k \in \mathbb{N} : \mathcal{X}_1 \cap B_k(o) = \mathcal{X}_2 \cap B_k(o)\})^{-1}.$$

The result follows by the weak convergence $\lambda^{1/d}(\Pi_{\lambda\kappa} - x_0) \xrightarrow{d} \Pi_{\kappa(x_0)}$ and the continuous mapping theorem (Theorem 2.7 of [69]. □

Recall that $\mathcal{X}_n := \{\eta_i\}_{i=1}^n$, where $\eta_1, ..., \eta_n$ are i.i.d. with density κ. Limit theorems for the sums $\sum_{x \in \Pi_{\lambda\kappa}} \xi_\lambda(x, \Pi_{\lambda\kappa})$ as well as for the weighted empirical measures

$$\mu_\lambda := \mu_\lambda^\xi := \sum_{x \in \Pi_{\lambda\kappa}} \xi_\lambda(x, \Pi_{\lambda\kappa})\delta_x \text{ and } \rho_n := \rho_n^\xi := \sum_{i=1}^n \xi_n(\eta_i, \mathcal{X}_n)\delta_{\eta_i} \tag{8.34}$$

naturally require moment conditions on the summands, thus motivating the next definition.

Definition 8.5. ξ has a moment of order $p > 0$ (with respect to κ and K) if

$$\sup_{\lambda \geq 1,\ x \in K, \mathcal{A} \in K} \mathbf{E}[|\xi_\lambda(x, \Pi_{\lambda\kappa} \cup \mathcal{A})|^p] < \infty, \tag{8.35}$$

where $\mathcal{A}$ ranges over all finite subsets of K.

Exercise 8.6. Show that the interaction function ξ from Examples 1 and 2 has moments of all orders when κ is bounded away from zero on its support.

Let $\mathbb{B}(K)$ denote the class of all bounded $f : K \to \mathbb{R}$ and for all measures μ on $\mathbb{R}^d$ let $\langle f, \mu \rangle := \int f d\mu$. Put $\bar{\mu} := \mu - \mathbf{E}\mu$. For all $f \in \mathbb{B}(K)$ we have by Palm theory for the Poisson process (see e.g Theorem 1.6 in [394]) that

$$\mathbf{E}[\langle f, \mu_\lambda \rangle] = \lambda \int_K f(x)\mathbf{E}[\xi_\lambda(x, \Pi_{\lambda\kappa})]\kappa(x)\, dx. \tag{8.36}$$

If (8.35) holds for some $p > 1$, then uniform integrability and Lemma 8.1 show that for all Lebesgue points x of κ one has $\mathbf{E}\xi_\lambda(x, \Pi_{\lambda\kappa}) \to \mathbf{E}\xi(o, \Pi_{\kappa(x)})$ as $\lambda \to \infty$. The set of points failing to be Lebesgue points has measure zero and so when the moment condition (8.35) holds for some $p > 1$, the bounded convergence theorem gives

$$\lim_{\lambda \to \infty} \lambda^{-1}\mathbf{E}[\langle f, \mu_\lambda \rangle] = \int_K f(x)\mathbf{E}[\xi(o, \Pi_{\kappa(x)})]\kappa(x)\, dx.$$

This simple convergence of means $\mathbf{E}[\langle f, \mu_\lambda \rangle]$ is now upgraded to convergence in L^q, $q = 1$ or 2.

Theorem 8.4. *Put $q = 1$ or 2. Let ξ be a homogeneously stabilizing (8.30) translation invariant functional satisfying the moment condition (8.35) for some $p > q$. Then for all $f \in \mathbb{B}(K)$ we have*

$$\lim_{n \to \infty} n^{-1}\langle f, \rho_n \rangle = \lim_{\lambda \to \infty} \lambda^{-1}\langle f, \mu_\lambda \rangle = \int_K f(x)\mathbf{E}[\xi(o, \Pi_{\kappa(x)})]\kappa(x)\, dx \text{ in } L^q. \tag{8.37}$$

If ξ is homogeneous of order p as defined at (8.16), then for all $\alpha \in (0, \infty)$ and $\tau \in (0, \infty)$ we have $\Pi_{\alpha\tau} \stackrel{d}{=} \alpha^{-1/d}\Pi_\tau$; see for example the mapping theorem on p. 18 of [298]. Consequently, if ξ is homogeneous of order p, it follows that $\mathbf{E}\xi(o, \Pi_{\kappa(x)}) = \kappa(x)^{-p/d}\mathbf{E}\xi(o, \Pi_1)$, whence the following weak law of large numbers.

Corollary 8.1. *Put $q = 1$ or 2. Let ξ be a homogeneously stabilizing (8.30) translation invariant functional satisfying the moment condition (8.35) for some $p > q$. If ξ is homogeneous of order p as at (8.16), then for all $f \in \mathbb{B}(K)$ we have*

$$\lim_{n\to\infty} n^{-1}\langle f,\rho_n\rangle = \lim_{\lambda\to\infty} \lambda^{-1}\langle f,\mu_\lambda\rangle = \mathbf{E}[\xi(o,\Pi_1)]\int_K f(x)\kappa(x)^{(d-p)/d}\,dx \qquad (8.38)$$

where the convergence is in the L^q sense.

Remarks.

1. The proofs of the above laws of large numbers are given in [394, 401].
2. The closed form limit (8.38) links the macroscopic limit behaviour of the point measures ρ_n and μ_λ with (i) the local interaction of ξ at a point at the origin inserted into the point process Π_1 and (ii) the underlying point density κ.
3. Going back to the minimal spanning tree at (8.4), the limiting constant $\alpha(\mathrm{MST}_B,d)$ can be found by putting ξ in (8.38) to be ξ_{MST}, letting $f\equiv 1$ in (8.38), and consequently deducing that $\alpha(\mathrm{MST}_B,d) = \mathbf{E}[\xi_{\mathrm{MST}}(o,\Pi_1)]$, where $\xi_{\mathrm{MST}}(x,\mathcal{X})$ is one half the sum of the lengths of the edges in the graph $\mathcal{G}_{\mathrm{MST}}(\mathcal{X}\cup\{x\})$ incident to x.
4. Donsker-Varadhan-style large deviation principles for stabilizing functionals are proved in [460] whereas moderate deviations for bounded stabilizing functionals are proved in [55].

8.3.4 Variance asymptotics and central limit theorems for stabilizing functionals

Asymptotic distribution results for $\langle f,\mu_\lambda\rangle$ and $\langle f,\rho_n\rangle$, $f\in\mathbb{B}(K)$, as λ and n tend to infinity respectively, require additional notation. For all $\tau > 0$, put

$$\begin{aligned}V^{\xi}(\tau) := {} & \mathbf{E}[\xi(o,\Pi_\tau)^2] + \\ & \tau\int_{\mathbb{R}^d}\{\mathbf{E}[\xi(o,\Pi_\tau\cup\{z\})\xi(z,\Pi_\tau\cup o)] - (\mathbf{E}[\xi(o,\Pi_\tau)])^2\}\,dz \qquad (8.39)\end{aligned}$$

and

$$\Delta^{\xi}(\tau) := \mathbf{E}[\xi(o,\Pi_\tau)] + \tau\int_{\mathbb{R}^d}\{\mathbf{E}[\xi(o,\Pi_\tau\cup\{z\}) - \mathbf{E}[\xi(o,\Pi_\tau)]\}\,dz. \qquad (8.40)$$

The scalars $V^{\xi}(\tau),\tau>0$, should be interpreted as mean pair correlation functions for the functional ξ on homogenous Poisson points Π_τ. By the translation invariance of ξ, the scalars $\Delta^{\xi}(\tau),\tau>0$, satisfy

$$\Delta^{\xi}(\tau) = \mathbf{E}[\xi(o,\Pi_\tau)] + \mathbf{E}\left[\sum_{x\in\Pi_\tau\cup\{z\}}\xi(x,\Pi_\tau\cup\{z\}) - \sum_{x\in\Pi_\tau}\xi(x,\Pi_\tau)\right],$$

which suggests that $\Delta^{\xi}(\tau)$ may be viewed as the expected 'add-one cost' for $\sum_{x \in \Pi_\tau} \xi(x, \Pi_\tau)$ when the point set Π_τ is augmented to $\Pi_\tau \cup \{z\}$.

By extending Lemma 8.1 to an analogous result giving the weak convergence of the joint distribution of $\xi_\lambda(x, \Pi_{\lambda\kappa})$ and $\xi_\lambda(x + \lambda^{-1/d} z, \Pi_{\lambda\kappa})$ for all pairs of points x and z in $\mathbb{R}^d$, we may show for exponentially stabilizing ξ and for bounded K that $\lambda^{-1}\,\mathbf{var}[\langle f, \mu_\lambda \rangle]$ converges as $\lambda \to \infty$ to a weighted average of the mean pair correlation functions.

Furthermore, recalling that $\overline{\mu}_\lambda := \mu_\lambda - \mathbf{E}[\mu_\lambda]$, and by using either Stein's method [395, 402] or the cumulant method [57], we may establish variance asymptotics and asymptotic normality of $\langle f, \lambda^{-1/2}\overline{\mu}_\lambda \rangle, f \in \mathbb{B}(K)$, as shown by the next result, proved in [57, 395, 402].

Theorem 8.5. *(Variance asymptotics and CLT for Poisson input) Assume that* κ *is Lebesgue-almost everywhere continuous. Let* ξ *be a homogeneously stabilizing (8.30) translation invariant functional satisfying the moment condition (8.35) for some* $p > 2$*. Suppose further that* K *is bounded and that* ξ *is* exponentially stabilizing *with respect to* κ *and* K *as in (8.32). Then for all* $f \in \mathbb{B}(K)$ *we have*

$$\lim_{\lambda \to \infty} \lambda^{-1}\,\mathbf{var}[\langle f, \mu_\lambda \rangle] = \sigma^2(f) := \int_K f(x)^2 V^{\xi}(\kappa(x))\kappa(x)\,dx < \infty \tag{8.41}$$

as well as convergence of the finite-dimensional distributions

$$(\langle f_1, \lambda^{-1/2}\overline{\mu}_\lambda \rangle, \ldots, \langle f_k, \lambda^{-1/2}\overline{\mu}_\lambda \rangle),$$

$f_1, \ldots, f_k \in \mathbb{B}(K)$, *to those of a mean zero Gaussian field with covariance kernel*

$$(f, g) \mapsto \int_K f(x)g(x)V^{\xi}(\kappa(x))\kappa(x)\,dx. \tag{8.42}$$

Extensions of Theorem 8.5

1. For an extension of Theorem 8.5 to manifolds, see [403]; for extensions to functionals of Gibbs point processes, see [460]. Theorems 8.4 and 8.5 also extend to treat functionals of point sets having i.i.d. marks [57, 395].
2. Rates of convergence. Suppose $\|\kappa\|_\infty < \infty$. Suppose that ξ is exponentially stabilizing and satisfies the moments condition (8.35) for some $p > 3$. If $\sigma^2(f) > 0$ for $f \in \mathbb{B}(K)$, then there exists a finite constant c depending on d, ξ, κ, p and f, such that for all $\lambda \geq 2$,

$$\sup_{t \in \mathbb{R}} \left| \mathbf{P}\left[\frac{\langle f, \mu_\lambda \rangle - \mathbf{E}[\langle f, \mu_\lambda \rangle]}{\sqrt{\mathbf{var}[\langle f, \mu_\lambda \rangle]}} \leq t \right] - \mathbf{P}(N(0,1) \leq t) \right| \leq c(\log \lambda)^{3d} \lambda^{-1/2}. \tag{8.43}$$

For details, see Corollary 2.1 in [402]. For rates of convergence in the multivariate central limit theorem, see [397].

3. Translation invariance. For ease of exposition, Theorems and 8.4 and 8.5 assume translation invariance of ξ. This assumption may be removed (see [57, 395, 394]), provided that we put $\xi_\lambda(x,\mathcal{X}) := \xi(x, x+\lambda^{1/d}(-x+\mathcal{X}))$ and provided that we replace $V^\xi(\tau)$ and $\Delta^\xi(\tau)$ defined at (8.39) and (8.40) respectively, by

$$V^\xi(x,\tau) := \mathbf{E}[\xi(x,\Pi_\tau)^2] \\ +\tau\int_{\mathbb{R}^d}\{\mathbf{E}[\xi(x,\Pi_\tau\cup\{z\})\xi(x,-z+(\Pi_\tau\cup o))]-(\mathbf{E}[\xi(x,\Pi_\tau)])^2\}\,dz \quad (8.44)$$

and

$$\Delta^\xi(x,\tau) := \mathbf{E}[\xi(x,\Pi_\tau)] + \tau\int_{\mathbb{R}^d}\{\mathbf{E}[\xi(x,\Pi_\tau\cup\{z\})-\mathbf{E}[\xi(x,\Pi_\tau)]\}\,dz. \quad (8.45)$$

4. The moment condition (8.35) may be weakened to one requiring only that $\mathcal{A}$ range over subsets of K having at most one element; see [395].

Proof of variance asymptotics (8.41)

The proof of (8.41) depends in part on the following generalization of Lemma 8.1, a proof of which appears in [395].

Lemma 8.2. *Let x be a Lebesgue point for κ. If ξ is homogeneously stabilizing as in (8.30), then for all $z\in\mathbb{R}^d$, we have as $\lambda\to\infty$*

$$(\xi_\lambda(x,\Pi_{\lambda\kappa}),\xi_\lambda(x+\lambda^{-1/d}z,\Pi_{\lambda\kappa})) \xrightarrow{d} (\xi(o,\Pi_{\kappa(x)}),\xi(z,\Pi_{\kappa(x)})). \quad (8.46)$$

Given Lemma 8.2 we sketch a proof of the variance convergence (8.41). For simplicity we assume that f is a.e. continuous. By Palm theory for the Poisson process $\Pi_{\lambda\kappa}$ we have

$$\lambda^{-1}\,\mathbf{var}[\langle f,\mu_\lambda\rangle] \\ = \lambda\int_K\int_K f(x)f(y)\{\mathbf{E}[\xi_\lambda(x,\Pi_{\lambda\kappa}\cup\{y\})\xi_\lambda(y,\Pi_{\lambda\kappa}\cup\{x\})] \\ -\mathbf{E}[\xi_\lambda(x,\Pi_{\lambda\kappa})]\mathbf{E}[\xi_\lambda(y,\Pi_{\lambda\kappa})]\}\kappa(x)\kappa(y)\,dx\,dy \\ +\int_K f(x)^2\mathbf{E}[\xi_\lambda^2(x,\Pi_{\lambda\kappa})]\kappa(x)\,dx. \quad (8.47)$$

Putting $y = x+\lambda^{-1/d}z$ in the right-hand side in (8.47) reduces the double integral to

$$\int_K\int_{-\lambda^{1/d}x+\lambda^{1/d}K} f(x)f(x+\lambda^{-1/d}z)\{\ldots\}\kappa(x)\kappa(x+\lambda^{-1/d}z)dz\,dx \quad (8.48)$$

where

$$\{...\} := \{\mathbf{E}[\xi_\lambda(x, \Pi_{\lambda\kappa} \cup \{x+\lambda^{-1/d}z\})\xi_\lambda(x+\lambda^{-1/d}z, \Pi_{\lambda\kappa} \cup \{x\})]$$

$$-\mathbf{E}[\xi_\lambda(x, \Pi_{\lambda\kappa})]\mathbf{E}[\xi_\lambda(x+\lambda^{-1/d}z, \Pi_{\lambda\kappa})]\}$$

is the two point correlation function for ξ_λ.

The moment condition and Lemma 8.2 imply for all Lebesgue points $x \in K$ that the two point correlation function for ξ_λ converges to the two point correlation function for ξ as $\lambda \to \infty$. Moreover, by exponential stabilization, the integrand in (8.48) is dominated by an integrable function of z over $\mathbb{R}^d$ (see Lemma 4.2 of [395]). The double integral in (8.47) thus converges to

$$\int_K \int_{\mathbb{R}^d} f(x)^2 \cdot \mathbf{E}[\xi(o, \Pi_{\kappa(x)} \cup \{z\})\xi(z, \Pi_{\kappa(x)} \cup o)] - (\mathbf{E}\xi(o, \Pi_{\kappa(x)}))^2 \kappa(x)^2 \, dz dx \quad (8.49)$$

by dominated convergence, the continuity of f, and the assumed moment bounds.

By Theorem 8.4, the assumed moment bounds, and dominated convergence, the single integral in (8.47) converges to

$$\int_K f(x)^2 \mathbf{E}[\xi^2(o, \Pi_{\kappa(x)})]\kappa(x)\, dx. \quad (8.50)$$

Combining (8.49) and (8.50) and using the definition of V^ξ, we obtain the variance asymptotics (8.41) for continuous test functions f. To show convergence for general $f \in \mathbb{B}(K)$ we refer to [395].

8.3.5 *Proof of asymptotic normality in Theorem 8.5; method of cumulants*

Now we sketch a proof of the central limit theorem part of Theorem 8.5. There are three distinct approaches to proving the central limit theorem:

1. Stein's method, in particular consequences of Stein's method for dependency graphs of random variables, as given by [120]. This approach, spelled out in [402], gives the rates of convergence to the normal law in (8.43).
2. Methods based on martingale differences are applicable when κ is the uniform density and when the functional H^ξ satisfies a stabilization criteria involving the insertion of single point into the sample; see [295] and [398] for details.
3. The method of cumulants may be used [57] to show that the k-th order cumulants c_λ^k of $\lambda^{-1/2}\langle f, \overline{\mu}_\lambda \rangle$, $k \geq 3$, vanish in the limit as $\lambda \to \infty$. This method makes use of the standard fact that if the cumulants c^k of a random variable ζ vanish for all $k \geq 3$, then ζ has a normal distribution. This approach assumes additionally that ξ has moments of all orders, i.e. (8.35) holds for all $p \geq 1$.

Here we describe the third method, which, when suitably modified yields moderate deviation principles [55] as well as limit theory for functionals over Gibbs point processes [460].

To show vanishing of cumulants of order three and higher, we follow the proof of Theorem 2.4 in section five of [57] and take the opportunity to correct a mistake in the exposition, which also carried over to [55], and which was first noticed by Mathew Penrose. We assume the test functions f belong to the class $C(K)$ of continuous functions on K and we will show for all continuous test functions f on K, that

$$\langle f, \lambda^{-1/2}\overline{\mu}_\lambda \rangle \xrightarrow{d} N(0, \sigma^2(f)), \tag{8.51}$$

where $\sigma^2(f)$ is at (8.41). The convergence of the finite-dimensional distributions (8.42) follows by standard methods involving the Cramér-Wold device.

We first recall the formal definition of cumulants. Put $K := [0,1]^d$ for simplicity. Write

$$\begin{aligned}
&\mathbf{E}\exp\left(\lambda^{-1/2}\langle -f, \overline{\mu}_\lambda\rangle\right) \\
&\quad = \exp\left(\lambda^{-1/2}\langle f, \mathbf{E}\mu_\lambda\rangle\right)\mathbf{E}\exp\left(\lambda^{-1/2}\langle -f, \mu_\lambda\rangle\right) \\
&\quad = \exp\left(\lambda^{-1/2}\langle f, \mathbf{E}\mu_\lambda\rangle\right)\left[1 + \sum_{k=1}^{\infty}\frac{\lambda^{-k/2}}{k!}\langle(-f)^k, M_\lambda^k\rangle\right],
\end{aligned} \tag{8.52}$$

where $f^k : \mathbb{R}^{dk} \to \mathbb{R}$, $k = 1, 2, ...$ is given by $f^k(v_1, \ldots, v_k) = f(v_1)\cdots f(v_k)$, and $v_i \in K$, $1 \le i \le k$. $M_\lambda^k := M_{\lambda\kappa}^k$ is a measure on $\mathbb{R}^{dk}$, the k-th moment measure (Chapter 9.5 of [140]), and has the property that

$$\langle f^k, M_\lambda^k\rangle = \int_{K^k}\mathbf{E}\left[\prod_{i=1}^k \xi_\lambda(x_i, \Pi_{\lambda\kappa})\right]\prod_{i=1}^k f(x_i)\kappa(x_i)\,d(\lambda^{1/d}x_i).$$

In general M_λ^k is not continuous with respect to Lebesgue measure on K^k, but rather it is continuous with respect to sums of Lebesgue measures on the diagonal subspaces of K^k, where two or more coordinates coincide.

In Section 5 of [57], the moment and cumulant measures considered there are with respect to the centered functional $\overline{\xi}$, whereas they should be with respect to the non-centered functional ξ. This requires corrections to the notation, which we provide here, but since higher order cumulants for centered and non-centered measures coincide, it does not change the arguments of [57], which we include for completeness and which go as follows.

We have

$$dM_\lambda^k(v_1, ..., v_k) = m_\lambda(v_1, ..., v_k)\prod_{i=1}^k \kappa(v_i)\,d(\lambda^{1/d}v_i),$$

where $m_\lambda(v_1, ..., v_k)$ is given by mixed moment

$$m_\lambda(v_1,...,v_k) := \mathbf{E}\left[\prod_{i=1}^{k} \xi_\lambda(v_i; \Pi_{\lambda\kappa} \cup \{v_j\}_{j=1}^k)\right]. \tag{8.53}$$

Due to the behaviour of M_λ^k on the diagonal subspaces we make the standing assumption that if the differential $d(\lambda_1^{1/d} v_1)\cdots d(\lambda_1^{1/d} v_k)$ involves repetition of certain coordinates, then it collapses into the corresponding lower order differential in which each coordinate occurs only once. For each $k \in \mathbb{N}$, by the assumed moment bounds (8.35), the mixed moment on the right hand side of (8.53) is bounded uniformly in λ by a constant $c(\xi,k)$. Likewise, the k-th summand in (8.52) is finite.

For all $i = 1,2,...$ we let K_i denote the i-th copy of K. For any subset T of the positive integers, we let

$$K^T := \prod_{i \in T} K_i.$$

If $|T| = l$, then for all $\lambda \geq 1$, by M_λ^T we mean a copy of the l-th moment measure on the l-fold product space K_λ^T. M_λ^T is equal to M_λ^l as defined above.

When the series (8.52) is convergent, the logarithm of the Laplace functional gives

$$\log\left[1 + \sum_{k=1}^{\infty} \frac{1}{k!}\lambda^{-k/2}\langle(-f)^k, M_\lambda^k\rangle\right] = \sum_{l=1}^{\infty} \frac{1}{l!}\lambda^{-l/2}\langle(-f)^l, c_\lambda^l\rangle; \tag{8.54}$$

the signed measures c_λ^l are cumulant measures. Regardless of the validity of (8.52), the existence of all cumulants c_λ^l, $l = 1,2,...$ follows from the existence of all moments in view of the representation

$$c_\lambda^l = \sum_{T_1,...,T_p} (-1)^{p-1}(p-1)! M_\lambda^{T_1} \cdots M_\lambda^{T_p},$$

where $T_1,...,T_p$ ranges over all unordered partitions of the set $1,...,l$ (see p. 30 of [341]). The first cumulant measure coincides with the expectation measure and the second cumulant measure coincides with the variance measure.

We follow the proof of Theorem 2.4 of [57], with these small changes: (i) replace the centered functional $\overline{\xi}$ with the non-centered ξ (ii) correspondingly, let all cumulants c_λ^l, $l = 1,2,...$ be the cumulant measures for the *non-centered* moment measures M_λ^k, $k = 1,2,....$ Since c_λ^1 coincides with the expectation measure, Theorem 8.4 gives for all $f \in C(K)$

$$\lim_{\lambda\to\infty} \lambda^{-1}\langle f, c_\lambda^1\rangle = \lim_{\lambda\to\infty} \lambda^{-1}\mathbf{E}[\langle f, \mu_\lambda^\xi\rangle] = \int_K f(x)\mathbf{E}[\xi(o, \Pi_{\kappa(x)})]\kappa(x)dx.$$

We already know from the variance convergence that

$$\lim_{\lambda\to\infty} \lambda^{-1}\langle f^2, c_\lambda^2\rangle = \lim_{\lambda\to\infty} \lambda^{-1}\,\mathbf{var}[\langle f, \mu_{\lambda\kappa}^\xi\rangle] = \int_K f(x)^2 V^\xi(\kappa(x))\kappa(x)dx.$$

Thus, to prove (8.51), it will be enough to show for all $k \geq 3$ and all $f \in C(K)$ that $\lambda^{-k/2}\langle f^k, c_\lambda^k \rangle \to 0$ as $\lambda \to \infty$. This will be done in Lemma 8.4 below, but first we recall some terminology from [57].

A cluster measure $U_\lambda^{S,T}$ on $K^S \times K^T$ for non-empty $S,T \subset \{1,2,...\}$ is defined by

$$U_\lambda^{S,T}(B \times D) = M_\lambda^{S \cup T}(B \times D) - M_\lambda^S(B) M_\lambda^T(D)$$

for all Borel B and D in K^S and K^T, respectively.

Let S_1, S_2 be a partition of S and let T_1, T_2 be a partition of T. A product of a cluster measure $U_\lambda^{S_1,T_1}$ on $K^{S_1} \times K^{T_1}$ with products of moment measures $M^{|S_2|}$ and $M^{|T_2|}$ on $K^{S_2} \times K^{T_2}$ will be called a (S,T) semi-cluster measure.

For each non-trivial partition (S,T) of $\{1,...,k\}$ the k-th cumulant c^k is represented as

$$c^k = \sum_{(S_1,T_1),(S_2,T_2)} \alpha((S_1,T_1),(S_2,T_2)) U^{S_1,T_1} M^{|S_2|} M^{|T_2|}, \tag{8.55}$$

where the sum ranges over partitions of $\{1,...,k\}$ consisting of pairings (S_1,T_1), (S_2,T_2), where $S_1, S_2 \subset S$ and $T_1, T_2 \subset T$, and where $\alpha((S_1,T_1),(S_2,T_2))$ are integer valued pre-factors. In other words, for any non-trivial partition (S,T) of $\{1,...,k\}$, c^k is a linear combination of (S,T) semi-cluster measures; see Lemma 5.1 of [57].

The following bound is critical for showing that $\lambda^{-k/2}\langle f, c_\lambda^k \rangle \to 0$ for $k \geq 3$ as $\lambda \to \infty$. This lemma appears as Lemma 5.2 in [57].

Lemma 8.3. *If ξ is exponentially stabilizing as in (8.32), then the functions m_λ cluster exponentially, that is there are positive constants $a_{j,l}$ and $c_{j,l}$ such that uniformly*

$$|m_\lambda(x_1,...x_j,y_1,...,y_l) - m_\lambda(x_1,...,x_j) m_\lambda(y_1,...,y_l)| \leq a_{j,l} \exp(-c_{j,l} \delta \lambda^{1/d}),$$

where $\delta := \min_{1 \leq i \leq j, 1 \leq p \leq l} |x_i - y_p|$ is the separation between the sets $\{x_i\}_{i=1}^j$ and $\{y_p\}_{p=1}^l$ of points in K.

The constants $a_{j,l}$, while independent of λ, may grow quickly in j and l, but this will not affect the decay of the cumulant measures in the scale parameter λ. The next lemma provides the desired decay of the cumulant measures; we provide a proof which is slightly different from that given for Lemma 5.3 of [57].

Lemma 8.4. *For all $f \in C(K)$ and $k = 2,3,...$ we have $\lambda^{-1}\langle f^k, c_\lambda^k \rangle = O\left(\|f\|_\infty^k\right)$.*

Proof. We need to estimate

$$\int_{K^k} f(v_1)...f(v_k)\, dc_\lambda^k(v_1,...,v_k).$$

We will modify the arguments in [57]. Given $v := (v_1,...,v_k) \in K^k$, let $D_k(v) := D_k(v_1,...,v_k) := \max_{i \leq k}(\|v_1 - v_i\| + ... + \|v_k - v_i\|)$ be the l^1 diameter for v. Let $\Xi(k)$ be the collection of all partitions of $\{1,...,k\}$ into exactly two subsets S and T. For all such partitions consider the subset $\sigma(S,T)$ of $K^S \times K^T$ having the property that $v \in \sigma(S,T)$ implies $d(x(v),y(v)) \geq D_k(v)/k^2$, where $x(v)$ and $y(v)$ are the projections of v onto K^S and K^T, respectively, and where $d(x(v),y(v))$ is the minimal Euclidean distance between pairs of points from $x(v)$ and $y(v)$. It is easy to see that for every $v := (v_1,...,v_k) \in K^k$, there is a splitting of v, say $x := x(v)$ and $y := y(v)$, such that $d(x,y) \geq D_k(v)/k^2$; if this were not the case then a simple argument shows that, given $v := (v_1,...,v_k)$ the distance between any pair of constituent components must be strictly less than $D_k(v)/k$, contradicting the definition of D_k. It follows that K^k is the union of the sets $\sigma(S,T),\ (S,T) \in \Xi(k)$. The key to the proof of Lemma 8.4 is to evaluate the cumulant c_λ^k over each $\sigma(S,T) \in \Xi(k)$, that is to write $\langle f, c_\lambda^k \rangle$ as a finite sum of integrals

$$\langle f, c_\lambda^k \rangle = \sum_{\sigma(S,T)\in\Xi(k)} \int_{\sigma(S,T)} f(v_1)\cdots f(v_k)\, dc_\lambda^k(v_1,...,v_k),$$

then appeal to the representation (8.55) to write the cumulant measure $dc_\lambda^k(v_1,...,v_k)$ on each $\sigma(S,T)$ as a linear combination of (S,T) semi-cluster measures, and finally to appeal to Lemma 8.3 to control the constituent cluster measures U^{S_1,T_1} by an exponentially decaying function of $\lambda^{1/d}D_k(v) := \lambda^{1/d}D_k(v_1,...,v_k)$.

Given $\sigma(S,T)$, $S_1 \subset S$ and $T_1 \subset T$, this goes as follows. Let $x \in K^S$ and $y \in K^T$ denote elements of K^S and K^T, respectively; likewise we let $\tilde{x}$ and $\tilde{y}$ denote elements of K^{S_1} and K^{T_1}, respectively. Let $\tilde{x}^c$ denote the complement of $\tilde{x}$ with respect to x and likewise with $\tilde{y}^c$. The integral of f against one of the (S,T) semi-cluster measures in (8.55), induced by the partitions (S_1,S_2) and (T_1,T_2) of S and T respectively, has the form

$$\int_{\sigma(S,T)} f(v_1)\cdots f(v_k)\, d\left(M_\lambda^{|S_2|}(\tilde{x}^c)U_\lambda^{i+j}(\tilde{x},\tilde{y})M_\lambda^{|T_2|}(\tilde{y}^c)\right).$$

Letting $u_\lambda(\tilde{x},\tilde{y}) := m_\lambda(\tilde{x},\tilde{y}) - m_\lambda(\tilde{x})m_\lambda(\tilde{y})$, the above equals

$$\int_{\sigma(S,T)} f(v_1)\cdots f(v_k) m_\lambda(\tilde{x}^c) u_\lambda(\tilde{x},\tilde{y}) m_\lambda(\tilde{y}^c) \prod_{i=1}^k \kappa(v_i)\, d(\lambda^{1/d}v_i). \tag{8.56}$$

We use Lemma 8.3 to control $u_\lambda(\tilde{x},\tilde{y}) := m_\lambda(\tilde{x},\tilde{y}) - m_\lambda(\tilde{x})m_\lambda(\tilde{y})$, we bound f and κ by their respective sup norms, we bound each mixed moment by $c(\xi,k)$, and we use $\sigma(S,T) \subset K^k$ to show that

$$\int_{\sigma(S,T)} f(v_1)\cdots f(v_k)\, d\left(M_\lambda^{|S_2|}(\tilde{x}^c)U_\lambda^{i+j}(\tilde{x},\tilde{y})M_\lambda^{|T_2|}(\tilde{y}^c)\right)$$

$$\leq D(k)c(\xi,k)^2 \|f\|_\infty^k \|\kappa\|_\infty^k \int_{K^k} \exp(-c\lambda^{1/d}D_k(v)/k^2)\, d(\lambda^{1/d}v_1)\cdots d(\lambda^{1/d}v_k).$$

Letting $z_i := \lambda^{1/d}v_i$ the above bound becomes

$$\lambda D(k)c(\xi,k)^2 \|f\|_\infty^k \|\kappa\|_\infty^k \int_{(\lambda^{1/d}K)^k} \exp(-cD_k(z)/k^2)\, dz_1\cdots dz_k$$

$$\leq \lambda D(k)c(\xi,k)^2 \|f\|_\infty^k \|\kappa\|_\infty^k \int_{(\mathbb{R}^d)^{k-1}} \exp(-cD_k(0,z_1,...,z_{k-1})/k^2)\, dz_1\cdots dz_k$$

where we use the translation invariance of $D_k(\cdot)$. Upon a further change of variable $w := z/k$ we have

$$\int_{\sigma(S,T)} f(v_1)\cdots f(v_k)\, d\left(M_\lambda^{|S_2|}(\tilde{x}^c)U_\lambda^{i+j}(\tilde{x},\tilde{y})M_\lambda^{|T_2|}(\tilde{y}^c)\right)$$

$$\leq \lambda \tilde{D}(k)c(\xi,k)^2\|f\|_\infty^k\|\kappa\|_\infty^k \int_{(\mathbb{R}^d)^{k-1}} \exp(-cD_k(0,w_1,...,w_{k-1}))\, dw_1\cdots dw_{k-1}.$$

Finally, since $D_k(0,w_1,...,w_{k-1}) \geq \|w_1\| + ... + \|w_{k-1}\|$ we obtain

$$\int_{\sigma(S,T)} f(v_1)\cdots f(v_k) d\left(M_\lambda^{|S_2|}(\tilde{x}^c)U_\lambda^{i+j}(\tilde{x},\tilde{y})M_\lambda^{|T_2|}(\tilde{y}^c)\right)$$

$$\leq \lambda \tilde{D}(k)c(\xi,k)^2\|f\|_\infty^k\|\kappa\|_\infty^k \left(\int_{\mathbb{R}^d} \exp(-\|w\|)\, dw\right)^{k-1} = O(\lambda)$$

as desired. □

8.3.6 Central limit theorem for functionals of binomial input

To obtain central limit theorems for functionals over binomial input $\mathcal{X}_n := \{\eta_i\}_{i=1}^n$ we need some more definitions. For all functionals ξ and $\tau \in (0,\infty)$, recall the 'add one cost' $\Delta^\xi(\tau)$ defined at (8.40). For all $j = 1,2,...$, let $\mathcal{S}_j$ be the collection of all subsets of $\mathbb{R}^d$ of cardinality at most j.

Definition 8.6. Say that ξ has a moment of order $p > 0$ (with respect to binomial input $\mathcal{X}_n$) if

$$\sup_{n\geq 1, x\in\mathbb{R}^d, \mathcal{D}\in\mathcal{S}_3}\ \sup_{(n/2)\leq m\leq(3n/2)} \mathbf{E}[|\xi_n(x,\mathcal{X}_m\cup\mathcal{D})|^p] < \infty. \tag{8.57}$$

Definition 8.7. ξ is *binomially exponentially stabilizing* for κ if for all $x \in \mathbb{R}^d, \lambda \geq 1$, and $\mathcal{D} \subset \mathcal{S}_2$ almost surely there exists $R := R_{\lambda,n}(x,\mathcal{D}) < \infty$ such that for all finite $\mathcal{A} \subset (\mathbb{R}^d \setminus B_{\lambda^{-1/d}R}(x))$, we have

$$\xi_\lambda\left(x,([\mathcal{X}_n\cup\mathcal{D}]\cap B_{\lambda^{-1/d}R}(x))\cup\mathcal{A}\right) = \xi_\lambda\left(x,[\mathcal{X}_n\cup\mathcal{D}]\cap B_{\lambda^{-1/d}R}(x)\right), \tag{8.58}$$

and moreover there is an $\varepsilon > 0$ such that the tail probability $\tau_\varepsilon(t)$ defined for $t > 0$ by

$$\tau_\varepsilon(t) := \sup_{\lambda\geq 1, n\in\mathbb{N}\cap((1-\varepsilon)\lambda,(1+\varepsilon)\lambda)}\ \sup_{x\in\mathbb{R}^d,\ \mathcal{D}\subset\mathcal{S}_2} \mathbf{P}(R_{\lambda,n}(x,\mathcal{D}) > t)$$

satisfies $\limsup_{t\to\infty} t^{-1}\log\tau_\varepsilon(t) < 0$.

If ξ is homogeneously stabilizing then in most examples of interest, similar methods can be used to show that ξ is binomially exponentially stabilizing whenever κ is bounded away from zero.

Exercise 8.7. Show that the interaction function ξ from Examples 1 and 2 is binomially exponentially stabilizing whenever κ is bounded away from zero on its support, assumed compact and convex.

Theorem 8.6. *(CLT for binomial input) Assume that κ is Lebesgue-almost everywhere continuous. Let ξ be a homogeneously stabilizing (8.30) translation invariant functional satisfying the moment conditions (8.35) and (8.57) for some $p > 2$. Suppose further that K is bounded and that ξ is* exponentially stabilizing *with respect to κ and K as in (8.32) and binomially exponentially stabilizing with respect to κ and K as in (8.58). Then for all $f \in \mathbb{B}(K)$ we have*

$$\lim_{n\to\infty} n^{-1}\,\mathbf{var}[\langle f,\rho_n\rangle] = \tau^2(f)$$
$$:= \int_K f(x)^2 V^{\xi}(\kappa(x))\kappa(x)\,dx - \left(\int_K f(x)\Delta^{\xi}(\kappa(x))\kappa(x)dx\right)^2 \qquad (8.59)$$

as well as convergence of the finite-dimensional distributions

$$(\langle f_1, n^{-1/2}\overline{\rho}_n\rangle, \ldots, \langle f_k, n^{-1/2}\overline{\rho}_n\rangle),$$

$f_1,\ldots,f_k \in \mathbb{B}(K)$, to a mean zero Gaussian field with covariance kernel

$$(f,g) \mapsto \int_K f(x)g(x)V^{\xi}(\kappa(x))\kappa(x)\,dx$$
$$- \int_K f(x)\Delta^{\xi}(\kappa(x))\kappa(x)\,dx \int_K g(x)\Delta^{\xi}(\kappa(x))\kappa(x)\,dx. \qquad (8.60)$$

Proof. We sketch the proof, borrowing heavily from coupling arguments appearing in the complete proofs given in [57, 398, 395]. Fix $f \in \mathbb{B}(K)$. Put $H_n := \langle f,\rho_n\rangle$, $H'_n := \langle f,\mu_n\rangle$, where μ_n is defined at (8.34) and assume that $\Pi_{n\kappa}$ is coupled to $\mathcal{X}_n$ by setting $\Pi_{n\kappa} = \bigcup_{i=1}^{N(n)} \eta_i$, where $N(n)$ is an independent Poisson random variable with mean n. Put

$$\alpha := \alpha(f) := \int_K f(x)\Delta^{\xi}(\kappa(x))\kappa(x)\,dx.$$

Conditioning on the random variable $N := N(n)$ and using that N is concentrated around its mean, it can be shown that as $n \to \infty$ we have

$$\mathbf{E}[(n^{-1/2}(H'_n - H_n - (N(n)-n)\alpha))^2] \to 0. \qquad (8.61)$$

The arguments are long and technical (cf. Section 5 of [395], Section 4 of [398]).

Let $\sigma^2(f)$ be as at (8.41) and let $\tau^2(f)$ be as at (8.59), so that $\tau^2(f) = \sigma^2(f) - \alpha^2$.

By Theorem 8.5 we have as $n \to \infty$ that $n^{-1}\,\mathbf{var}[H'_n] \to \sigma^2(f)$ and $n^{-1/2}(H'_n - \mathbf{E}H'_n) \xrightarrow{d} N(0,\sigma^2(f))$. We now deduce Theorem 8.6, following verbatim by now standard arguments (see for example p. 1020 of [398], p. 251 of [57]), included here for sake of completeness.

To prove convergence of $n^{-1}\,\mathbf{var}[H_n]$, we use the identity

$$n^{-1/2}H'_n = n^{-1/2}H_n + n^{-1/2}(N(n)-n)\alpha + n^{-1/2}[H'_n - H_n - (N(n)-n)\alpha]. \quad (8.62)$$

The variance of the third term on the right-hand side of (8.62) goes to zero by (8.61), whereas the second term has variance α^2 and is independent of the first term. It follows that with $\sigma^2(f)$ defined at (8.41), we have

$$\sigma^2(f) = \lim_{n\to\infty} n^{-1}\,\mathbf{var}[H'_n] = \lim_{n\to\infty} n^{-1}\,\mathbf{var}[H_n] + \alpha^2,$$

so that $\sigma^2(f) \geq \alpha^2$ and $n^{-1}\,\mathbf{var}[H_n] \to \tau^2(f)$. This gives (8.59).

Now to prove Theorem 8.6 we argue as follows. By Theorem 8.5, we have $n^{-1/2}(H'_n - \mathbf{E}H'_n) \xrightarrow{d} N(0,\sigma^2(f))$. Together with (8.61), this yields

$$n^{-1/2}[H_n - \mathbf{E}H'_n + (N(n)-n)\alpha] \xrightarrow{d} N(0,\sigma^2(f)).$$

However, since $n^{-1/2}(N(n)-n)\alpha$ is independent of H_n and is asymptotically normal with mean zero and variance α^2, it follows by considering characteristic functions that

$$n^{-1/2}(H_n - \mathbf{E}H'_n) \xrightarrow{d} N(0,\sigma^2(f)-\alpha^2). \quad (8.63)$$

By (8.61), the expectation of $n^{-1/2}(H'_n - H_n - (N(n)-n)\alpha)$ tends to zero, so in (8.63) we can replace $\mathbf{E}H'_n$ by $\mathbf{E}H_n$, which gives us

$$n^{-1/2}(H_n - \mathbf{E}H_n) \xrightarrow{d} N(0,\tau^2(f)).$$

To obtain convergence of finite-dimensional distributions (8.60) we use the Cramér-Wold device.

□

8.4 Applications

Consider a linear statistic $H^\xi(\mathcal{X})$ of a large geometric structure on $\mathcal{X}$. If we are interested in the limit behavior of H^ξ on random point sets, then the results of the previous section suggest checking whether the interaction function ξ is stabilizing. Verifying the stabilization of ξ is sometimes non-trivial and may involve discretization methods. Here we describe four non-trivial statistics H^ξ for which one may show stabilization/localization of ξ. Our list is non-exhaustive and primarily focusses on the problems described in Section 8.1.1.

8.4.1 Random packing

Given $d \in \mathbb{N}$ and $\lambda \geq 1$, let $\eta_{1,\lambda}, \eta_{2,\lambda}, \ldots$ be a sequence of independent random d-vectors uniformly distributed on the cube $Q_\lambda := [0, \lambda^{1/d})^d$. Let $\tau_i, i \geq 1$, be i.i.d. time marks, independent of $\eta_i, i \geq 1$, and uniformly distributed on $[0,1]$. Equip each vector η_i with the time mark τ_i and re-order the indices so that τ_i are increasing. Let S be a fixed bounded closed convex set in $\mathbb{R}^d$ with non-empty interior (i.e., a 'solid') with centroid at the origin o of $\mathbb{R}^d$ (for example, the unit ball), and for $i \in \mathbb{N}$, let $S_{i,\lambda}$ be the translate of S having centroid at $\eta_{i,\lambda}$ and arrival time τ_i. Thus $\mathcal{S}_\lambda := (S_{i,\lambda})_{i \geq 1}$ is an infinite sequence of solids sequentially arriving at uniform random positions in Q_λ at arrival times $\tau_i, i \geq 1$ (the centroids lie in Q_λ but the solids themselves need not lie wholly inside Q_λ).

Let the first solid $S_{1,\lambda}$ be packed (i.e., accepted), and recursively for $i = 2, 3, \ldots$, let the i-th solid $S_{i,\lambda}$ be packed if it does not overlap any solid in $\{S_{1,\lambda}, \ldots, S_{i-1,\lambda}\}$ which has already been packed. If not packed, the i-th solid is discarded. This process, known as *random sequential adsorption (RSA) with infinite input*, is irreversible and terminates when it is not possible to accept additional solids. At termination, we say that the sequence of solids $\mathcal{S}_\lambda$ *jams* Q_λ or *saturates* Q_λ. The number of solids accepted in Q_λ at termination is denoted by the *jamming* number $N_\lambda := N_{\lambda,d} := N_{\lambda,d}(\mathcal{S}_\lambda)$.

There is a large literature of experimental results concerning the jamming numbers, but a limited collection of rigorous mathematical results, especially in $d \geq 2$. The short range interactions of arriving particles lead to complicated long range spatial dependence between the status of particles. Dvoretzky and Robbins [163] show in $d = 1$ that the jamming numbers $N_{\lambda,1}$ are asymptotically normal.

By writing the jamming number as a linear statistic involving a stabilizing interaction ξ on marked point sets, and recalling Remark 1 following Theorem 8.5, one may establish [458] that $N_{\lambda,d}$ are asymptotically normal for all $d \geq 1$. This puts the experimental results and Monte Carlo simulations of Quintanilla and Torquato [410] and Torquato (ch. 11.4 of [494])) on rigorous footing.

Theorem 8.7. *Let $\mathcal{S}_\lambda$ and $N_\lambda := N_\lambda(\mathcal{S}_\lambda)$ be as above. There are constants $\mu := \mu(S,d) \in (0,\infty)$ and $\sigma^2 := \sigma^2(S,d) \in (0,\infty)$ such that as $\lambda \to \infty$ we have*

$$\left|\lambda^{-1}\mathbf{E}N_\lambda - \mu\right| = O(\lambda^{-1/d}) \tag{8.64}$$

and $\lambda^{-1}\,\mathbf{var}[N_\lambda] \to \sigma^2$ with

$$\sup_{t \in \mathbb{R}} \left| \mathbf{P}\left(\frac{N_\lambda - \mathbf{E}N_\lambda}{\sqrt{\mathbf{var}[N_\lambda]}} \leq t \right) - \mathbf{P}(N(0,1) \leq t) \right| = O((\log\lambda)^{3d}\lambda^{-1/2}). \tag{8.65}$$

To prove this, one could enumerate the arriving solids in $\mathcal{S}_\lambda$, by (x_i, t_i), where $x_i \in \mathbb{R}^d$ is the spatial coordinate of the i-th solid and $t_i \in [0,\infty)$ is its temporal coordinate, i.e. the arrival time. Furthermore, letting $\mathcal{X} := \{(x_i,t_i)\}_{i=1}^\infty$ be a marked point process, one could set $\xi((x,t),\mathcal{X})$ to be one or zero depending on whether the solid with center at $x \in \mathcal{S}_\lambda$ is accepted or not; $H^\xi(\mathcal{X})$ is the total number of solids accepted. Thus ξ is defined on elements of the marked point process $\mathcal{X}$. A natural way to prove Theorem 8.7 would then be to show that ξ satisfies the conditions of Theorem 8.5. The moment conditions (8.35) are clearly satisfied as ξ is bounded by 1. To show stabilization it turns out that it is easier to *discretize* as follows.

For any $A \subset \mathbb{R}^d$, let $A_+ := A \times \mathbb{R}_+$. Let $\zeta(\mathcal{X},A)$ be the number of solids with centers in $\mathcal{X} \cap A$ which are packed according to the packing rules. Abusing notation, let Π denote a homogeneous Poisson point process in $\mathbb{R}^d \times \mathbb{R}_+$ with intensity $dx \times ds$, with dx denoting Lebesgue measure on $\mathbb{R}^d$ and ds denoting Lebesgue measure on $\mathbb{R}_+$. Abusing the terminology at (8.30), ζ is *homogeneously stabilizing* since it may be shown that almost surely there exists $R < \infty$ (a radius of homogeneous stabilization for ζ) such that for all $\mathcal{X} \subset (\mathbb{R}^d \setminus B_R)_+$ we have

$$\zeta((\Pi \cap (B_R)_+) \cup \mathcal{X}, Q_1) = \zeta(\Pi \cap (B_R)_+, Q_1). \tag{8.66}$$

Since ζ is homogeneously stabilizing it follows that the limit

$$\zeta(\Pi, i+Q_1) := \lim_{r\to\infty} \zeta\left(\Pi \cap (B_r(i))_+, i+Q_1\right)$$

exists almost surely for all $i \in \mathbb{Z}^d$. The random variables $(\zeta(\Pi, i+Q_1),\ i \in \mathbb{Z}^d)$ form a stationary random field. It may be shown that the tail probability for R decays exponentially fast.

Given ζ, for all $\lambda > 0$, all $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, and all Borel $A \subset \mathbb{R}^d$ we let $\zeta_\lambda(\mathcal{X},A) := \zeta(\lambda^{1/d}\mathcal{X}, \lambda^{1/d}A)$. Let Π_λ, $\lambda \geq 1$, denote a homogeneous Poisson point process in $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\lambda\, dx \times ds$. Define the random measure μ_λ^ζ on $\mathbb{R}^d$ by

$$\mu_\lambda^\zeta(\,\cdot\,) := \zeta_\lambda(\Pi_\lambda \cap Q_1, \cdot) \tag{8.67}$$

and the centered version $\overline{\mu}_\lambda^\zeta := \mu_\lambda^\zeta - \mathbf{E}[\mu_\lambda^\zeta]$. Modification of the stabilization methods of Section 8.3 then yield Theorem 8.7; this is spelled out in [458].

For companion results for RSA packing with *finite input per unit volume* we refer to [400].

8.4.2 Convex hulls

Let $K \subset \mathbb{R}^d$ be a compact convex body with non-empty interior and having a C^3 boundary of positive Gaussian curvature $x \mapsto H_{d-1}(x)$, with $x \in \partial K$. Letting Π_λ be a Poisson point process in $\mathbb{R}^d$ of intensity λ we let K_λ be the convex hull of $K \cap \Pi_\lambda$.

The random polytope K_λ, together with the analogous polytope K_n obtained by considering n i.i.d. uniformly distributed points in $B_1(o)$, are well-studied objects in stochastic geometry, with a long history originating with the work of Rényi and Sulanke [421]. See the surveys of Affentranger [3], Buchta [88], Gruber [207], Schneider [444, 446], and Weil and Wieacker [513]), together with Chapter 8.2 in Schneider and Weil [451]. See the overview in Section 7.1.

Functionals of K_λ of interest include its volume, here denoted $V_d(K_\lambda)$ and the number of k-dimensional faces of K_λ, here denoted $f_k(K_\lambda)$, $k \in \{0,1,...,d-1\}$. Note that $f_0(K_\lambda)$ is the number of vertices of K_λ. The k-th intrinsic volumes of K_λ are denoted by $V_k(K_\lambda)$, $k \in \{1,...,d-1\}$.

As seen in Section 7.1, we have for all $d \geq 2$ and all $k \in \{0,\dots,d-1\}$ that there are constants $D_{k,d}$ such that

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d+1)} \mathbf{E} f_k(K_\lambda) = D_{k,d} \int_{\partial K} H_{d-1}(x)^{1/(d+1)} dx.$$

and one may wonder whether there exist similar asymptotics for limiting variances. This is indeed the case, which may be seen as follows.

Define the functional $\xi(x,\mathcal{X})$ to be one or zero, depending on whether $x \in \mathcal{X}$ is a vertex in the convex hull of $\mathcal{X}$. When $K = B_1(o)$ the unit ball in $\mathbb{R}^d$, by reformulating functionals of convex hulls in terms of functionals of re-scaled parabolic growth processes in space and time, it may be shown that ξ is exponentially localizing [111]. The arguments are non-trivial and we refer to [111] for details. Taking into account the proper scaling in space-time, a modification of Theorem 8.5 yields variance asymptotics for $V_d(K_\lambda)$, namely

$$\lim_{\lambda\to\infty} \lambda^{(d+3)/(d+1)} \mathbf{var}[V_d(K_\lambda)] = \sigma_V^2, \tag{8.68}$$

where $\sigma_V^2 \in (0,\infty)$ is a constant. This adds to Reitzner's central limit theorem (Theorem 1 of [419]), his variance approximation $\mathbf{var}[V_d(K_\lambda)] \approx \lambda^{-(d+3)/(d+1)}$ (Theorem 3 and Lemma 1 of [419]), and Hsing [248], which is confined to $d = 2$. The stabilization methods of Theorem 8.5 yield a central limit theorem for $V_d(K_\lambda)$.

Let $k \in \{0,1,...,d-1\}$. Consider the functional $\xi_k(x,\mathcal{X})$, defined to be zero if x is not a vertex in the convex hull of $\mathcal{X}$ and otherwise defined to be the product of $(k+1)^{-1}$ and the number of k-dimensional faces containing x. Consideration of the parabolic growth processes and the stabilization of ξ_k in the context of such processes (cf. [111]) yield variance asymptotics and a central limit theorem for the number of k-dimensional faces of K_λ, yielding for all $k \in \{0,1,\dots,d-1\}$

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d+1)} \mathbf{var}[f_k(K_\lambda)] = \sigma_{f_k}^2, \tag{8.69}$$

where $\sigma_{f_k}^2 \in (0,\infty)$ is given as a closed form expression described in terms of paraboloid growth processes. For the case $k = 0$, this is proved in [459], whereas [111] handles the cases $k > 0$. This adds to Reitzner (Lemma 2 of [419]), whose breakthrough paper showed $\mathbf{var}[f_k(K_\lambda)] \approx \lambda^{(d-1)/(d+1)}$.

Theorem 8.5 also yields variance asymptotics for the intrinsic volumes $V_k(K_\lambda)$ of K_λ for all $k \in \{1,\dots,d-1\}$, namely

$$\lim_{\lambda\to\infty} \lambda^{(d+3)/(d+1)} \mathbf{var}[V_k(K_\lambda)] = \sigma_{V_k}^2, \tag{8.70}$$

where again $\sigma^2_{V_k}$ is explicitly described in terms of paraboloid growth processes. This adds to Bárány et al. (Theorem 1 of [46]), which shows $\mathbf{var}[V_k(K_n)] \approx n^{-(d+3)/(d+1)}$.

8.4.3 Intrinsic dimension of high dimensional data sets

Given a finite set of samples taken from a multivariate distribution in $\mathbb{R}^d$, a fundamental problem in learning theory involves determining the intrinsic dimension of the sample [156, 299, 427, 492]. Multidimensional data ostensibly belonging to a high-dimensional space $\mathbb{R}^d$ often are concentrated on a smooth submanifold $\mathcal{M}$ or hypersurface with intrinsic dimension m, where $m < d$. The problem of determining the intrinsic dimension of a data set is of fundamental interest in machine learning, signal processing, and statistics and it can also be handled via analysis of the sums (8.1).

Discerning the intrinsic dimension m allows one to reduce dimension with minimal loss of information and to consequently avoid difficulties associated with the 'curse of dimensionality'. When the data structure is linear there are several methods available for dimensionality reduction, including principal component analysis and multidimensional scaling, but for non-linear data structures, mathematically rigorous dimensionality reduction is more difficult. One approach to dimension estimation, inspired by Levina and Bickel [328] uses probabilistic methods involving the k-nearest neighbour graph $G^N(k, \mathcal{X})$ defined in Section 8.1.2.

For all $k = 3, 4, \ldots$, the Levina and Bickel estimator of the dimension of a data cloud $\mathcal{X} \subset \mathcal{M}$, is given by

$$\hat{m}_k(\mathcal{X}) := (\mathrm{card}(\mathcal{X}))^{-1} \sum_{x \in \mathcal{X}} \xi_k(x, \mathcal{X}),$$

where for all $x \in \mathcal{X}$ we have

$$\xi_k(x, \mathcal{X}) := (k-2) \left(\sum_{j=1}^{k-1} \log \frac{D_k(x)}{D_j(x)} \right)^{-1},$$

where $D_j(x) := D_j(x, \mathcal{X}),\ 1 \le j \le k$, are the distances between x and its j-th nearest neighbour in $\mathcal{X}$. We also define for all $\rho > 0$ the functionals

$$\xi_{k,\rho}(x, \mathcal{X}) := (k-2) \left(\sum_{j=1}^{k-1} \log \frac{D_k(x)}{D_j(x)} \right)^{-1} \mathbf{1}(D_k(x) < \rho)$$

and we put

$$\hat{m}_{k,\rho}(\mathcal{X}) := (\mathrm{card}(\mathcal{X}))^{-1} \sum_{x \in \mathcal{X}} \xi_{k,\rho}(x, \mathcal{X}).$$

Let $\{\eta_i\}_{i=1}^n$ be i.i.d. random variables with values in a submanifold $\mathcal{M}$ and put $\mathcal{X}_n := \{\eta_i\}_{i=1}^n$. Levina and Bickel [328] argue that $\hat{m}_k(\mathcal{X}_n)$ approximates the intrinsic dimension of $\mathcal{X}_n$, i.e., the dimension of $\mathcal{M}$. Indeed, $\hat{m}_k$ is an unbiased estimator when the underlying sample is a homogeneous Poisson point process on $\mathbb{R}^m$, as seen by the next exercise.

Exercise 8.8. Recall that Π_1 is a homogeneous Poisson point process on $\mathbb{R}^m$ of intensity 1. Conditional on D_k, the collection $\{(\frac{D_j(o,\Pi_1)}{D_k(o,\Pi_1)})^m\}_{j=1}^{k-1}$ is a sample from a Unif$[0,1]$-distribution. Deduce that

$$\mathbf{E}\xi_k(o,\Pi_1) = m(k-2)\mathbf{E}\left(\sum_{j=1}^{k-1}\log(1/U_j)\right)^{-1} = m.$$

Subject to regularity conditions on $\mathcal{M}$ and the density κ, the papers [403] and [526] substantiate the arguments of Levina and Bickel and show (i) consistency of the dimension estimator $\hat{m}_k(\mathcal{X}_n)$ and (ii) a central limit theorem for $\hat{m}_{k,\rho}(\mathcal{X}_n)$, ρ fixed and small, together with a rate of convergence. This goes as follows.

For all $\tau > 0$, recall that Π_τ is a homogeneous Poisson point process on $\mathbb{R}^m$ of intensity τ. Recalling the notation (8.39) and (8.40), we put

$$V^{\xi_k}(\tau,m) := \mathbf{E}[\xi_k(o,\Pi_\tau)^2] + \\ + \tau\int_{\mathbb{R}^m}\left[\mathbf{E}[\xi_k(o,\Pi_\tau\cup\{u\})\xi_k(u,\Pi_\tau\cup o)] - (\mathbf{E}[\xi_k(o,\Pi_\tau)])^2\right] du \tag{8.71}$$

and

$$\delta^{\xi_k}(\tau,m) := \mathbf{E}[\xi_k(o,\Pi_\tau)] + \tau\int_{\mathbb{R}^m}\mathbf{E}[\xi_k(o,\Pi_\tau\cup\{u\}) - \xi_k(o,\Pi_\tau)]\,du. \tag{8.72}$$

We put $\delta^{\xi_k}(m) := \delta^{\xi_k}(1,m)$. Let $\Pi_{\lambda\kappa}$ be the collection $\{\eta_1,...,\eta_{N(\lambda)}\}$, where η_i are i.i.d. with density κ and $N(\lambda)$ is an independent Poisson random variable with parameter λ. Thus $\Pi_{\lambda\kappa}$ is a Poisson point process on $\mathcal{M}$ with intensity $\lambda\kappa$. By extending Theorems 8.4 and 8.5 to C^1 submanifolds $\mathcal{M}$ as in [403], we obtain the following limit theory for the Levina and Bickel estimator.

Theorem 8.8. *Let κ be bounded away from zero and infinity on $\mathcal{M}$. We have for all $k \geq 4$*

$$\lim_{\lambda\to\infty}|\hat{m}_k(\Pi_{\lambda\kappa}) - m| = \lim_{n\to\infty}|\hat{m}_k(\mathcal{X}_n) - m| = 0, \tag{8.73}$$

where $m = dim(\mathcal{M})$ and where the convergence holds in probability. If κ is a.e. continuous then there exists $\rho_1 > 0$ such that if $\rho \in (0,\rho_1)$ and $k \geq 7$, then

$$\lim_{n\to\infty} n\,\mathbf{var}[\hat{m}_{k,\rho}(\mathcal{X}_n)] = \sigma_k^2(m) := \frac{m^2}{k-3} - (\delta^{\xi_k}(m))^2 \tag{8.74}$$

and as $n \to \infty$,

$$n^{1/2}(\hat{m}_{k,\rho}(\mathcal{X}_n) - \mathbf{E}\hat{m}_{k,\rho}(\mathcal{X}_n)) \xrightarrow{d} N(0,\sigma_k^2(m)). \tag{8.75}$$

Remark. Theorem 8.8 adds to Chatterjee [116], who does not provide variance asymptotics (8.74) and who considers convergence rates with respect to the weaker Kantorovich-Wasserstein distance. Bickel and Yan (Theorems 1 and 3 of Section 4 of [67]) establish a central limit theorem for $\hat{m}_k(\mathcal{X}_n)$ for linear $\mathcal{M}$.

8.4.4 Clique counts, Vietoris-Rips complex

A central problem in data analysis involves discerning and counting clusters. Geometric graphs and the Vietoris-Rips complex play a central role and both are amenable to asympototic analysis via stabilization techniques. The Vietoris-Rips complex is studied in connection with the statistical analysis of high-dimensional data sets [118], manifold reconstruction [119], and it has also received attention amongst topologists in connection with clustering and connectivity questions of data sets [112].

If $\mathcal{X} \subset \mathbb{R}^d$ is finite and $\beta > 0$, then the *Vietoris-Rips complex* $\mathcal{R}^\beta(\mathcal{X})$ is the abstract simplicial complex whose k-simplices (cliques of order $k+1$) correspond to unordered $(k+1)$ tuples of points of $\mathcal{X}$ which are pairwise within Euclidean distance β of each other. Thus, if there is a subset S of $\mathcal{X}$ of size $k+1$ with all points of S distant at most β from each other, then S is a k-simplex in the complex.

Given $\mathcal{R}^\beta(\mathcal{X})$ and $k \in \mathbb{N}$, let $N_k^\beta(\mathcal{X})$ be the cardinality of k-simplices in $\mathcal{R}^\beta(\mathcal{X})$. Let $\xi_k^\beta(x,\mathcal{X})$ be the product of $(k+1)^{-1}$ and the cardinality of k-simplices containing x in $\mathcal{R}^\beta(\mathcal{X})$. Thus $N_k^\beta(\mathcal{X}) = \sum_{x\in\mathcal{X}} \xi_k^\beta(x,\mathcal{X})$. The value of $\xi_k^\beta(x,\mathcal{X})$ depends only on points distant at most β from x, showing that β is a radius of stabilization for ξ_k^β and thus ξ_k^β is trivially exponentially stabilizing (8.32) and binomially exponentially stabilizing (8.58).

The next scaling result, which holds for C^1 submanifolds $\mathcal{M}$, links the large scale behavior of the clique count with the density κ of the underlying point set. Let η_i be i.i.d. with density κ on the manifold $\mathcal{M}$. Put $\mathcal{X}_n := \{\eta_i\}_{i=1}^n$. Let Π_τ be a homogeneous Poisson point process on $\mathbb{R}^m$ of constant intensity τ, dx the volume measure on $\mathcal{M}$, and let $V^{\xi_k^\beta}$ and $\delta^{\xi_k^\beta}$ be defined as in (8.39) and (8.40), respectively, with ξ replaced by ξ_k^β. It is shown in [403] that a generalization of Theorems 8.4 and 8.6 to binomial input on manifolds yields:

Theorem 8.9. *Let κ be bounded on $\mathcal{M}$;* $\dim \mathcal{M} = m$. *For all $k \in \mathbb{N}$ and all $\beta > 0$ we have*

$$\lim_{n\to\infty} n^{-1} N_k^\beta(n^{1/m}\mathcal{X}_n) = \int_{\mathcal{M}} \mathbf{E}[\xi_k^\beta(o,\Pi_{\kappa(x)})]\kappa(x)\,dx \ \text{ in } L^2. \tag{8.76}$$

If κ is a.e. continuous and bounded away from zero on its support, assumed to be a compact subset of $\mathcal{M}$, then

$$\lim_{n\to\infty} n^{-1}\,\mathbf{var}[N_k^{\beta}(n^{1/m}\mathcal{X}_n)]$$
$$= \sigma_k^2(m) := \int_{\mathcal{M}} V^{\xi_k^{\beta}}(\kappa(x))\kappa(x)\,dx - \left(\int_{\mathcal{M}} \delta^{\xi_k^{\beta}}(\kappa(x))\kappa(x)\,dx\right)^2 \quad (8.77)$$

and, as $n \to \infty$

$$n^{-1/2}(N_k^{\beta}(n^{1/m}\mathcal{X}_n) - \mathbf{E}N_k^{\beta}(n^{1/m}\mathcal{X}_n)) \xrightarrow{d} N(0,\sigma_k^2(m)). \quad (8.78)$$

This result extends Proposition 3.1, Theorem 3.13, and Theorem 3.17 of [394]. For more details and for further simplification of the limits (8.76) and (8.77) we refer to [403].