Chapter 1
Limit theorems in discrete stochastic geometry

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Abstract This overview surveys two general methods for establishing limit theorems for functionals in discrete stochastic geometry. The functionals of interest are linear statistics with the general representation $\sum_{x \in \mathcal{X}} \xi(x, \mathcal{X})$, where $\mathcal{X}$ is locally finite and where the interactions of x with respect to $\mathcal{X}$, given by $\xi(x, \mathcal{X})$, exhibit spatial dependence. We focus on subadditive methods and stabilization methods as a way to obtain weak laws of large numbers and central limit theorems for normalized and re-scaled versions of $\sum_{i=1}^{n} \xi(X_i, \{X_j\}_{j=1}^{n})$, where $X_j, j \geq 1$, are i.i.d. random variables. The general theory is applied to particular problems in Euclidean combinatorial optimization, convex hulls, random sequential packing, and dimension estimation.

1.1 Introduction

This overview surveys two general methods for establishing limit theorems, including weak laws of large numbers and central limit theorems, for functionals of large random geometric structures. By geometric structures, we mean for example networks arising in computational geometry, graphs arising in Euclidean optimization problems, models for random sequential packing, germ-grain models, and the convex hull of high density point sets. Such diverse structures share only the common feature that they are defined in terms of random points belonging to Euclidean space $\mathbb{R}^d$. The points are often the realization of i.i.d. random variables, but they could also be the realization of Poisson point processes or even Gibbs point processes. There is scope here for generalization to point processes in more general spaces, including manifolds and general metric spaces, but for ease of exposition we restrict attention to point processes in $\mathbb{R}^d$. As such, this introductory overview makes few demands

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involving prior familiarity with the literature. Our goals are to provide an accessible survey of asymptotic methods involving (i) subadditivity and (ii) stabilization and to illustrate the applicability of these methods to problems in discrete stochastic geometry.

Functionals of geometric structures are often formulated as *linear statistics* on locally finite point sets $\mathcal{X}$ of $\mathbb{R}^d$, that is to say consist of sums represented as

$$H(\mathcal{X}) := H^{\xi}(\mathcal{X}) := \sum_{x \in \mathcal{X}} \xi(x, \mathcal{X}), \tag{1.1}$$

where the function ξ, defined on all pairs $(x, \mathcal{X})$, $x \in \mathcal{X}$, represents the *interaction* of x with respect to $\mathcal{X}$. In nearly all problems of interest, the values of $\xi(x, \mathcal{X})$ and $\xi(y, \mathcal{X})$, $x \neq y$, are not unrelated but, loosely speaking, become more related as the Euclidean distance $||x - y||$ becomes smaller. This 'spatial dependency' is the chief source of difficulty when developing the limit theory for H^{ξ} on random point sets. Despite this inherent spatial dependency, relatively simple subadditive methods originating in the landmark paper of Beardwood, Halton, and Hammersley [8], and developed further in [66] and [70], yield mean and a.s. asymptotics of the normalized sums

$$n^{-1} H^{\xi}(\{x_i\}_{i=1}^n), \tag{1.2}$$

where x_i are i.i.d. with values in $[0,1]^d$. Subadditive methods lean heavily on the self-similarity of the unit cube, but to obtain distributional results, variance asymptotics, and explicit limiting constants in laws of large numbers, one needs tools going beyond subadditivity. When the spatial dependency may be localized, in a sense to be made precise, then this localization yields distributional and second order results, and it also shows that the *large scale macroscopic behaviour of* H^{ξ} on random point sets, e.g. laws of large numbers and central limit theorems, *is governed by the local interactions described by* ξ.

Typical questions motivating this survey, which may all be framed in terms of the linear statistics (1.1), include the following:

1. Given i.i.d. points $x_1, ..., x_n$ in the unit cube $[0,1]^d$, what is the asymptotic length of the shortest tour through $x_1, ..., x_n$?
2. Given i.i.d. points $x_1, ...x_n$ in the unit d-dimensional ball, what is the asymptotic distribution of the number of k-dimensional faces, $k \in \{0, 1, ..., d-1\}$, in the random polytope given by the convex hull of $x_1, ..., x_n$?
3. Open balls $B_1, B_2, ..., B_n$ of volume n^{-1} arrive sequentially and uniformly at random in $[0,1]^d$. The first ball B_1 is *packed*, and recursively for $i = 2, 3, ...$, the i-th ball B_i is packed iff B_i does not overlap any ball in $B_1, ..., B_{i-1}$ which has already been packed. If not packed, the i-th ball is discarded. The process continues until no more balls can be packed. As $n \to \infty$, what is the asymptotic distribution of the number of balls which are packed in $[0,1]^d$?

To see that such questions fit into the framework of (1.1) it suffices to make these corresponding choices for ξ:

1'. $\xi(x,\mathcal{X})$ is one half the sum of the lengths of edges incident to x in the shortest tour on $\mathcal{X}$; $H^{\xi}(\mathcal{X})$ is the length of the shortest tour through $\mathcal{X}$,
2'. $\xi_k(x,\mathcal{X})$ is defined to be zero if x is not a vertex in the convex hull of $\mathcal{X}$ and otherwise defined to be the product of $(k+1)^{-1}$ and the number of k-dimensional faces containing x; $H^{\xi}(\mathcal{X})$ is the number of k-faces in the convex hull of $\mathcal{X}$,
3'. $\xi(x,\mathcal{X})$ is equal to one or zero depending on whether the ball with center at $x \in \mathcal{X}$ is accepted or not; $H^{\xi}(\mathcal{X})$ is the total number of balls accepted.

When $\mathcal{X}$ is a growing point set of random variables, the large scale asymptotic analysis of the sums (1.1) is sometimes handled by M-dependent methods, ergodic theory, or mixing methods. However, these classical methods, when applicable, may not give explicit asymptotics in terms of the underlying interaction and point densities, they may not yield second order results, or they may not easily yield explicit rates of convergence. Our goal here is to provide an abridged treatment of two alternate methods suited to the asymptotic theory of the sums (1.2), namely to discuss (i) subadditivity and (ii) stabilization.

The sub-additive approach, described in detail in the monographs [66], [70], yields a.s. laws of large numbers for problems in Euclidean combinatorial optimization, including the length of minimal spanning trees, minimal matchings, and shortest tours on random point sets. Formal definitions of these archetypical problems are given below. Sub-additive methods also yield the a.s. limit theory of problems in computational geometry, including the total edge length of nearest neighbour graphs, the Voronoi and Delaunay graphs, the sphere of influence graph, as well as graphs graphs arising in minimal triangulations and the k-means problem. The approach based on stabilization, originating in Penrose and Yukich [41] and further developed in [6, 38, 39, 42, 45], is useful in proving laws of large numbers, central limit theorems, and variance asymptotics for many of these functionals; as such it provides closed form expressions for the limiting constants arising in the mean and variance asymptotics. This approach has been used to study linear statistics arising in random packing [42], convex hulls [59], ballistic deposition models [6, 42], quantization [60, 72], loss networks [60], high-dimensional spacings [7], distributed inference in random networks [2], and geometric graphs in Euclidean combinatorial optimization [41, 43].

Recalling that $\mathcal{X}$ is a locally finite point set in $\mathbb{R}^d$, functionals and graphs of interest include:

1. **Traveling salesman functional; TSP.** A closed tour on $\mathcal{X}$ or closed Hamiltonian tour is a closed path traversing each vertex in $\mathcal{X}$ exactly once. Let TSP($\mathcal{X}$) be the length of the shortest closed tour T on $\mathcal{X}$. Thus

$$TSP(\mathcal{X}) := \min_T \sum_{e \in T} |e|, \tag{1.3}$$

where the minimum is over all tours T and where $|e|$ denotes the Euclidean edge length of the edge e. Thus,

$$TSP(\mathcal{X}) := \min_{\sigma} \left\{ \|x_{\sigma(n)} - x_{\sigma(1)}\| + \sum_{i=1}^{n-1} \|x_{\sigma(i)} - x_{\sigma(i+1)}\| \right\},$$

where the minimum is taken over all permutations σ of the integers $1,2,...,n$.

2. **Minimum spanning tree; MST.** Let $\mathrm{MST}(\mathcal{X})$ be the length of the shortest spanning tree on $\mathcal{X}$, namely

$$MST(\mathcal{X}) := \min_{T} \sum_{e \in T} |e|, \tag{1.4}$$

where the minimum is over all spanning trees T of $\mathcal{X}$.

3. **Minimal matching**. The *minimal matching* on $\mathcal{X}$ has length given by

$$MM(\mathcal{X}) := \min_{\sigma} \sum_{i=1}^{n/2} \|x_{\sigma(2i-1)} - x_{\sigma(2i)}\|, \tag{1.5}$$

where the minimum is over all permutations of the integers $1,2,...,n$. If n has odd parity, then the minimal matching on $\mathcal{X}$ is the minimum of the minimal matchings on the n distinct subsets of $\mathcal{X}$ of size $n-1$.

4. **k-nearest neighbours graph.** Let $k \in \mathbb{N}$. The k-nearest neighbours (undirected) graph on $\mathcal{X}$, here denoted $G^N(k,\mathcal{X})$, is the graph with vertex set $\mathcal{X}$ obtained by including $\{x,y\}$ as an edge whenever y is one of the k nearest neighbours of x and/or x is one of the k nearest neighbours of y. The k-nearest neighbours (directed) graph on $\mathcal{X}$, denoted $\mathsf{G}^N(k,\mathcal{X})$, is the graph with vertex set $\mathcal{X}$ obtained by placing an edge between each point and its k nearest neighbours. Let $NN(k,\mathcal{X})$ denote the total edge length of $G^N(k,\mathcal{X})$, i.e.,

$$NN(k,\mathcal{X}) := \sum_{e \in G^N(k,\mathcal{X})} |e|, \tag{1.6}$$

with a similar definition for the total edge length of $\mathsf{G}^N(k,\mathcal{X})$.

5. **Steiner minimal spanning tree.** A *Steiner tree* on $\mathcal{X}$ is a connected graph containing the vertices in $\mathcal{X}$. The graph may include vertices other than those in $\mathcal{X}$. The total edge length of the Steiner minimal spanning tree on $\mathcal{X}$ is

$$ST(\mathcal{X}) := \min_{S} \sum_{e \in S} |e|, \tag{1.7}$$

where the minimum ranges over all Steiner trees S on $\mathcal{X}$.

6. **Minimal semi-matching.** A semi-matching on $\mathcal{X}$ is a graph in which all vertices have degree 2, with the understanding that an isolated edge between two vertices represents two copies of that edge. The graph thus contains tours with an odd number of edges as well as isolated edges. The minimal semi-matching functional on $\mathcal{X}$ is

$$SM(\mathcal{X}) := \min_{SM} \sum_{e \in SM} |e|, \tag{1.8}$$

where the minimum ranges over all semi-matchings SM on $\mathcal{X}$.

7. k**-TSP functional.** Fix $k \in \mathbb{N}$. Let $\mathcal{C}$ be a collection of k sub-tours on points of $\mathcal{X}$, each sub-tour containing a distinguished vertex x_0 and such that each $x \in \mathcal{X}$ belongs to exactly one sub-tour. $T(k;\mathcal{C},\mathcal{X})$ is the sum of the combined lengths of the k sub-tours in $\mathcal{C}$. The k-TSP functional is the infimum
$$T(k;\mathcal{X}) := \inf_{\mathcal{C}} T(k;\mathcal{C},\mathcal{X}). \tag{1.9}$$
Power-weighted edge versions of these functionals are found in [70].

1.2 Subadditivity

Sub-additive functionals

Let $x_n \in \mathbb{R},\ n \geq 1$, satisfy the 'sub-additive inequality'
$$x_{m+n} \leq x_m + x_n \ \text{ for all } m,\, n \in \mathbb{N}. \tag{1.10}$$
Sub-additive sequences are nearly additive in the sense that they satisfy the *sub-additive limit theorem*, namely $\lim_{n\to\infty} x_n/n = \alpha$ where $\alpha := \inf\{x_m/m : m \geq 1\} \in [-\infty,\infty)$. This classic result, proved in Hille (1948), may be viewed as a limit result about sub-additive functions indexed by intervals.

For certain choices of the interaction ξ, the functionals H^ξ defined at (1.1) satisfy geometric subadditivity over rectangles and, as we will see, consequently satisfy a sub-additive limit theorem analogous to the classic one just mentioned. To allow greater generality we henceforth allow the interaction ξ to depend on a parameter $p \in (0,\infty)$ and we will write $\xi(\cdot,\cdot) := \xi_p(\cdot,\cdot)$. For example, $\xi_p(\cdot,\cdot)$ could denote the sum of the pth powers of lengths of edges incident to x, where the edges belong to some specified graph on $\mathcal{X}$.

We henceforth work in this context, but to lighten the notation we will suppress mention of p.

Let $\mathcal{R} := \mathcal{R}(d)$ denote the collection of d-dimensional rectangles in $\mathbb{R}^d$. Write $H^\xi(\mathcal{X},R)$ for $H^\xi(\mathcal{X}\cap R)$, $R \in \mathcal{R}$. Say that H^ξ is *geometrically sub-additive*, or simply *sub-additive*, if there is a constant $c_1 := c_1(p) < \infty$ such that for all $R \in \mathcal{R}$, all partitions of R into rectangles R_1 and R_2, and all finite point sets $\mathcal{X}$ we have
$$H^\xi(\mathcal{X},R) \leq H^\xi(\mathcal{X},R_1) + H^\xi(\mathcal{X},R_2) + c_1(\mathrm{diam}\,(R))^p. \tag{1.11}$$
Unlike scalar subadditivity (1.10), the relation (1.11) carries an error term.

Classic optimization problems as well as certain functionals of Euclidean graphs, satisfy geometric subadditivity (1.11). For example, the length of the minimal spanning tree defined at (1.4) satisfies (1.11) when p is set to 1, which may be seen as follows. Put $\mathrm{MST}(\mathcal{X},R)$ to be the length of the minimal spanning tree on $\mathcal{X}\cap R$. Given a finite set $\mathcal{X}$ and a rectangle $R := R_1 \cup R_2$, let $\mathcal{T}_i$ denote the minimal spanning tree on $\mathcal{X} \cap R_i$, $1 \le i \le 2$. Tie together the two spanning trees $\mathcal{T}_1$ and $\mathcal{T}_2$ with an edge having a length bounded by the sum of the diameters of the rectangles R_1 and R_2. Performing this operation generates a feasible spanning tree on $\mathcal{X}$ at a total cost bounded by $\mathrm{MST}(\mathcal{X}, R_1)$ + $\mathrm{MST}(\mathcal{X}, R_2)$ + $\mathrm{diam}(R)$. Putting $p = 1$, (1.11) follows by minimality. We may similarly show that the TSP (1.3), minimal matching (1.5), and nearest neighbour functionals (1.6) satisfy geometric subadditivity (1.11) with $p = 1$.

Super-additive functionals

Were geometric functionals H^ξ to simultaneously satisfy a super-additive relation analogous to (1.11), then the resulting 'near additivity' of H^ξ would lead directly to laws of large numbers. This is too much to hope for. On the other hand, many geometric functionals $H^\xi(\cdot, R)$ admit a 'dual' version - one which essentially treats the boundary of the rectangle R as a single point, that is to say edges on the boundary ∂R have zero length or 'zero cost'. This boundary version, introduced in [48] and used in [49] and [50] and here denoted $H_B^\xi(\cdot, R)$, closely approximates $H^\xi(\cdot, R)$ in a sense to be made precise (see (1.17) below) and is *super-additive without any error term*. More exactly, the boundary version $H_B^\xi(\cdot, R)$ satisfies

$$H_B^\xi(\mathcal{X}, R) \ge H_B^\xi(\mathcal{X} \cap R_1, R_1) + H_B^\xi(\mathcal{X} \cap R_2, R_2). \tag{1.12}$$

By way of illustration we define the boundary minimal spanning tree functional. For all rectangles $R \in \mathcal{R}$ and finite sets $\mathcal{X} \subset R$ put

$$MST_B(\mathcal{X}, R) := \min \left(MST(\mathcal{X}, R), \ \inf \sum_i MST(\mathcal{X}_i \cup a_i) \right),$$

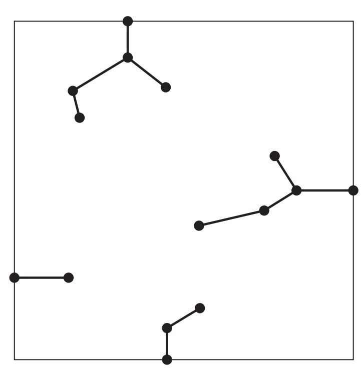

Fig. 1.1 The boundary MST graph; edges on boundary have zero cost.

where the infimum ranges over all partitions $(\mathcal{X}_i)_{i\geq 1}$ of $\mathcal{X}$ and all sequences of points $(a_i)_{i\geq 1}$ belonging to ∂R. When $\mathrm{MST}_B(\mathcal{X},R) \neq \mathrm{MST}(\mathcal{X},R)$ the graph realizing the boundary functional $\mathrm{MST}_B(\mathcal{X},R)$ may be thought of as a collection of small trees connected via the boundary ∂R into a single large tree, where the connections on ∂R incur no cost. See Figure 1.1. It is a simple matter to see that the boundary MST functional satisfies sub-additivity (1.11) with $p = 1$ and is also super-additive (1.12). Later we will see that the boundary MST functional closely approximates the standard MST functional.

The traveling salesman (shortest tour) graph, minimal matching graph, and nearest neighbour graph all satisfy (1.11) and have boundary versions which are super-additive (1.12); see [70] for details.

Sub-additive and super-additive Euclidean functionals

Recall that $\xi(\cdot,\cdot) := \xi_p(\cdot,\cdot)$. The following conditions endow the functional $H^{\xi}(\cdot,\cdot)$ with a *Euclidean structure*:

$$H^{\xi}(\mathcal{X},R) = H^{\xi}(\mathcal{X}+y,R+y) \tag{1.13}$$

for all $y \in \mathbb{R}^d$, $R \in \mathcal{R}$, $\mathcal{X} \subset R$ and

$$H^{\xi}(\alpha\mathcal{X},\alpha R) = \alpha^p H^{\xi}(\mathcal{X},R) \tag{1.14}$$

for all $\alpha > 0$, $R \in \mathcal{R}$ and $\mathcal{X} \subset R$. By αB we understand the set $\{\alpha x, x \in B\}$ and by $y+\mathcal{X}$ we mean $\{y+x:\ x \in \mathcal{X}\}$. Conditions (1.13) and (1.14) express the *translation invariance* and *homogeneity of order p* of H^{ξ}, respectively. Homogeneity (1.14) is satisfied whenever the interaction ξ is itself homogeneous of order p, that is to say whenever

$$\xi(\alpha x,\alpha\mathcal{X}) = \alpha^p \xi(x,\mathcal{X}), \quad \alpha > 0. \tag{1.15}$$

Functionals satisfying translation invariance and homogeneity of order 1 include the total edge length of graphs, including those defined at (1.3)-(1.9).

If a functional $H^{\xi}(\mathcal{X},R)$, $(\mathcal{X},R) \in \mathsf{N} \times \mathcal{R}$, is super-additive over rectangles and has a Euclidean structure over $\mathsf{N} \times \mathcal{R}$, where N is the space of locally set of finite point sets in $\mathbb{R}^d$, then we say that H^{ξ} is a *super-additive Euclidean functional*, formally defined as follows:

Definition 1. Let $H^{\xi}(\emptyset,R) = 0$ for all $R \in \mathcal{R}$ and suppose H^{ξ} satisfies (1.13) and (1.14). If H^{ξ} satisfies

$$H^{\xi}(\mathcal{X},R) \geq H^{\xi}(\mathcal{X}\cap R_1,R_1) + H^{\xi}(\mathcal{X}\cap R_2,R_2), \tag{1.16}$$

whenever $R \in \mathcal{R}$ is partitioned into rectangles R_1 and R_2 then H^{ξ} is a *super-additive Euclidean functional*. *Sub-additive Euclidean functionals* satisfy (1.13), (1.14), and geometric subadditivity (1.11).

It may be shown that the functionals TSP, MST and MM are sub-additive Euclidean functionals and that they admit dual boundary versions which are super-additive Euclidean functionals; see Chapter 2 of [70]. To be useful in establishing asymptotics, dual boundary functionals must closely approximate the corresponding functional. The following closeness condition is sufficient for these purposes. Recall that we suppress the dependence of ξ on p, writing $\xi(\cdot,\cdot) := \xi_p(\cdot,\cdot)$.

Definition 2. Say that H^ξ and H^ξ_B are *pointwise close* if for all finite subsets $\mathcal{X} \subset [0,1]^d$ we have

$$|H^\xi(\mathcal{X},[0,1]^d) - H^\xi_B(\mathcal{X},[0,1]^d)| = o\left(\mathrm{card}(\mathcal{X})\right)^{(d-p)/d}. \tag{1.17}$$

The TSP, MST, MM and nearest neighbour functionals all admit respective boundary versions which are pointwise close in the sense of (1.17); see Lemma 3.7 of [70]. See [70] for description of other functionals having boundary versions which are pointwise close in the sense of (1.17).

Iteration of geometric subadditivity (1.11) leads to growth bounds on sub-additive Euclidean functionals H^ξ, namely for all $p \in (0,d)$ there is a constant $c_2 := c_2(\xi_p, d)$ such that for all rectangles $R \in \mathcal{R}$ and all $\mathcal{X} \subset R$, $\mathcal{X} \in \mathsf{N}$, we have

$$H^\xi(\mathcal{X},R) \le c_2(\mathrm{diam}\,(R))^p(\mathrm{card}\,\mathcal{X})^{(d-p)/d}. \tag{1.18}$$

Subadditivity (1.11) and growth bounds (1.18) by themselves do not provide enough structure to yield the limit theory for Euclidean functionals; one also needs control on the oscillations of these functionals as points are added or deleted. Some functionals, such as TSP, clearly increase with increasing argument size, whereas others, such as MST, may decrease. A useful continuity condition goes as follows.

Definition 3. A Euclidean functional H^ξ is *smooth of order p* if there is a finite constant $c_3 := c_3(\xi_p, d)$ such that for all finite sets $\mathcal{X}_1, \mathcal{X}_2 \subset [0,1]^d$ we have

$$|H^\xi(\mathcal{X}_1 \cup \mathcal{X}_2) - H^\xi(\mathcal{X}_1)| \le c_3(\mathrm{card}\,(\mathcal{X}_2))^{(d-p)/d}. \tag{1.19}$$

Examples of functionals satisfying smoothness (1.19)

1. Let TSP be as in (1.3). For all finite sets $\mathcal{X}_1$ and $\mathcal{X}_2 \subset [0,1]^d$ we have

$$TSP(\mathcal{X}_1) \le TSP(\mathcal{X}_1 \cup \mathcal{X}_2) \le TSP(\mathcal{X}_1) + TSP(\mathcal{X}_2),$$

where the first inequality follows by the monotonicity of the TSP functional and the second by subadditivity (1.11). Since by (1.18) we have $\mathrm{TSP}(\mathcal{X}_2) \le c_2\sqrt{d}(\mathrm{card}\,\mathcal{X}_2)^{(d-1)/d}$, it follows that the TSP is smooth of order 1.
2. Let MST be as in (1.4). Subadditivity (1.11) and the growth bounds (1.18) imply that for all sets $\mathcal{X}_1, \mathcal{X}_2 \subset [0,1]^d$ we have $\mathrm{MST}(\mathcal{X}_1 \cup \mathcal{X}_2) \le \mathrm{MST}(\mathcal{X}_1) + (c_1\sqrt{d} + c_2\sqrt{d}(\mathrm{card}\,\mathcal{X}_2)^{(d-1)/d} \le \mathrm{MST}(\mathcal{X}_1) + c(\mathrm{card}\,\mathcal{X}_2)^{(d-1)/d}$. It follows that the MST

is smooth of order 1 once we show the reverse inequality

$$MST(\mathcal{X}_1 \cup \mathcal{X}_2) \geq MST(\mathcal{X}) - c(\mathrm{card}\,\mathcal{X}_2)^{(d-1)/d}. \tag{1.20}$$

To show (1.20) let $\mathcal{T}$ denote the graph of the minimal spanning tree on $\mathcal{X}_1 \cup \mathcal{X}_2$. Remove the edges in $\mathcal{T}$ which contain a vertex in $\mathcal{X}_2$. Since each vertex has bounded degree, say D, this generates a subgraph $\mathcal{T}_1 \setminus \mathcal{T}$ which has at most $D \cdot \mathrm{card}\,\mathcal{X}_2$ components. Choose one vertex from each component and form the minimal spanning tree $\mathcal{T}_2$ on these vertices. Since the union of the trees $\mathcal{T}_1$ and $\mathcal{T}_2$ is a feasible spanning tree on $\mathcal{X}_1$, it follows that

$$MST(\mathcal{X}_1) \leq \sum_{e \in \mathcal{T}_1 \cup \mathcal{T}_2} |e| \leq MST(\mathcal{X}_1 \cup \mathcal{X}_2) + c(D \cdot \mathrm{card}\,\mathcal{X}_2)^{(d-1)/d}$$

by the growth bounds (1.18). Thus smoothness (1.19) holds for the MST functional.

We may similarly show that the minimal matching functional MM defined at (1.5) is smooth of order 1 (Chapter 3.3 of [70]). Likewise, the semi-matching, nearest neighbour, and k-TSP functionals are smooth of order 1, as shown in Sections 8.2, 8.3 and 8.4 of [70]), respectively. A modification of the Steiner functional (1.7) is smooth of order 1 (see Ch. 10 of [70]). We thus see that the functionals TSP, MST and MM defined at (1.3)-(1.5) are all smooth sub-additive Euclidean functionals which are pointwise close to a canonical boundary functional. The functionals (1.6)-(1.9) satisfy the same properties. Now we give some limit theorems for such functionals.

Laws of large numbers

We state a basic law of large numbers for Euclidean functionals on i.i.d. uniform random variables $\eta_1, ..., \eta_n$ in $[0,1]^d$. Recall that a sequence of random variables ζ_n converges completely, here denoted c.c., to a limit random variable ζ, if for all $\varepsilon > 0$, we have $\sum_{n=1}^{\infty} \mathbf{P}(|\zeta_n - \zeta| > \varepsilon) < \infty$.

Theorem 1. *Let $p \in [1,d)$. If $H_B^\xi := H_B^{\xi_p}$ is a smooth super-additive Euclidean functional of order p on $\mathbb{R}^d$, then*

$$\lim_{n \to \infty} n^{(p-d)/d} H_B^\xi(\eta_1, ..., \eta_n) = \alpha(H_B^\xi, d) \ \ c.c., \tag{1.21}$$

where $\alpha(H_B^\xi, d)$ is a positive constant. If H^ξ is a Euclidean functional which is pointwise close to H_B^ξ as in (1.17), then

$$\lim_{n \to \infty} n^{(p-d)/d} H^\xi(\eta_1, ..., \eta_n) = \alpha(H_B^\xi, d) \ \ c.c. \tag{1.22}$$

Remarks.

1. Theorem 1 gives c.c. laws of large numbers for the functionals (1.3)-(1.9); see [70] for details.
2. Smooth sub-additive Euclidean functionals which are point-wise close to smooth super-additive Euclidean functionals are 'nearly additive' and consequently satisfy Donsker-Varadhan-style *large deviation principles*, as shown in [64].
3. The papers [25] and [30] provide further accounts of the limit theory for subadditive Euclidean functionals.

Rates of convergence of Euclidean functionals

If a sub-additive Euclidean functional H^{ξ} is *close in mean* (cf. Definition 3.9 in [70]) to the associated super-additive Euclidean functional H_B^{ξ}, namely if

$$|\mathbf{E}[H^{\xi}(\eta_1,...,\eta_n)] - \mathbf{E}[H_B^{\xi}(\eta_1,...,\eta_n)]| = o(n^{(d-p)/d}), \tag{1.23}$$

where we recall that η_i are i.i.d. uniform on $[0,1]^d$, then we may upper bound $|\mathbf{E}[H^{\xi}(\eta_1,...,\eta_n)] - \alpha(H_B^{\xi},d)n^{(d-p)/d}|$, thus yielding rates of convergence of

$$\mathbf{E}[n^{(p-d)/d}H^{\xi}(\eta_1,...,\eta_n)]$$

to its mean. Since the TSP, MST, and MM functionals satisfy closeness in mean $(p \neq d-1,\ d \geq 3)$ the following theorem immediately provides rates of convergence for our prototypical examples.

Theorem 2. *(Rates of convergence of means) Let H^{ξ} and H_B^{ξ} be sub-additive and super-additive Euclidean functionals, respectively, satisfying the close in mean approximation (1.23). If H^{ξ} is smooth of order $p \in [1,d)$ as defined at (1.19), then for $d \geq 2$ and for $\alpha(H_B^{\xi},d)$ as at (1.21), we have*

$$|\mathbf{E}[H^{\xi}(\eta_1,...,\eta_n)] - \alpha(H_B^{\xi},d)n^{(d-p)/d}| \leq c\left(n^{(d-p)/2d} \vee n^{(d-p-1)/d}\right). \tag{1.24}$$

Koo and Lee [30] give conditions under which Theorem 2 can be improved.

General umbrella theorem for Euclidean functionals

Here is the main result of this section. Let $X_1,...,X_n$ be i.i.d. random variables with values in $[0,1]^d$, $d \geq 2$ and put $\mathcal{X}_n := \{X_i\}_{i=1}^n$.

Theorem 3. *(Umbrella theorem for Euclidean functionals) Let H^{ξ} and H_B^{ξ} be sub-additive and super-additive Euclidean functionals, respectively, both smooth of order $p \in [1,d)$. Assume that H^{ξ} and H_B^{ξ} are close in mean (1.23). Then*

$$\lim_{n\to\infty} n^{(p-d)/d}H^{\xi}(\mathcal{X}_n) = \alpha(H_B^{\xi},d)\int_{[0,1]^d} \kappa(x)^{(d-p)/d}\,dx \quad c.c., \tag{1.25}$$

where κ *is the density of the absolutely continuous part of the law of* η_1.

Remarks.

1. There exists an umbrella type of theorem for Euclidean functionals satisfying monotonicity and other assumptions not pertaining to boundary functionals, see e.g. Theorem 2 of [65]. Theorem 3 has its origins in [48] and [49].
2. Theorem 3 is used by Baltz et al. [3] to analyze asymptotics for the multiple vehicle routing problem; Costa and Hero [18] show asymptotics similar to Theorem 3 for the MST on suitably regular Riemannian manifolds and they apply their results to estimation of Rényi entropy and manifold dimension. Costa and Hero [19], using the theory of sub-additive and superadditive Euclidean functionals, called by them 'entropic graphs', obtain asymptotics for the total edge length of k-nearest neighbour graphs on manifolds. The paper [25] provides further applications of entropic graphs to imaging and clustering.
3. The TSP functional satisfies the conditions of Theorem 3 and we thus recover as a corollary the Beardwood-Halton-Hammersley theorem [8]. It can likewise be shown that Theorem 3 also establishes the limit theory for total edge length of the functionals defined at (1.4)-(1.9); see [70] for details.
4. If the X_i fail to have a density then the right-hand side of (1.25) vanishes. On the other hand, Hölder's inequality shows that the right-hand side of (1.25) is largest when κ is uniform on $[0,1]^d$.
5. See Chapter 7 of [70] for extensions of Theorem 3 to functionals of random variables on unbounded domains.

Proof. (Sketch of proof of Theorem 3) The proof of Theorem 3 is simplified by using the Azuma-Hoeffding concentration inequality to show that it is enough to prove convergence of means in (1.25). Smoothness then shows that it is enough to prove convergence of $\mathbf{E}[H^\xi(\mathcal{X}_n)/n^{(d-p)/d}]$ for the so-called blocked distributions, i.e. those whose absolutely continuous part is a linear combination of indicators over congruent sub-cubes forming a partition of $[0,1]^d$. To establish convergence for the blocked distributions, one combines Theorem 1 with the sub-additive and superadditive relations. These methods are standard and we refer to [70] for complete details. □

The limit (1.25) exhibits the asymptotic dependency of the total edge length of graphs on the underlying point density κ. Still, (1.25) is unsatisfying in that we don't have a closed form expression for the constant $\alpha(H_B^\xi, d)$. Stabilization methods, described below, are used to explicitly identify $\alpha(H_B^\xi, d)$.

1.3 Stabilization

Sub-additive methods yield a.s. limit theory for the functionals H^ξ defined at (1.2) but they do not express the macroscopic behaviour of H^ξ in terms of the local interactions described by ξ. Stabilization methods overcome this limitation, they yield second order and distributional results, and they also provide limit results for the empirical measures

$$\sum_{x \in \mathcal{X}} \xi(x, \mathcal{X}) \delta_x, \tag{1.26}$$

where δ_x is the point mass at x. The empirical measure (1.26) has total mass given by H^{ξ}.

We will often assume that the interaction or 'score' function ξ, defined on pairs $(x, \mathcal{X})$, with $\mathcal{X}$ locally finite in $\mathbb{R}^d$, is translation invariant, i.e. $\xi(x+y, \mathcal{X}+y) = \xi(x, \mathcal{X}),\ y \in \mathbb{R}^d$.

When $\mathcal{X}$ is random the range of spatial dependence of ξ at $x \in \mathcal{X}$ is random and the purpose of stabilization is to quantify this range in a way useful for asymptotic analysis. There are several notions of stabilization, with the simplest being that of stabilization of ξ with respect to a rate τ homogeneous Poisson point process Π_τ on $\mathbb{R}^d$, defined as follows. Let $B_r(x)$ denote the Euclidean ball centered at x with radius r and let $\mathbf{0}$ denote a point at the origin of $\mathbb{R}^d$.

Homogeneous stabilization

We say that a translation invariant ξ is *homogeneously stabilizing* if for all $\tau > 0$ there exists an almost surely finite random variable $R := R(\Pi_\tau)$ such that

$$\xi(\mathbf{0}, (\Pi_\tau \cap B_R(\mathbf{0})) \cup \mathcal{A}) = \xi(\mathbf{0}, \Pi_\tau \cap B_R(\mathbf{0})) \tag{1.27}$$

for all locally finite $\mathcal{A} \subset \mathbb{R}^d \setminus B_R(\mathbf{0})$. Thus the value of ξ at $\mathbf{0}$ is unaffected by changes in the configuration outside $B_R(\mathbf{0})$. The random range of dependency given by R depends on the realization of Π_τ.

Examples.

1. Nearest neighbour distances. Recalling (1.6), consider the nearest neighbour graph $G^N(1, \mathcal{X})$ on the point set $\mathcal{X}$ and let $\xi(x, \mathcal{X})$ denote one half the sum of the lengths of edges in $G^N(1, \mathcal{X})$ which are incident to x. Thus $H^{\xi}(\mathcal{X})$ is the sum of edge lengths in $G^N(1, \mathcal{X})$. Partition $\mathbb{R}^2$ with six congruent cones with apex at the origin of $\mathbb{R}^2$ and put R_i to be the distance between the origin and the nearest point in $\Pi_\tau \cap K_i,\ 1 \le i \le 6$. It is easy to see that $R := \max_{1 \le i \le 6} R_i$ is a radius of stabilization, i.e,. points in $B_R^c(\mathbf{0})$ do not change the value of $\xi(\mathbf{0}, \Pi_\tau)$. Indeed, any point w in $B_R^c(\mathbf{0})$ is closer to a point in $\Pi_\tau \cap B_R(\mathbf{0})$ than it is to the origin and so edges incident to w will not affect the value of $\xi(\mathbf{0}, \Pi_\tau)$.
2. Let $V(\mathcal{X})$ be the graph of the Voronoi tessellation of $\mathcal{X}$ and let $\xi(x, \mathcal{X})$ be one half the sum of the lengths of the edges in the Voronoi cell $C(x)$ around x. The Voronoi flower around x, or fundamental region, is the union of those balls having as center a vertex of $C(x)$ and exactly two points of $\mathcal{X}$ on their boundary and no points of $\mathcal{X}$ inside. Then it may be shown (see Zuyev [73]) that the geometry of $C(x)$ is completely determined by the Voronoi flower and thus the radius of a ball centered at x containing the Voronoi flower qualifies as a stabilization radius.

3. Minimal spanning trees. Recall from (1.4) that $\mathrm{MST}(\mathcal{X})$ is the total edge length of the minimal spanning tree on $\mathcal{X}$; let $\xi(x,\mathcal{X})$ be one half the sum of the lengths of the edges in the MST which are incident to x. Then ξ is homogeneously stabilizing, which follows from arguments involving the uniqueness of the infinite component in continuum percolation [44].

Given $\mathcal{X} \subset \mathbb{R}^d$, $a > 0$ and $y \in \mathbb{R}^d$, recall that $a\mathcal{X} := \{ax : \ x \in \mathcal{X}\}$. For all $\lambda > 0$ define the λ *re-scaled version* of ξ by

$$\xi_\lambda(x,\mathcal{X}) := \xi(\lambda^{1/d}x, \lambda^{1/d}\mathcal{X}). \tag{1.28}$$

Re-scaling is natural when considering point sets in compact sets K having cardinality roughly λ; dilation by $\lambda^{1/d}$ means that unit volume subsets of $\lambda^{1/d}K$ host on the average one point. When $x \in \mathbb{R}^d \setminus \mathcal{X}$, we abbreviate notation and write $\xi(x,\mathcal{X})$ instead of $\xi(x,\mathcal{X}\cup\{x\})$.

It is useful to consider point processes on $\mathbb{R}^d$ more general than the homogeneous Poisson point processes. Let κ be a probability density function on $\mathbb{R}^d$ with support $K \subseteq \mathbb{R}^d$. For all $\lambda > 0$, let $\Pi_{\lambda\kappa}$ denote a Poisson point process in $\mathbb{R}^d$ with intensity measure $\lambda\kappa(x)\,dx$. We shall assume throughout that κ is bounded with supremum denoted $\|\kappa\|_\infty$.

Homogeneous stabilization is an example of ‘point stabilization’ [56] in that ξ is required to stabilize around a given point $x \in \mathbb{R}^d$ with respect to homogeneously distributed Poisson points Π_τ. A related ‘point stabilization’ requires that ξ stabilize around x, but now with respect to $\Pi_{\lambda\kappa}$ uniformly in $\lambda \in [1,\infty)$.

Stabilization with respect to κ

ξ is *stabilizing with respect to* κ *and* K if for all $\lambda \in [1,\infty)$ and all $x \in K$, there exists an almost surely finite random variable $R := R(x,\lambda)$ (*a radius of stabilization* for ξ_λ at x) such that for all finite $\mathcal{A} \subset (\mathbb{R}^d \setminus B_{\lambda^{-1/d}R}(x))$, we have

$$\xi_\lambda\left(x, [\Pi_{\lambda\kappa} \cap B_{\lambda^{-1/d}R}(x)] \cup \mathcal{A}\right) = \xi_\lambda\left(x, \Pi_{\lambda\kappa} \cap B_{\lambda^{-1/d}R}(x)\right). \tag{1.29}$$

If the tail probability $\tau(t)$ defined for $t > 0$ by $\tau(t) := \sup_{\lambda\geq 1,\ x\in K} P(R(x,\lambda) > t)$ satisfies $\limsup_{t\to\infty} t^{-1}\log\tau(t) < 0$ then we say that ξ is *exponentially stabilizing with respect to* κ *and* K.

Roughly speaking, $R := R(x,\lambda)$ is a radius of stabilization if for all $\lambda \in [1,\infty)$, the value of $\xi_\lambda(x,\Pi_{\lambda\kappa})$ is unaffected by changes to the points outside $B_{\lambda^{-1/d}R}(x)$. In most examples of interest, methods showing that functionals homogeneously stabilize are easily modified to show stabilization with respect to densities κ.

Returning to our examples 1-3, it may be shown that the interaction function ξ from examples 1 and 2 stabilizes exponentially fast when κ is bounded away from zero on its support whereas the interaction ξ from example 3 is not known to stabilize exponentially fast.

We may weaken homogeneous stabilization by requiring that the point sets $\mathcal{A}$ in (1.27) belong to the homogeneous Poisson point process Π_τ. This weaker version of stabilization, called *localization*, is used in [13] and [59] to establish variance asymptotics and central limit theorems for functionals of convex hulls of random samples in the unit ball. Given $r > 0$, let $\xi^r(x, \mathcal{X}) := \xi(x, \mathcal{X} \cap B_r(x))$.

Localization

Say that $\hat{R} := \hat{R}(x, \Pi_\tau)$ is a *radius of localization* for ξ at x with respect to Π_τ if $\xi(x, \Pi_\tau) = \xi^{\hat{R}}(x, \Pi_\tau)$ and for all $s > \hat{R}$ we have $\xi^s(x, \Pi_\tau) = \xi^{\hat{R}}(x, \Pi_\tau)$.

Benefits of Stabilization

Recall that $\Pi_{\lambda\kappa}$ is the Poisson point process on $\mathbb{R}^d$ with intensity measure $\lambda\kappa(x)dx$. It is easy to show that $\lambda^{1/d}(\Pi_{\lambda\kappa} - x_0)$ converges to $\Pi_{\kappa(x_0)}$ as $\lambda \to \infty$, where convergence is in the sense of weak convergence of point processes. If $\xi(\cdot,\cdot)$ is a functional defined on $\mathbb{R}^d \times \mathsf{N}$, where we recall that N is the space of locally finite point sets in $\mathbb{R}^d$, one might hope that ξ is *continuous* on the pairs $(\mathbf{0}, \lambda^{1/d}(\Pi_{\lambda\kappa} - x_0))$ in the sense that $\xi(\mathbf{0}, \lambda^{1/d}(\Pi_{\lambda\kappa} - x_0))$ converges in distribution to $\xi(\mathbf{0}, \Pi_{\kappa(x_0)})$ as $\lambda \to \infty$. This turns out to be the case whenever ξ is homogeneously stabilizing as in (1.27). This is the content of the next lemma; for a complete proof see [37]. Recall that almost every $x \in \mathbb{R}^d$ is a *Lebesgue point* of κ, that is to say for almost all $x \in \mathbb{R}^d$ we have that $\varepsilon^{-d}\int_{B_\varepsilon(x)} |\kappa(y) - \kappa(x)|\,dy$ tends to zero as ε tends to zero.

Lemma 1. *Let x_0 be a Lebesgue point for κ. If ξ is homogeneously stabilizing as in (1.27), then as $\lambda \to \infty$*

$$\xi_\lambda(x_0, \Pi_{\lambda\kappa}) \xrightarrow{d} \xi(\mathbf{0}, \Pi_{\kappa(x_0)}). \tag{1.30}$$

Proof. (Sketch of the proof) By translation invariance of ξ, we have $\xi_\lambda(x_0, \Pi_{\lambda\kappa}) = \xi(\mathbf{0}, \lambda^{1/d}(\Pi_{\lambda\kappa} - x_0))$. By the stabilization of ξ, it may be shown that $(\mathbf{0}, \Pi_{\kappa(x_0)})$ is a continuity point for ξ with respect to the product topology on $\mathbb{R}^d \times \mathsf{N}$, where the space of locally finite point sets N in $\mathbb{R}^d$ is equipped with metric $d(\mathcal{X}_1, \mathcal{X}_2) := (\max\{k \in \mathbb{N} : \mathcal{X}_1 \cap B_k(\mathbf{0}) = \mathcal{X}_2 \cap B_k(\mathbf{0})\})^{-1}$ [37]. The result follows by the weak convergence $\lambda^{1/d}(\Pi_{\lambda\kappa} - x_0) \xrightarrow{d} \Pi_{\kappa(x_0)}$ and the continuous mapping theorem (Theorem 5.5. of [10]). $\square$

Recall that $\eta_1, ..., \eta_n$ are i.i.d. with density κ and put $\mathcal{X}_n := \{\eta_i\}_{i=1}^n$. Limit theorems for the sums $\sum_{x \in \Pi_{\lambda\kappa}} \xi_\lambda(x, \Pi_{\lambda\kappa})$ as well as for the associated random point measures

$$\mu_\lambda := \mu_\lambda^\xi := \sum_{x \in \Pi_{\lambda\kappa}} \xi_\lambda(x, \Pi_{\lambda\kappa})\delta_x \quad \text{and} \quad \rho_n := \rho_n^\xi := \sum_{i=1}^n \xi_n(\eta_i, \mathcal{X}_n)\delta_{\eta_i} \tag{1.31}$$

naturally require moment conditions on the summands, thus motivating the next definition.

Definition 4. ξ has a moment of order $p > 0$ (with respect to κ and K) if

$$\sup_{\lambda \geq 1,\ x \in K, \mathcal{A} \in K} \mathbf{E}[|\xi_\lambda(x, \Pi_{\lambda\kappa} \cup \mathcal{A})|^p] < \infty, \tag{1.32}$$

where A ranges over all finite subsets of K.

Let $\mathbb{B}(K)$ denote the class of all bounded $f : K \to \mathbb{R}$ and for all measures μ on $\mathbb{R}^d$ let $\langle f, \mu \rangle := \int f d\mu$. Put $\bar{\mu} := \mu - \mathbf{E}\mu$. For all $f \in \mathbb{B}(K)$ we have by Campbell's theorem that

$$\mathbf{E}[\langle f, \mu_\lambda \rangle] = \lambda \int_K f(x) \mathbf{E}[\xi_\lambda(x, \Pi_{\lambda\kappa})] \kappa(x)\, dx. \tag{1.33}$$

If (1.32) holds for some $p > 1$, then uniform integrability and Lemma 1 show that for all Lebesgue points x of κ one has $\mathbf{E}[\xi_\lambda(x, \Pi_{\lambda\kappa})] \to \mathbf{E}[\xi(\mathbf{0}, \Pi_{\kappa(x)})]$ as $\lambda \to \infty$. The set of points failing to be Lebesgue points has measure zero and by the bounded convergence theorem it follows that

$$\lim_{\lambda \to \infty} \lambda^{-1} \mathbf{E}[\langle f, \mu_\lambda \rangle] = \int_K f(x) \mathbf{E}[\xi(\mathbf{0}, \Pi_{\kappa(x)})] \kappa(x)\, dx.$$

This simple convergence of means $\mathbf{E}[\langle f, \mu_\lambda \rangle]$ is now upgraded to one providing convergence in L^q, $q = 1$ or 2.

Theorem 4. *(WLLN [37, 44]) Put $q = 1$ or 2. Let ξ be a homogeneously stabilizing (1.27) translation invariant functional satisfying the moment condition (1.32) for some $p > q$. Then for all $f \in \mathbb{B}(K)$ we have*

$$\lim_{n \to \infty} n^{-1} \langle f, \rho_n \rangle = \lim_{\lambda \to \infty} \lambda^{-1} \langle f, \mu_\lambda \rangle = \int_K f(x) \mathbf{E}[\xi(\mathbf{0}, \Pi_{\kappa(x)})] \kappa(x)\, dx \ \text{ in } \ L^q. \tag{1.34}$$

If ξ is homogeneous of order p as defined at (1.15), then for all $\alpha \in (0, \infty)$ and $\tau \in (0, \infty)$ we have $\Pi_{\alpha\tau} \stackrel{d}{=} \alpha^{-1/d} \Pi_\tau$; see e.g. the mapping theorem on p. 18 of [29]. Consequently, if ξ is homogeneous of order p, it follows that $\mathbf{E}[\xi(\mathbf{0}, \Pi_{\kappa(x)})] = \kappa(x)^{-p/d} \mathbf{E}[\xi(\mathbf{0}, \Pi_1)]$, whence the following weak law of large numbers.

Corollary 1. *Put $q = 1$ or 2. Let ξ be a homogeneously stabilizing (1.27) translation invariant functional satisfying the moment condition (1.32) for some $p > q$. If ξ is homogeneous of order p as at (1.15), then for all $f \in \mathbb{B}(K)$ we have*

$$\lim_{n \to \infty} n^{-1} \langle f, \rho_n \rangle = \lim_{\lambda \to \infty} \lambda^{-1} \langle f, \mu_\lambda \rangle = \mathbf{E}[\xi(\mathbf{0}, \Pi_1)] \int_K f(x) \kappa^{(d-p)/d}(x)\, dx \ \text{ in } \ L^q. \tag{1.35}$$

Remarks.

1. The closed form limit (1.35) explicitly links the macroscopic limit behaviour of the point measures ρ_n and μ_λ with (i) the local interaction of ξ at a point at the origin inserted into the point process Π_1 and (ii) the underlying point density κ.

2. Going back to the minimal spanning tree treated at (1.4), we see that the limiting constant $\alpha(MST_B, d)$ can be found by putting ξ in (1.35) to be ξ_{MST}, letting $f \equiv 1$ in (1.35), and consequently deducing that $\alpha(MST_B, d) = \mathbf{E}[\xi_{MST}(\mathbf{0}, \Pi_1)]$, where $\xi_{MST}(x, \mathcal{X})$ is one half the sum of the lengths of the edges in the minimal spanning tree graph on $\{x\} \cup \mathcal{X}$ incident to x.
3. Donsker-Varadhan-style large deviation principles for stabilizing functionals are proved in [60] whereas moderate deviations for bounded stabilizing functionals are proved in [5].

Asymptotic distribution results for $\langle f, \mu_\lambda \rangle$ and $\langle f, \rho_n \rangle$, $f \in \mathbb{B}(K)$, as λ and n tend to infinity respectively, require additional notation. For all $\tau > 0$, put

$$V^\xi(\tau) := \mathbf{E}[\xi(\mathbf{0}, \Pi_\tau)^2] + \tau \int_{\mathbb{R}^d} \{\mathbf{E}[\xi(\mathbf{0}, \Pi_\tau \cup \{z\}) \xi(z, \Pi_\tau \cup \mathbf{0})] - (\mathbf{E}[\xi(\mathbf{0}, \Pi_\tau)])^2\} \, dz \quad (1.36)$$

and

$$\Delta^\xi(\tau) := \mathbf{E}[\xi(\mathbf{0}, \Pi_\tau)] + \tau \int_{\mathbb{R}^d} \{\mathbf{E}[\xi(\mathbf{0}, \Pi_\tau \cup \{z\}) - \mathbf{E}[\xi(\mathbf{0}, \Pi_\tau)]\} \, dz. \quad (1.37)$$

The scalars $V^\xi(\tau)$ should be interpreted as mean pair correlation functions for the functional ξ on homogenous Poisson points Π_τ. On the other hand, since the translation invariance of ξ gives $\mathbf{E}\left[\sum_{x \in \Pi_\tau \cup \{z\}} \xi(x, \Pi_\tau \cup \{z\}) - \sum_{x \in \Pi_\tau} \xi(x, \Pi_\tau)\right] = \Delta^\xi(\tau)$, we may view $\Delta^\xi(\tau)$ as an expected 'add-one cost'.

By extending Lemma 1 to an analogous result giving the weak convergence of the joint distribution of $\xi_\lambda(x, \Pi_{\lambda\kappa})$ and $\xi_\lambda(x + \lambda^{-1/d} z, \Pi_{\lambda\kappa})$ for all pairs of points x and z in $\mathbb{R}^d$, we may show for exponentially stabilizing ξ and for bounded K that $\lambda^{-1}\mathbf{var}[\langle f, \mu_\lambda \rangle]$ converges as $\lambda \to \infty$ to a weighted average of the mean pair correlation functions.

Furthermore, recalling that $\overline{\mu}_\lambda := \mu_\lambda - \mathbf{E}[\mu_\lambda]$, and by using either Stein's method [39, 45] or the cumulant method [6], we may establish variance asymptotics and asymptotic normality of $\langle f, \lambda^{-1/2} \overline{\mu}_\lambda \rangle$, $f \in \mathbb{B}(K)$, as shown by:

Theorem 5. *(Variance asymptotics and CLT for Poisson input) Assume that κ is Lebesgue-almost everywhere continuous. Let ξ be a homogeneously stabilizing (1.27) translation invariant functional satisfying the moment condition (1.32) for some $p > 2$. Suppose further that K is bounded and that ξ is* exponentially stabilizing *with respect to κ and K as in (1.29). Then for all $f \in \mathbb{B}(K)$ we have*

$$\lim_{\lambda \to \infty} \lambda^{-1} \mathbf{var}[\langle f, \mu_\lambda \rangle] = \sigma^2(f) := \int_K f^2(x) V^\xi(\kappa(x)) \kappa(x) \, dx \quad (1.38)$$

as well as convergence of the finite-dimensional distributions

$$(\langle f_1, \lambda^{-1/2} \overline{\mu}_\lambda \rangle, \ldots, \langle f_k, \lambda^{-1/2} \overline{\mu}_\lambda \rangle),$$

$f_1, \ldots, f_k \in \mathbb{B}(K)$, *to a Gaussian field with covariance kernel*

$$(f,g) \mapsto \int_K f(x)g(x)V^{\xi}(\kappa(x))\kappa(x)\,dx. \tag{1.39}$$

Remarks

1. Theorem 5 is proved in [6, 39, 45]. In [39], it is shown the moment condition (1.32) can be weakened to one requiring only that $\mathcal{A}$ range over subsets of K having at most one element.
2. Extensions of Theorem 5. For an extension of Theorem 5 to manifolds, see [46]; for extensions to functionals of Gibbs point processes, see [60]. Theorem 5 also easily extends to treat functionals of marked point sets [6, 39], provided the marks are i.i.d.
3. Rates of convergence. Suppose $\|\kappa\|_\infty < \infty$. Suppose that ξ is exponentially stabilizing and satisfies the moments condition (1.32) for some $p > 3$. If $\sigma^2(f) > 0$ for $f \in \mathbb{B}(K)$, then there exists a finite constant c depending on d, ξ, κ, p and f, such that for all $\lambda \geq 2$,

$$\sup_{t\in\mathbb{R}} \left| P\left[\frac{\langle f,\mu_\lambda\rangle - \mathbf{E}[\langle f,\mu_\lambda\rangle]}{\sqrt{\mathbf{var}[\langle f,\mu_\lambda\rangle]}} \leq t \right] - P(N(0,1) \leq t) \right| \leq c(\log\lambda)^{3d}\lambda^{-1/2}. \tag{1.40}$$

   For details, see Corollary 2.1 in [45]. For rates of convergence in the multivariate central limit theorem, see [40].
4. Translation invariance. For ease of exposition, Theorems and 4 and 5 assume translation invariance of ξ. This assumption may be removed (see [6, 39, 37]), provided that we put $\xi_\lambda(x,\mathcal{X}) := \xi(x, x+\lambda^{1/d}(-x+\mathcal{X}))$ and provided that we replace $V^{\xi}(\tau)$ and $\Delta^{\xi}(\tau)$ defined at (1.36) and (1.37) respectively, by

$$\begin{aligned} V^{\xi}(x,\tau) &:= \mathbf{E}[\xi(x,\Pi_\tau)^2] \\ &\quad + \tau\int_{\mathbb{R}^d} \{\mathbf{E}[\xi(x,\Pi_\tau\cup\{z\})\xi(x,-z+(\Pi_\tau\cup\mathbf{0}))] - (\mathbf{E}[\xi(x,\Pi_\tau)])^2\}\,dz \end{aligned} \tag{1.41}$$

   and

$$\Delta^{\xi}(x,\tau) := \mathbf{E}[\xi(x,\Pi_\tau)] + \tau\int_{\mathbb{R}^d} \{\mathbf{E}[\xi(x,\Pi_\tau\cup\{z\}) - \mathbf{E}[\xi(x,\Pi_\tau)]\}\,dz. \tag{1.42}$$

We now consider the proof of Theorem 5. The proof of (1.38) depends in part on the following generalization of Lemma 1, a proof of which appears in [39]. Let $\tilde{\Pi}_\tau$ represent an independent copy of Π_τ.

Lemma 2. *Let x_0 and x_1 be distinct Lebesgue points for κ. If ξ is homogeneously stabilizing as in (1.27), then as $\lambda \to \infty$*

$$(\xi_\lambda(x_0,\Pi_{\lambda\kappa}), \xi_\lambda(x_1,\Pi_{\lambda\kappa})) \xrightarrow{d} (\xi(\mathbf{0},\Pi_{\kappa(x_0)}), \xi(\mathbf{0},\tilde{\Pi}_{\kappa(x_1)})). \tag{1.43}$$

Given Lemma 2 we sketch a proof of the variance convergence (1.38)). For simplicity we assume that f is a.e. continuous. By Campbell's theorem we have

$$
\begin{aligned}
&\lambda^{-1}\mathbf{var}[\langle f,\mu_\lambda\rangle] \\
&\qquad = \lambda \int_K \int_K f(x)f(y)\{\mathbf{E}[\xi_\lambda(x,\Pi_{\lambda\kappa}\cup\{y\})\xi_\lambda(y,\Pi_{\lambda\kappa}\cup\{x\})] \\
&\qquad - \mathbf{E}[\xi_\lambda(x,\Pi_{\lambda\kappa})]\mathbf{E}[\xi_\lambda(y,\Pi_{\lambda\kappa})]\}\kappa(x)\kappa(y)\,dx\,dy \\
&\qquad + \int_K f^2(x)\mathbf{E}[\xi_\lambda^2(x,\Pi_{\lambda\kappa})]\kappa(x)\,dx. \qquad (1.44)
\end{aligned}
$$

Putting $y = x + \lambda^{-1/d}z$ in the right-hand side in (1.44) reduces the double integral to

$$
= \int_K \int_{-\lambda^{1/d}x+\lambda^{1/d}K} f(x)f(x+\lambda^{-1/d}z)\{...\}\kappa(x)\kappa(x+\lambda^{-1/d}z)dz\,dx \qquad (1.45)
$$

where

$$
\begin{aligned}
\{...\} := \{&\mathbf{E}[\xi_\lambda(x,\Pi_{\lambda\kappa}\cup\{x+\lambda^{-1/d}z\})\xi_\lambda(x+\lambda^{-1/d}z,\Pi_{\lambda\kappa}\cup\{x\})] \\
&-\mathbf{E}[\xi_\lambda(x,\Pi_{\lambda\kappa})]E[\xi_\lambda(x+\lambda^{-1/d}z,\Pi_{\lambda\kappa})]\}
\end{aligned}
$$

is the two point correlation function for ξ_λ.

The moment condition and Lemma 2 imply that for all Lebesgue points $x \in K$ that the two point correlation function for ξ_λ converges to the two point correlation function for ξ. Moreover, by exponential stabilization, the integrand in (1.45) is dominated by an integrable function of z over $\mathbb{R}^d$ (see Lemma 4.2 of [39]). The double integral in (1.44) thus converges to

$$
\begin{aligned}
\int_K \int_{\mathbb{R}^d} f^2(x)\cdot \mathbf{E}[\xi(\Pi_{\kappa(x)}\cup\{z\})\xi(-z+(\Pi_{\kappa(x)}\cup\mathbf{0}))] \\
-(\mathbf{E}[\xi(\Pi_{\kappa(x)})])^2\kappa^2(x)\,dz\,dx \qquad (1.46)
\end{aligned}
$$

by dominated convergence, the continuity of f, and the assumed moment bounds.

By Theorem 4, the assumed moment bounds, and dominated convergence, the single integral in (1.44) converges to

$$
\int_K f^2(x)\mathbf{E}[\xi^2(\mathbf{0},\Pi_{\kappa(x)})]\kappa(x)\,dx. \qquad (1.47)
$$

Combining (1.46) and (1.47) and using the definition of V^ξ, we obtain the variance asymptotics (1.38) for continuous test functions f. To show convergence for general $f \in \mathbb{B}(K)$ we refer to [39].

Now we sketch a proof of the central limit theorem part of Theorem 5. There are three distinct approaches to proving the central limit theorem:

1. Stein's method, in particular consequences of Stein's method for dependency graphs of random variables, as given by [17]. This approach, spelled out in [45], gives the rates of convergence to the normal in (1.40).

2. Methods based on martingale differences are applicable when κ is the uniform density and when the functional H^{ξ} satisfies a stabilization criteria involving the insertion of single point into the sample; see [41] and [30] for details.
3. The method of cumulants may be used [6] to show that the k-th order cumulants c_{λ}^{k} of $\lambda^{-1/2}\langle f, \overline{\mu}_{\lambda}\rangle,\ k \geq 3$, vanish in the limit as $\lambda \to \infty$. We make use of the standard fact that if the cumulants c^k of a random variable ζ vanish for all $k \geq 3$, then ζ has a normal distribution. This method assumes additionally that ξ has moments of all orders, i.e. (1.32) holds for all $p \geq 1$.

Here we describe the third method, which, when suitably modified yields moderate deviation principles [5] as well as limit theory for functionals over Gibbs point processes [60].

To show vanishing of cumulants of order three and higher, we follow the proof of Theorem 2.4 in section five of [6] and take the opportunity to correct a mistake in the exposition, which also carried over to [5], and which was first noticed by Mathew Penrose. We assume the test functions f belong to the class $C(K)$ of continuous functions on K.

Method of cumulants

We will use the method of cumulants to show for all continuous test functions f on K, that

$$\langle f, \lambda^{-1/2}\overline{\mu}_{\lambda}\rangle \xrightarrow{d} N(0, \sigma^2(f)), \tag{1.48}$$

where $\sigma^2(f)$ is at (1.38). The convergence of the finite-dimensional distributions (1.39) follows by standard methods involving the Cramér-Wold device.

We first recall the formal definition of cumulants. Put $K := [0,1]^d$ for simplicity. Write

$$\begin{aligned}
&\mathbf{E}\exp\left(\lambda^{-1/2}\langle -f, \overline{\mu}_{\lambda}\rangle\right) \\
&\quad = \exp\left(\lambda^{-1/2}\langle f, \mathbf{E}\mu_{\lambda}\rangle\right)\mathbf{E}\exp\left(\lambda^{-1/2}\langle -f, \mu_{\lambda}\rangle\right) \\
&\quad = \exp\left(\lambda^{-1/2}\langle f, \mathbf{E}\mu_{\lambda}\rangle\right)\left[1 + \sum_{k=1}^{\infty}\frac{\lambda^{-k/2}}{k!}\langle(-f)^k, M_{\lambda}^k\rangle\right],
\end{aligned} \tag{1.49}$$

where $f^k : \mathbb{R}^{dk} \to \mathbb{R},\ k = 1,2,...$ is given by $f^k(v_1,\dots,v_k) = f(v_1)\cdots f(v_k)$, and $v_i \in K,\ 1 \leq i \leq k$. $M_{\lambda}^k := M_{\lambda\kappa}^k$ is a measure on $\mathbb{R}^{dk}$, the k-th moment measure (p. 130 of [21]), and has the property that

$$\langle f^k, M_{\lambda}^k\rangle = \int_{K^k}\mathbf{E}\left[\prod_{i=1}^{k}\xi(x_i, \Pi_{\lambda\kappa})\right]\prod_{i=1}^{k} f(x_i)\kappa(x_i)\,d(\lambda x_i).$$

In general M_λ^k is not continuous with respect to Lebesgue measure on K^k, but rather it is continuous with respect to sums of Lebesgue measures on the diagonal subspaces of K^k, where two or more coordinates coincide.

In Section 5 of [6], the moment and cumulant measures considered there are with respect to the centered functional $\overline{\xi}$, whereas they should be with respect to the non-centered functional ξ. This requires corrections to the notation, which we provide here, but, since higher order cumulants for centered and non-centered measures coincide, it does not change the arguments of [6], which we include for completeness and which go as follows.

We have

$$dM_\lambda^k(v_1,...,v_k) = m_\lambda(v_1,...,v_k)\prod_{i=1}^k \kappa(v_i)\, d(\lambda^{1/d}v_i),$$

where the Radon-Nikodym derivative $m_\lambda(v_1,...,v_k)$ of M_λ^k with respect to $\prod_{i=1}^k \mu_\lambda$ is given by mixed moment

$$m_\lambda(v_1,...,v_k) := \mathbf{E}\left[\prod_{i=1}^k \xi_\lambda(v_i;\Pi_{\lambda\kappa}\cup\{v_j\}_{j=1}^k)\right]. \tag{1.50}$$

Due to the behaviour of M_λ^k on the diagonal subspaces we make the standing assumption that if the differential $d(\lambda_1^{1/d}v_1)\cdots d(\lambda_1^{1/d}v_k)$ involves repetition of certain coordinates, then it collapses into the corresponding lower order differential in which each coordinate occurs only once. For each $k\in\mathbb{N}$, by the assumed moment bounds (1.32), the mixed moment on the right hand side of (1.50) is bounded uniformly in λ by a constant $c(\xi,k)$. Likewise, the k-th summand in (1.49) is finite.

For all $i=1,2,...$ we let K_i denote the i-th copy of K. For any subset T of the positive integers, we let

$$K^T := \prod_{i\in T} K_i.$$

If $|T| = l$, then for all $\lambda \ge 1$, by M_λ^T we mean a copy of the l-th moment measure on the l-fold product space K_λ^T. M_λ^T is equal to M_λ^l as defined above.

When the series (1.49) is convergent, the logarithm of the Laplace functional gives

$$\log\left[1+\sum_{k=1}^\infty \frac{1}{k!}\lambda^{-k/2}\langle(-f)^k, M_\lambda^k\rangle\right] = \sum_{l=1}^\infty \frac{1}{l!}\lambda^{-l/2}\langle(-f)^l, c_\lambda^l\rangle; \tag{1.51}$$

the signed measures c_λ^l are cumulant measures. Regardless of the validity of (1.49), the existence of all cumulants c_λ^l, $l=1,2,...$ follows from the existence of all moments in view of the representation

$$c_\lambda^l = \sum_{T_1,...,T_p} (-1)^{p-1}(p-1)!M_\lambda^{T_1}\cdots M_\lambda^{T_p},$$

where $T_1,...,T_p$ ranges over all unordered partitions of the set $1,...,l$ (see p. 30 of [33]). The first cumulant measure coincides with the expectation measure and the second cumulant measure coincides with the variance measure.

We follow the proof of Theorem 2.4 of [6], with these small changes: (i) replace the centered functional $\overline{\xi}$ with the non-centered ξ (ii) correspondingly, let all cumulants $c_\lambda^l,\ l=1,2,...$ be the cumulant measures for the *non-centered* moment measures $M_\lambda^k,\ k=1,2,....$ Since c_λ^1 coincides with the expectation measure, Theorem 4 gives for all $f \in C(K)$

$$\lim_{\lambda\to\infty} \lambda^{-1}\langle f, c_\lambda^1\rangle = \lim_{\lambda\to\infty} \lambda^{-1}\mathbf{E}[\langle f, \mu_\lambda^\xi\rangle] = \int_K f(x)\mathbf{E}[\xi(\mathbf{0},\Pi_{\kappa(x)})]\kappa(x)dx.$$

We already know from the variance convergence that

$$\lim_{\lambda\to\infty} \lambda^{-1}\langle f^2, c_\lambda^2\rangle = \lim_{\lambda\to\infty} \lambda^{-1}\mathbf{var}[\langle f, \mu_{\lambda\kappa}^\xi\rangle] = \int_K f^2(x)V^\xi(\kappa(x))\kappa(x)dx.$$

Thus, to prove (1.48), it will be enough to show for all $k \geq 3$ and all $f \in C(K)$ that $\lambda^{-k/2}\langle f^k, c_\lambda^k\rangle \to 0$ as $\lambda \to \infty$. This will be done in Lemma 4 below, but first we recall some terminology from [6].

A cluster measure $U_\lambda^{S,T}$ on $K^S \times K^T$ for non-empty $S,T \subset \{1,2,...\}$ is defined by

$$U_\lambda^{S,T}(B\times D) = M_\lambda^{S\cup T}(B\times D) - M_\lambda^S(B)M_\lambda^T(D)$$

for all Borel B and D in K^S and K^T, respectively.

Let S_1,S_2 be a partition of S and let T_1,T_2 be a partition of T. A product of a cluster measure $U_\lambda^{S_1,T_1}$ on $K^{S_1}\times K^{T_1}$ with products of moment measures $M^{|S_2|}$ and $M^{|T_2|}$ on $K^{S_2}\times K^{T_2}$ will be called a (S,T) semi-cluster measure.

For each non-trivial partition (S,T) of $\{1,...,k\}$ the k-th cumulant c^k is represented as

$$c^k = \sum_{(S_1,T_1),(S_2,T_2)} \alpha((S_1,T_1),(S_2,T_2))U^{S_1,T_1}M^{|S_2|}M^{|T_2|}, \tag{1.52}$$

where the sum ranges over partitions of $\{1,...,k\}$ consisting of pairings (S_1,T_1), (S_2,T_2), where $S_1,S_2 \subset S$ and $T_1,T_2 \subset T$, and where $\alpha((S_1,T_1),(S_2,T_2))$ are integer valued pre-factors. In other words, for any non-trivial partition (S,T) of $\{1,...,k\}$, c^k is a linear combination of (S,T) semi-cluster measures; see Lemma 5.1 of [6].

The following bound is critical for showing that $\lambda^{-k/2}\langle f, c_\lambda^k\rangle \to 0$ for $k \geq 3$ as $\lambda \to \infty$.

Lemma 3. *If ξ is exponentially stabilizing as in (1.29), then the functions m_λ cluster exponentially, that is there are positive constants $a_{j,l}$ and $c_{j,l}$ such that uniformly*

$$|m_\lambda(x_1,...x_j,y_1,...,y_l) - m_\lambda(x_1,...,x_j)m_\lambda(y_1,...,y_l)| \leq a_{j,l}\exp(-c_{j,l}\delta\lambda^{1/d}),$$

where $\delta := \min_{1\leq i\leq j, 1\leq p\leq l}|x_i - y_p|$ is the separation between the sets $\{x_i\}_{i=1}^j$ and $\{y_p\}_{p=1}^l$ of points in K.

The constants $a_{j,l}$, while independent of λ, may grow quickly in j and l, but this will not affect the decay of the cumulant measures in the scale parameter λ. The next lemma provides the desired decay of the cumulant measures; we provide a proof which is slightly different from that given for Lemma 5.3 of [6].

Lemma 4. *For all $f \in C(K)$ and $k = 2,3,...$ we have $\lambda^{-1}\langle f^k, c_\lambda^k\rangle = O\left(||f||_\infty^k\right)$.*

Proof. We need to estimate

$$\int_{K^k} f(v_1)...f(v_k)\,dc_\lambda^k(v_1,...,v_k).$$

We will modify the arguments in [6], borrowing from [57]. Given $v := (v_1,...,v_k) \in K^k$, let $D_k(v) := D_k(v_1,...,v_k) := \max_{i\le k}(||v_1 - v_i|| + ... + ||v_k - v_i||)$ be the l^1 diameter for v. Let $\Xi(k)$ be the collection of all partitions of $\{1,...,k\}$ into exactly two subsets S and T. For all such partitions consider the subset $\sigma(S,T)$ of $K^S \times K^T$ having the property that $v \in \sigma(S,T)$ implies $d(x(v), y(v)) \ge D_k(v)/k^2$, where $x(v)$ and $y(v)$ are the projections of v onto K^S and K^T, respectively, and where $d(x(v), y(v))$ is the minimal Euclidean distance between pairs of points from $x(v)$ and $y(v)$. It is easy to see that for every $v := (v_1,...,v_k) \in K^k$, there is a splitting of v, say $x := x(v)$ and $y := y(v)$, such that $d(x,y) \ge D_k(v)/k^2$; if this were not the case then a simple argument shows that, given $v := (v_1,...,v_k)$ the distance between any pair of constituent components must be strictly less than $D_k(v)/k$, contradicting the definition of D_k. It follows that K^k is the union of the sets $\sigma(S,T),\ (S,T) \in \Xi(k)$. The key to the proof of Lemma 4 is to evaluate the cumulant c_λ^k over each $\sigma(S,T) \in \Xi(k)$, that is to write $\langle f, c_\lambda^k\rangle$ as a finite sum of integrals

$$\langle f, c_\lambda^k\rangle = \sum_{\sigma(S,T)\in\Xi(k)} \int_{\sigma(S,T)} f(v_1)\cdots f(v_k)\,dc_\lambda^k(v_1,...,v_k),$$

then appeal to the representation (1.52) to write the cumulant measure $dc_\lambda^k(v_1,...,v_k)$ on each $\sigma(S,T)$ as a linear combination of (S,T) semi-cluster measures, and finally to appeal to Lemma 3 to control the constituent cluster measures U^{S_1,T_1} by an exponentially decaying function of $\lambda^{1/d}D_k(v) := \lambda^{1/d}D_k(v_1,...,v_k)$.

Given $\sigma(S,T)$, $S_1 \subset S$ and $T_1 \subset T$, this goes as follows. Let $x \in K^S$ and $y \in K^T$ denote elements of K^S and K^T, respectively; likewise we let $\tilde{x}$ and $\tilde{y}$ denote elements of K^{S_1} and K^{T_1}, respectively. Let $\tilde{x}^c$ denote the complement of $\tilde{x}$ with respect to x and likewise with $\tilde{y}^c$. The integral of f against one of the (S,T) semi-cluster measures in (1.52), induced by the partitions (S_1,S_2) and (T_1,T_2) of S and T respectively, has the form

$$\int_{\sigma(S,T)} f(v_1)\cdots f(v_k)\,d\left(M_\lambda^{|S_2|}(\tilde{x}^c)U_\lambda^{i+j}(\tilde{x},\tilde{y})M_\lambda^{|T_2|}(\tilde{y}^c)\right).$$

Letting $u_\lambda(\tilde{x},\tilde{y}) := m_\lambda(\tilde{x},\tilde{y}) - m_\lambda(\tilde{x})m_\lambda(\tilde{y})$, the above equals

$$\int_{\sigma(S,T)} f(v_1)\cdots f(v_k)m_\lambda(\tilde{x}^c)u_\lambda(\tilde{x},\tilde{y})m_\lambda(\tilde{y}^c)\prod_{i=1}^k \kappa(v_i)\,d(\lambda^{1/d}v_i). \tag{1.53}$$

We use Lemma 3 to control $u_\lambda(\tilde{x},\tilde{y}) := m_\lambda(\tilde{x},\tilde{y}) - m_\lambda(\tilde{x})m_\lambda(\tilde{y})$, we bound f and κ by their respective sup norms, we bound each mixed moment by $c(\xi,k)$, and we use $\sigma(S,T) \subset K^k$ to show that

$$\int_{\sigma(S,T)} f(v_1)\cdots f(v_k)\, d\left(M_\lambda^{|S_2|}(\tilde{x}^c) U_\lambda^{i+j}(\tilde{x},\tilde{y}) M_\lambda^{|T_2|}(\tilde{y}^c)\right)$$

$$\leq D(k)c(\xi,k)^2 ||f||_\infty^k ||\kappa||_\infty^k \int_{K^k} \exp(-c\lambda^{1/d} D_k(v)/k^2)\, d(\lambda^{1/d} v_1)\cdots d(\lambda^{1/d} v_k).$$

Letting $z_i := \lambda^{1/d} v_i$ the above bound becomes

$$\lambda D(k)c(\xi,k)^2 ||f||_\infty^k ||\kappa||_\infty^k \int_{(\lambda^{1/d}K)^k} \exp(-cD_k(z)/k^2)\, dz_1\cdots dz_k$$

$$\leq \lambda D(k)c(\xi,k)^2 ||f||_\infty^k ||\kappa||_\infty^k \int_{(\mathbb{R}^d)^{k-1}} \exp(-cD_k(0,z_1,...,z_{k-1})/k^2)\, dz_1\cdots dz_k$$

where we use the translation invariance of $D_k(\cdot)$. Upon a further change of variable $w := z/k$ we have

$$\int_{\sigma(S,T)} f(v_1)\cdots f(v_k)\, d\left(M_\lambda^{|S_2|}(\tilde{x}^c) U_\lambda^{i+j}(\tilde{x},\tilde{y}) M_\lambda^{|T_2|}(\tilde{y}^c)\right)$$

$$\leq \lambda \tilde{D}(k)c(\xi,k)^2 ||f||_\infty^k ||\kappa||_\infty^k \int_{(\mathbb{R}^d)^{k-1}} \exp(-cD_k(0,w_1,...,w_{k-1}))\, dw_1\cdots dw_{k-1}.$$

Finally, since $D_k(0,w_1,...,w_{k-1}) \geq ||w_1|| + ... + ||w_{k-1}||$ we obtain

$$\int_{\sigma(S,T)} f(v_1)\cdots f(v_k) d\left(M_\lambda^{|S_2|}(\tilde{x}^c) U_\lambda^{i+j}(\tilde{x},\tilde{y}) M_\lambda^{|T_2|}(\tilde{y}^c)\right)$$

$$\leq \lambda \tilde{D}(k)c(\xi,k)^2 ||f||_\infty^k ||\kappa||_\infty^k \left(\int_{\mathbb{R}^d} \exp(-||w||)\, dw\right)^{k-1} = O(\lambda)$$

as desired. □

Central limit theorem for functionals over binomial input

To obtain central limit theorems for functionals over binomial input $\mathcal{X}_n$ we need some more definitions. For all functionals ξ and $\tau \in (0,\infty)$, recall the 'add one cost' defined at (1.37). For all $j = 1,2,...$, let $\mathcal{S}_j$ be the collection of all subsets of $\mathbb{R}^d$ of cardinality at most j.

Definition 5. Say that ξ has a moment of order $p > 0$ (with respect to binomial input $\mathcal{X}_n$) if

$$\sup_{n\geq 1, x\in\mathbb{R}^d, \mathcal{D}\in\mathcal{S}_3}\ \sup_{(n/2)\leq m\leq(3n/2)} \mathbf{E}[|\xi_n(x,\mathcal{X}_m\cup\mathcal{D})|^p] < \infty. \tag{1.54}$$

Definition 6. ξ is *binomially exponentially stabilizing* for κ if for all $x \in \mathbb{R}^d, \lambda \geq 1$, and $\mathcal{D} \subset \mathcal{S}_2$ there exists an almost surely finite random variable $R := R_{\lambda,n}(x,\mathcal{D})$ such that for all finite $\mathcal{A} \subset (\mathbb{R}^d \setminus B_{\lambda^{-1/d}R}(x))$, we have

$$\xi_\lambda \left(x, ([\mathcal{X}_n \cup \mathcal{D}] \cap B_{\lambda^{-1/d}R}(x)) \cup \mathcal{A}\right) = \xi_\lambda \left(x, [\mathcal{X}_n \cup \mathcal{D}] \cap B_{\lambda^{-1/d}R}(x)\right), \tag{1.55}$$

and moreover there is an $\varepsilon > 0$ such that the tail probability $\tau_\varepsilon(t)$ defined for $t > 0$ by

$$\tau_\varepsilon(t) := \sup_{\lambda \geq 1, n \in \mathbb{N} \cap ((1-\varepsilon)\lambda, (1+\varepsilon)\lambda)} \ \sup_{x \in \mathbb{R}^d,\ \mathcal{D} \subset \mathcal{S}_2} \mathbf{P}(R_{\lambda,n}(x,\mathcal{D}) > t)$$

satisfies $\limsup_{t \to \infty} t^{-1} \log \tau_\varepsilon(t) < 0$.

If ξ is homogeneously stabilizing then in most examples of interest, similar methods can be used to show that ξ is binomially exponentially stabilizing whenever κ is bounded away from zero.

Theorem 6. *(CLT for binomial input) Assume that κ is Lebesgue-almost everywhere continuous. Let ξ be a homogeneously stabilizing (1.27) translation invariant functional satisfying the moment conditions (1.32) and (1.54) for some $p > 2$. Suppose further that K is bounded and that ξ is* exponentially stabilizing *with respect to κ and K as in (1.29) and binomially exponentially stabilizing with respect to κ and K as in (1.55). Then for all $f \in \mathbb{B}(K)$ we have*

$$\lim_{n \to \infty} n^{-1} \mathbf{var}[\langle f, \rho_n \rangle] = \tau^2(f) := \int_K f^2(x) V^\xi(\kappa(x))\kappa(x)\, dx - \left(\int_K \Delta^\xi(\kappa(x))\kappa(x) dx \right)^2 \tag{1.56}$$

as well as convergence of the finite-dimensional distributions

$$(\langle f_1, n^{-1/2}\overline{\rho}_n \rangle, \ldots, \langle f_k, n^{-1/2}\overline{\rho}_n \rangle),$$

$f_1, \ldots, f_k \in \mathbb{B}(K)$, *to a Gaussian field with covariance kernel*

$$(f,g) \mapsto \int_K f(x)g(x)V^\xi(\kappa(x))\kappa(x)\, dx - \int_K f(x)\Delta^\xi(\kappa(x))\kappa(x)\, dx \int_K g(x)\Delta^\xi(\kappa(x))\kappa(x)\, dx. \tag{1.57}$$

Proof. We sketch the proof, borrowing heavily from coupling arguments appearing in [6, 41, 39]. Fix $f \in \mathbb{B}(K)$. Put $H_n := \langle f, \rho_n \rangle$, $H_n' := \langle f, \mu_n \rangle$, where μ_n is defined at (1.31) and assume that $\Pi_{n\kappa}$ is coupled to $\mathcal{X}_n$ by setting $\Pi_{n\kappa} = \bigcup_{i=1}^{\eta(n)} X_i$, where $\eta(n)$ is an independent Poisson random variable with mean n. Put

$$\alpha := \alpha(f) := \int_K f(x)\Delta^\xi(\kappa(x))\kappa(x)\, dx.$$

Conditioning on the random variable $\eta := \eta(n)$ and using that η is concentrated around its mean, it can be shown that as $n \to \infty$ we have

$$\mathbf{E}[(n^{-1/2}(H_n' - H_n - (\eta(n) - n)\alpha))^2] \to 0. \tag{1.58}$$

The arguments are long and technical (cf. Section 5 of [39], Section 4 of [41]).

Let $\sigma^2(f)$ be as at (1.38) and let $\tau^2(f)$ be as at (1.56), so that $\tau^2(f) = \sigma^2(f) - \alpha^2$.

By Theorem 5 we have as $n \to \infty$ that $\mathbf{var}[H_n'] \to \sigma^2(f)$ and $n^{-1/2}(H_n' - \mathbf{E}H_n') \xrightarrow{d} N(0, \sigma^2(f))$. We now deduce Theorem 6, following verbatim by now standard arguments (see e.g. p. 1020 of [41], p. 251 of [6]), included here for sake of completeness.

To prove convergence of $n^{-1}\mathbf{var}[H_n]$, we use the identity

$$n^{-1/2}H_n' = n^{-1/2}H_n + n^{-1/2}(\eta(n) - n)\alpha + n^{-1/2}[H_n' - H_n - (\eta(n) - n)\alpha]. \tag{1.59}$$

The variance of the third term on the right-hand side of (1.59) goes to zero by (1.58), whereas the second term has variance α^2 and is independent of the first term. It follows that with $\sigma^2(f)$ defined at (1.38), we have

$$\sigma^2(f) = \lim_{n\to\infty} n^{-1}\mathbf{var}[H_n'] = \lim_{n\to\infty} n^{-1}\mathbf{var}[H_n] + \alpha^2,$$

so that $\sigma^2(f) \geq \alpha^2$ and $n^{-1}\mathbf{var}[H_n] \to \tau^2(f)$. This gives (1.56).

Now to prove Theorem 6 we argue as follows. By Theorem 5, we have $n^{-1/2}(H_n' - \mathbf{E}H_n') \xrightarrow{d} N(0, \sigma^2)$. Together with (1.58), this yields

$$n^{-1/2}[H_n - \mathbf{E}H_n' + (\eta(n) - n)\alpha] \xrightarrow{d} N(0, \sigma^2(f)).$$

However, since $n^{-1/2}(\eta(n) - n)\alpha$ is independent of H_n and is asymptotically normal with mean zero and variance α^2, it follows by considering characteristic functions that

$$n^{-1/2}(H_n - \mathbf{E}H_n') \xrightarrow{d} N(0, \sigma^2(f) - \alpha^2). \tag{1.60}$$

By (1.58), the expectation of $n^{-1/2}(H_n' - H_n - (\eta_n - n)\alpha)$ tends to zero, so in (1.60) we can replace $\mathbf{E}H_n'$ by $\mathbf{E}H_n$, which gives us

$$n^{-1/2}(H_n - \mathbf{E}H_n) \xrightarrow{d} N(0, \tau^2(f)).$$

To obtain convergence of finite-dimensional distributions (1.57) we use the Cramér-Wold device.

□

1.4 Applications

Consider a linear statistic $H^\xi(\mathcal{X})$ of a large geometric structure on $\mathcal{X}$. If we are interested in the limit behavior of H^ξ on random point sets, then the results of the previous section suggest checking whether the interaction function ξ is stabilizing.

Verifying the stabilization of ξ is sometimes non-trivial and may involve discretization methods. Here we describe four non-trivial statistics H^{ξ} for which one may show stabilization/localization of ξ. Our list is non-exhaustive and primarily focusses on the problems described in Section 1.1.

Random packing

[55] Given $d \in \mathbb{N}$ and $\lambda \geq 1$, let $\eta_{1,\lambda}, \eta_{2,\lambda}, \ldots$ be a sequence of independent random d-vectors uniformly distributed on the cube $Q_\lambda := [0, \lambda^{1/d})^d$. Let S be a fixed bounded closed convex set in $\mathbb{R}^d$ with non-empty interior (i.e., a 'solid') with centroid at the origin $\mathbf{0}$ of $\mathbb{R}^d$ (for example, the unit ball), and for $i \in \mathbb{N}$, let $S_{i,\lambda}$ be the translate of S with centroid at $\eta_{i,\lambda}$. So $\mathcal{S}_\lambda := (S_{i,\lambda})_{i \geq 1}$ is an infinite sequence of solids arriving at uniform random positions in Q_λ (the centroids lie in Q_λ but the solids themselves need not lie wholly inside Q_λ).

Let the first solid $S_{1,\lambda}$ be packed (i.e., accepted), and recursively for $i = 2, 3, \ldots$, let the i-th solid $S_{i,\lambda}$ be packed if it does not overlap any solid in $\{S_{1,\lambda}, \ldots, S_{i-1,\lambda}\}$ which has already been packed. If not packed, the i-th solid is discarded. This process, known as *random sequential adsorption (RSA) with infinite input*, is irreversible and terminates when it is not possible to accept additional solids. At termination, we say that the sequence of solids $\mathcal{S}_\lambda$ *jams* Q_λ or *saturates* Q_λ. The number of solids accepted in Q_λ at termination is denoted by the *jamming* number $N_\lambda := N_{\lambda,d} := N_{\lambda,d}(\mathcal{S}_\lambda)$.

There is a large literature of experimental results concerning the jamming numbers, but a limited collection of rigorous mathematical results, especially in $d \geq 2$. The short range interactions of arriving particles lead to complicated long range spatial dependence between the status of particles. Dvoretzky and Robbins [23] show in $d = 1$ that the jamming numbers $N_{\lambda,1}$ are asymptotically normal.

By writing the jamming number as a linear statistic involving a stabilizing interaction ξ, one may establish [55] that $N_{\lambda,d}$ are asymptotically normal for all $d \geq 1$. This puts the experimental results and Monte Carlo simulations of Quintanilla and Torquato [47] and Torquato (ch. 11.4 of [67])) on rigorous footing.

Theorem 7. *Let $\mathcal{S}_\lambda$ and $N_\lambda := N_\lambda(\mathcal{S}_\lambda)$ be as above. There are constants $\mu := \mu(S,d) \in (0,\infty)$ and $\sigma^2 := \sigma^2(S,d) \in (0,\infty)$ such that as $\lambda \to \infty$ we have*

$$\left|\lambda^{-1}\mathbf{E}N_\lambda - \mu\right| = O(\lambda^{-1/d}) \tag{1.61}$$

and $\lambda^{-1}\mathbf{var}[N_\lambda] \to \sigma^2$ with

$$\sup_{t\in\mathbb{R}}\left|\left[\frac{N_\lambda - \mathbf{E}N_\lambda}{\sqrt{\mathbf{var}[N_\lambda]}} \leq t\right] - \mathbf{P}(N(0,1) \leq t)\right| = O((\log\lambda)^{3d}\lambda^{-1/2}). \tag{1.62}$$

To prove this, one could enumerate the arriving solids in $\mathcal{S}_\lambda$, by (x_i, t_i), where $x_i \in \mathbb{R}^d$ is the spatial coordinate of the i-th solid and $t_i \in [0, \infty)$ is its temporal coordinate, i.e. the arrival time. Furthermore, letting $\mathcal{X} := \{(x_i, t_i)\}_{i=1}^\infty$ be a marked point process, one could set $\xi((x,t), \mathcal{X})$ to be one or zero depending on whether the solid with center at $x \in \mathcal{S}_\lambda$ is accepted or not; $H^\xi(\mathcal{X})$ is the total number of solids accepted. Thus ξ is defined on elements of the marked point process $\mathcal{X}$. A natural way to prove Theorem 7 would then be to show that ξ satisfies the conditions of Theorem 5. The moment conditions (1.32) are clearly satisfied as ξ is bounded by 1. To show stabilization it turns out that it is easier to *discretize* as follows.

For any $A \subset \mathbb{R}^d$, let $A_+ := A \times \mathbb{R}_+$. Let $\xi(\mathcal{X}, A)$ be the number of solids with centers in $\mathcal{X} \cap A$ which are packed according to the packing rules. Abusing notation, let Π denote a homogeneous Poisson point process in $\mathbb{R}^d \times \mathbb{R}_+$ with intensity $dx \times ds$, with dx denoting Lebesgue measure on $\mathbb{R}^d$ and ds denoting Lebesgue measure on $\mathbb{R}_+$. Abusing the terminology at (1.27), ξ is *homogeneously stabilizing* since it may be shown that there exists an almost surely finite random variable R (a *radius of homogeneous stabilization* for ξ) such that for all $\mathcal{X} \subset (\mathbb{R}^d \setminus B_R)_+$ we have

$$\xi((\Pi \cap (B_R)_+) \cup \mathcal{X}, Q_1) = \xi(\Pi \cap (B_R)_+, Q_1). \tag{1.63}$$

Since ξ is homogeneously stabilizing it follows that the limit

$$\xi(\Pi, i + Q_1) := \lim_{r \to \infty} \xi\left(\Pi \cap (B_R(i))_+, i + Q_1\right)$$

exists almost surely for all $i \in \mathbb{Z}^d$. The random variables $(\xi(\Pi, i + Q_1),\ i \in \mathbb{Z}^d)$ form a stationary random field. It may be shown that the tail probability for R decays exponentially fast.

Given ξ, for all $\lambda > 0$, all $\mathcal{X} \subset \mathbb{R}^d \times \mathbb{R}_+$, and all Borel $A \subset \mathbb{R}^d$ we let $\xi_\lambda(\mathcal{X}, A) := \xi(\lambda^{1/d}\mathcal{X}, \lambda^{1/d}A)$. Let Π_λ, $\lambda \geq 1$, denote a homogeneous Poisson point process in $\mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\lambda\, dx \times ds$. Define the random measure μ_λ^ξ on $\mathbb{R}^d$ by

$$\mu_\lambda^\xi(\,\cdot\,) := \xi_\lambda(\Pi_\lambda \cap Q_1, \cdot) \tag{1.64}$$

and the centered version $\overline{\mu}_\lambda^\xi := \mu_\lambda^\xi - \mathbf{E}[\mu_\lambda^\xi]$. Modification of the stabilization methods of Section 1.3 then yield Theorem 7; this is spelled out in [55].

For companion results for RSA packing with *finite input per unit volume* we refer to [42].

Convex hulls

Let B_d denote the d-dimensional unit ball. Letting Π_λ be a Poisson point process in $\mathbb{R}^d$ of intensity λ we let K_λ be the convex hull of $B_d \cap \Pi_\lambda$. The random polytope K_λ, together with the analogous polytope K_n obtained by considering n i.i.d. uniformly distributed points in B_d, are well-studied objects in stochastic geometry, with a long history originating with the work of Rényi and Sulanke [53]. See the surveys of Affentranger [1], Buchta [12], Gruber [24], Schneider [61, 62], and Weil and Wieacker [69]), together with Chapter 8.2 in Schneider and Weil [63].

Functionals of K_λ of interest include its volume, here denoted $V(K_\lambda)$ and the number of k-dimensional faces of K_λ, here denoted $f_k(K_\lambda),\ k \in \{0,1,...,d-1\}$. Note that $f_0(K_\lambda)$ is the number of vertices of K_λ. The k-th intrinsic volumes of K_λ are here denoted by $V_k(K_\lambda),\ k \in \{1,...,d-1\}$.

Define the functional $\xi(x,\mathcal{X})$ to be one or zero, depending on whether $x \in \mathcal{X}$ is a vertex in the convex hull of $\mathcal{X}$. By reformulating functionals of convex hulls in terms of functionals of re-scaled parabolic growth processes in space and time, it may be shown that ξ is exponentially localizing [13]. The arguments are non-trivial and we refer to [13] for details. Taking into account the proper scaling in space-time, a modification of Theorem 5 yields variance asymptotics for $V(K_\lambda)$, namely

$$\lim_{\lambda\to\infty} \lambda^{(d+3)/(d+1)}\mathbf{var}[V(K_\lambda)] = \sigma_V^2, \tag{1.65}$$

where $\sigma_V^2 \in (0,\infty)$ is a constant. This adds to Reitzner's central limit theorem (Theorem 1 of [51]), his variance approximation $\mathbf{var}[V(K_\lambda)] \approx \lambda^{-(d+3)/(d+1)}$ (Theorem 3 and Lemma 1 of [51]), and Hsing [26], which is confined to $d = 2$. The stabilization methods of Theorem 5 yield a central limit theorem for $V(K_\lambda)$.

Let $k \in \{0,1,...,d-1\}$. Consider the functional $\xi_k(x,\mathcal{X})$, defined to be zero if x is not a vertex in the convex hull of $\mathcal{X}$ and otherwise defined to be the product of $(k+1)^{-1}$ and the number of k-dimensional faces containing x. Consideration of the parabolic growth processes and the stabilization of ξ_k in the context of such processes (cf. [13]) yield variance asymptotics and a central limit theorem for the number of k-dimensional faces of K_λ, yielding for all $k \in \{0,1,...,d-1\}$

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d+1)}\mathbf{var}[f_k(K_\lambda)] = \sigma_{f_k}^2, \tag{1.66}$$

where $\sigma_{f_k}^2 \in (0,\infty)$ is given as a closed form expression described in terms of paraboloid growth processes. For the case $k = 0$, this is proved in [59], whereas [13] handles the cases $k > 0$. This adds to Reitzner (Lemma 2 of [51]), whose breakthrough paper showed $\mathbf{var}[f_k(K_\lambda)] \approx \lambda^{(d-1)/(d+1)}$.

Theorem 5 also yields variance asymptotics for the intrinsic volumes $V_k(K_\lambda)$ of K_λ for all $k \in \{1,...,d-1\}$, namely

$$\lim_{\lambda\to\infty} \lambda^{(d+3)/(d+1)}\mathbf{var}[V_k(K_\lambda)] = \sigma_{V_k}^2, \tag{1.67}$$

where again $\sigma_{V_k}^2$ is explicitly described in terms of paraboloid growth processes. This adds to Bárányi et al. (Theorem 1 of [4]), which shows $\mathbf{var}[V_k(K_n)] \approx n^{-(d+3)/(d+1)}$.

Intrinsic dimension of high dimensional data sets

Given a finite set of samples taken from a multivariate distribution in $\mathbb{R}^d$, a fundamental problem in learning theory involves determining the intrinsic dimension of the sample [22, 29, 54, 68]. Multidimensional data ostensibly belonging to a high-dimensional space $\mathbb{R}^d$ often are concentrated on a smooth submanifold $\mathcal{M}$ or hypersurface with intrinsic dimension m, where $m < d$. The problem of determining the intrinsic dimension of a data set is of fundamental interest in machine learning, signal processing, and statistics and it can also be handled via analysis of the sums (1.1).

Discerning the intrinsic dimension m allows one to reduce dimension with minimal loss of information and to consequently avoid difficulties associated with the 'curse of dimensionality'. When the data structure is linear there are several methods available for dimensionality reduction, including principal component analysis and multidimensional scaling, but for non-linear data structures, mathematically rigorous dimensionality reduction is more difficult. One approach to dimension estimation, inspired by Bickel and Levina [32] uses probabilistic methods involving the k-nearest neighbour graph $G^N(k,\mathcal{X})$ defined in the paragraph containing (1.6).

For all $k = 3,4,...$, the Levina and Bickel estimator of the dimension of a data cloud $\mathcal{X} \subset \mathcal{M}$, is given by

$$\hat{m}_k(\mathcal{X}) := (\mathrm{card}\,(\mathcal{X}))^{-1} \sum_{y\in\mathcal{X}} \xi_k(y,\mathcal{X}),$$

where for all $y \in \mathcal{X}$ we have

$$\xi_k(y,\mathcal{X}) := (k-2)\left(\sum_{j=1}^{k-1} \log \frac{D_k(y)}{D_j(y)}\right)^{-1},$$

where $D_j(y) := D_j(y,\mathcal{X}),\ 1 \le j \le k$, are the distances between y and its j-th nearest neighbour in $\mathcal{X}$.

Let $\{\eta_i\}_{i=1}^n$ be i.i.d. random variables with values in a submanifold $\mathcal{M}$; let $\mathcal{X}_n := \{\eta_i\}_{i=1}^n$. Levina and Bickel [32] argue that $\hat{m}_k(\mathcal{X}_n)$ estimates the intrinsic dimension of $\mathcal{X}_n$, i.e., the dimension of $\mathcal{M}$.

Subject to regularity conditions on $\mathcal{M}$ and the density κ, the papers [46] and [71] substantiate this claim and show (i) consistency of the dimension estimator $\hat{m}_k(\mathcal{X}_n)$ and (ii) a central limit theorem for $\hat{m}_k(\mathcal{X}_n)$ together with a rate of convergence. This goes as follows.

For all $\tau > 0$, recall that Π_τ is a homogeneous Poisson point process on $\mathbb{R}^m$. Recalling the notation of Section 1.3 we put

$$V^{\xi_k}(\tau,m) := \mathbf{E}[\xi_k(\mathbf{0},\Pi_\tau)^2] +$$

$$+\ \tau \int_{\mathbb{R}^m} \left[\mathbf{E}[\xi_k(\mathbf{0},\Pi_\tau \cup \{u\})\xi_k(u,\Pi_\tau \cup \mathbf{0})] - (\mathbf{E}[\xi_k(\mathbf{0},\Pi_\tau)])^2\right] du \qquad (1.68)$$

and

$$\delta^{\xi_k}(\tau, m) := \mathbf{E}[\xi_k(\mathbf{0}, \Pi_\tau)] + \tau \int_{\mathbb{R}^m} \mathbf{E}[\xi_k(\mathbf{0}, \Pi_\tau \cup \{u\}) - \xi_k(\mathbf{0}, \Pi_\tau)]\, du. \tag{1.69}$$

We put $V^{\xi_k}(m) := V^{\xi_k}(1,m)$ and $\delta^{\xi_k}(m) := \delta^{\xi_k}(1,m)$. Let $\Pi_{\lambda\kappa}$ be the collection $\{X_1, ..., X_{N(\lambda)}\}$, where X_i are i.i.d. with density κ and $N(\lambda)$ is an independent Poisson random variable with parameter λ. By extending Theorems 4 and 5 to manifolds, it may be shown [46] that for manifolds $\mathcal{M}$ which are regular, we have the following

Theorem 8. *Let κ be bounded away from zero and infinity on $\mathcal{M}$. We have for all $k \geq 4$*

$$\lim_{\lambda\to\infty} \hat{m}_k(\Pi_{\lambda\kappa}) = \lim_{n\to\infty} \hat{m}_k(\mathcal{X}_n) = m = dim(\mathcal{M}), \tag{1.70}$$

where the convergence holds in L^2. If κ is a.e. continuous and $k \geq 5$, then

$$\lim_{n\to\infty} n^{-1}\mathbf{var}[\hat{m}_k(\mathcal{X}_n)] = \sigma_k^2(m) := V^{\xi_k}(m) - (\delta^{\xi_k}(m))^2 \tag{1.71}$$

and there is a constant $c := c(\mathcal{M}) \in (0,\infty)$ such that for all $k \geq 6$ and all $\lambda \geq 2$ we have

$$\sup_{t\in\mathbb{R}} \left| P\left[\frac{\hat{m}_k(\Pi_{\lambda\kappa}) - \mathbf{E}\hat{m}_k(\Pi_{\lambda\kappa})}{\sqrt{\mathbf{var}[\hat{m}_k(\Pi_{\lambda\kappa})]}} \leq t \right] - \Phi(t) \right| \leq c(\log\lambda)^{3m}\lambda^{-1/2}. \tag{1.72}$$

Finally, for $k \geq 7$ we have as $n \to \infty$,

$$n^{-1/2}(\hat{m}_k(\mathcal{X}_n) - \mathbf{E}\hat{m}_k(\mathcal{X}_n)) \xrightarrow{d} N(0, \sigma_k^2(m)). \tag{1.73}$$

Remark. Theorem 8 adds to Chatterjee [16], who does not provide variance asymptotics (1.71) and who considers convergence rates with respect to the weaker Kantorovich-Wasserstein distance. Bickel and Yan (Theorems 1 and 3 of Section 4 of [9]) establish a central limit theorem for $\hat{m}_k(\mathcal{X}_n)$ for linear $\mathcal{M}$.

Clique counts, Vietoris-Rips complex

A central problem in data analysis involves discerning and counting clusters. Geometric graphs and the Vietoris-Rips complex play a central role and both are amenable to asympototic analysis via stabilization techniques. The Vietoris-Rips complex is studied in connection with the statistical analysis of high-dimensional data sets [15], manifold reconstruction [20], and it has also received attention amongst topologists in connection with clustering and connectivity questions of data sets [14].

If $\mathcal{X} \subset \mathbb{R}^d$ is finite and $\delta > 0$, then the Vietoris-Rips complex $\mathcal{R}^\delta(\mathcal{X})$ is the abstract simplicial complex whose k-simplices (cliques of order $k+1$) correspond to unordered $(k+1)$ tuples of points of $\mathcal{X}$ which are pairwise within Euclidean distance δ of each other. Thus, if there is a subset S of $\mathcal{X}$ of size $k+1$ with all points of S distant at most δ from each other, then S is a k-simplex in the complex.

Given $\mathcal{R}^\delta(\mathcal{X})$ and $k \in \mathbb{N}$, let $N_k^\delta(\mathcal{X})$ be the cardinality of k-simplices in $\mathcal{R}^\delta(\mathcal{X})$. Let $\xi_k(\eta, \mathcal{X})$ be the cardinality of k-simplices containing y in $\mathcal{R}^\delta(\mathcal{X})$. Since the value of ξ_k depends only on points distant at most δ, it follows that δ is a radius of stabilization for ξ_k and that ξ_k is trivially exponentially stabilizing (1.29) and binomially exponentially stabilizing (1.55).

The next scaling result, which holds for suitably regular manifolds $\mathcal{M}$, links the large scale behaviour of the clique count with the density κ of the underlying point set. Let η_i be i.i.d. with density κ on the manifold $\mathcal{M}$. Put $\mathcal{X}_n := \{\eta_i\}_{i=1}^n$. Letting Π_1 be a homogeneous Poisson point process on $\mathbb{R}^m$, dy the volume measure on $\mathcal{M}$, and recalling (1.68) and (1.69), it may be shown [46] that a generalization of Theorems 4 and 5 to manifolds yields:

Theorem 9. *Let κ be bounded on $\mathcal{M}$; $\dim \mathcal{M} = m$. For all $k \in \mathbb{N}$ and all $\delta > 0$ we have*

$$\lim_{n\to\infty} n^{-1} N_k^\delta(n^{1/m}\mathcal{X}_n) = \mathbf{E}[\xi_k(\mathbf{0}, \mathcal{R}^\delta(\Pi_1))] \int_{\mathcal{M}} \kappa^{k+1}(y)\, dy \quad \text{in } L^2. \tag{1.74}$$

If κ is a.e. continuous then

$$\begin{aligned} &\lim_{n\to\infty} n^{-1} \mathbf{var}[N_k^\delta(n^{1/m}\mathcal{X}_n)] \\ &\quad = \sigma_k^2(m) := V^{\xi_k}(m) \int_{\mathcal{M}} \kappa^{2k+1}(y)\, dy - \left(\delta^{\xi_k}(m) \int_{\mathcal{M}} \kappa^{k+1}(y)\, dy\right)^2 \end{aligned} \tag{1.75}$$

and, as $n \to \infty$

$$n^{-1/2}(N_k^\delta(n^{1/m}\mathcal{X}_n) - \mathbf{E}[N_k^\delta(n^{1/m}\mathcal{X}_n)]) \xrightarrow{d} N(0, \sigma_k^2(m)). \tag{1.76}$$

This result extends Proposition 3.1, Theorem 3.13, and Theorem 3.17 of [37]. For more details, we refer to [46].

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