Large Deviations for Functionals of Spatial Point Processes with Applications to Random Packing and Spatial Graphs

T. Schreiber[1] and J. E. Yukich[2]

April 20, 2005

Abstract

Functionals of spatial point process often satisfy a weak spatial dependence condition known as stabilization. We prove general Donsker-Varadhan large deviation principles (LDP) for such functionals and show that the general result can be applied to prove LDPs for various particular functionals, including those concerned with random packing, nearest neighbor graphs, and lattice versions of the Voronoi and sphere of influence graphs.

1 Main results

This paper studies the large deviations of functionals of spatial point processes indexed by multidimensional cubes. When functionals of spatial point processes are approximately additive over their index sets and satisfy a weak regularity condition, then Donsker-Varadhan large deviation principles (LDP) follow [24]. However many functionals of spatial point processes are not known to be approximately additive, but instead satisfy a weak spatial dependence property termed stabilization [3, 17, 18, 19].

American Mathematical Society 2000 subject classifications. Primary 60F05, Secondary 60D05

Key words and phrases. Large deviation principles, stabilizing functionals, random Euclidean graphs, random sequential packing

[1] Research partially supported by the Foundation for Polish Science (FNP) and by the Polish Minister of Scientific Research and Information Technology (MNiI) grant 1 P03A 018 28 (2005-2007)

[2] Research supported in part by NSA grant MDA904-03-1-0070 and NSF grant DMS-0203720

Stabilization, which quantifies the local dependence structure, is a useful unifying concept which yields general laws of large numbers and central limit theorems for functionals in geometric probability. Stabilization helps describe the large scale limit behavior of random functionals and random measures in terms of the underlying density of points [3, 4, 15, 16, 18, 19].

The purpose of this paper is to show that stabilization also yields general Donsker-Varadhan large deviation principles for functionals of spatial point processes (Theorem 1.1) as well as for the measures induced by such functionals (Theorem 1.2). The general LDPs are then applied to deduce large deviation principles for functionals of spatial point processes which do not have an obvious subadditive or additive structure. This includes functionals concerned with random sequential packing, nearest neighbor graphs, and lattice versions of the Voronoi and sphere of influence graphs. In this way we obtain the LDP counterparts for some functionals known to satisfy laws of large numbers and central limit theorems [3, 16, 17, 18, 19].

A key simplifying idea involves the approximation of stabilizing functionals by a finite range correction, namely by a functional whose value at a point depends only on points within a finite deterministic distance. By considering stabilizing functionals admitting a finite range correction approximation, we effectively study a process which is nearly additive over its index set, allowing us to draw on the general LDP results of [24].

1.1 Stabilizing functionals

To fix our ideas, throughout we will consider a translation-invariant non-negative function $\xi(x, \mathcal{X})$ defined for all pairs $(x, \mathcal{X})$, where $\mathcal{X}$ is a locally finite subset of $\mathbb{R}^d$ containing x. We will write $\xi(x, \mathcal{X}) := \xi(x, \mathcal{X} \cup \{x\})$ if $x \notin \mathcal{X}$. Define

$$H^{\xi}(\mathcal{X}) := \sum_{x \in \mathcal{X}} \xi(x, \mathcal{X})$$

and for Borel $D \subseteq \mathbb{R}^d$ put

$$H^{\xi}(\mathcal{X}; D) := \sum_{x \in \mathcal{X} \cap D} \xi(x, \mathcal{X}).$$

Fix $\tau > 0$ and let $\mathcal{P}_\tau$ denote an intensity τ homogeneous Poisson point process in $\mathbb{R}^d$. We will assume that ξ is stabilizing at intensity τ, that is for each $x \in \mathbb{R}^d$ there exists an a.s. finite random variable $R(x) := R^{\xi}(x, \mathcal{P}_\tau)$ (a radius of stabilization) such that

$$\xi(x, (\mathcal{P}_\tau \cap B_{R(x)}(x)) \cup \mathcal{A}) = \xi(x, \mathcal{P}_\tau \cap B_{R(x)}(x))$$

for all locally finite $\mathcal{A} \subseteq \mathbb{R}^d \setminus B_{R(x)}(x)$. Here and elsewhere for all $r > 0$ and $x \in \mathbb{R}^d$, $B_r(x)$ denotes the ball of radius r centered at x. More generally, for a finite point configuration $\mathcal{X} \subseteq \mathbb{R}^d$ we consider the stabilization radius $R(x) := R^\xi(x, \mathcal{X})$ of ξ at x defined so that $\xi(x, (\mathcal{X} \cap B_{R(x)}(x)) \cup \mathcal{A})$ takes the same value for all locally finite $\mathcal{A} \subseteq \mathbb{R}^d \setminus B_{R(x)}(x)$. To shorten the notation, we put

$$H_\lambda^\xi := H_{\lambda;\tau}^\xi := H^\xi(\mathcal{P}_\tau \cap Q_\lambda; Q_\lambda),$$

where $Q_\lambda := [0, \lambda]^d, \ \lambda > 0$. Letting δ_x denote the Dirac point mass at x, for $\lambda > 0$ we define the (re-scaled) weighted measure on $[0, 1]^d$

$$\mu_\lambda^\xi := \mu_{\lambda;\tau}^\xi := \sum_{x \in \mathcal{P}_\tau \cap Q_\lambda} \xi(x, \mathcal{P}_\tau \cap Q_\lambda) \delta_{x/\lambda},$$

so that in particular $H_\lambda^\xi = \mu_\lambda^\xi([0, 1]^d)$. We keep the intensity τ fixed and make no explicit reference to the dependence on τ as long as it does not lead to confusion.

We will write $\mathcal{M}^+([0, 1]^d)$ for the space of non-negative Borel measures on $[0, 1]^d$, endowed with the usual weak topology and we fix on $\mathcal{M}^+([0, 1]^d)$ a metric ϱ compatible with this topology. Throughout all random variables are defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

1.2 Finite range corrections

By their definition, stabilizing functionals involve asymptotic decoupling of the behavior exhibited by the process in distant regions. A natural idea, to be exploited below in the context of large deviations, is to approximate the original stabilizing functional ξ by its *finite range correction* constructed so as to stabilize within deterministic finite distances. It turns out that quite natural regularity conditions on the process H_λ^ξ, required for the large deviation principle to hold, can be then formulated in terms of the quality of such finite range approximations.

To put these ideas in formal terms, for $r > 0$ we say that a non-negative functional $\xi^{[r]}$ is an r-stabilizing *finite range correction* of ξ if the following conditions are satisfied for all locally finite subsets $\mathcal{X}$ in $\mathbb{R}^d$:

(C1) $\xi^{[r]}(x, \mathcal{X}) = \xi(x, \mathcal{X})$ whenever $R^\xi(x, \mathcal{X}) < r$, with $R^\xi(x, \mathcal{X})$ standing for the stabilization radius of ξ at x for the configuration $\mathcal{X}$, and

(C2) $\xi^{[r]}$ is stabilizing at intensity τ with a stabilization radius bounded a.s. by r.

In applications, we will often put

$$\xi^{[r]}(x, \mathcal{X}) := \begin{cases} \xi(x, \mathcal{X}), & \text{if } R^\xi(x, \mathcal{X}) < r, \\ 0, & \text{otherwise,} \end{cases} \tag{1.1}$$

but we note that alternative definitions fulfilling **(C1)** and **(C2)** can be considered on equal rights provided that the upcoming conditions **(L1)** and **(L2)** are satisfied. In general we only assume the existence of a deterministic procedure constructing $\xi^{[r]}$ out of ξ.

Given a finite range correction $\xi^{[r]}$ satisfying **(C1)** and **(C2)** we construct in the obvious way the empirical measure $\mu_\lambda^{\xi^{[r]}}$ as well as the total mass functional $H_\lambda^{\xi^{[r]}} := \mu_\lambda^{\xi^{[r]}}([0,1]^d)$. Further, we define also the (possibly signed) *difference measure* $\delta_\lambda^{\xi^{[r]}} := \mu_\lambda^{\xi} - \mu_\lambda^{\xi^{[r]}}$. However, defining $\xi^{[r]}$ as in (1.1), the difference measure is non-negative and its total variation $||\delta_\lambda^{\xi^{[r]}}||_{TV}$, coinciding in this case with the total mass, decreases in r for fixed λ. It should be emphasized that Theorems 1.1 and 1.2 below hold for general finite range corrections not necessarily given by (1.1).

1.3 Large deviation principles: main results

We assume that the finite range correction $\xi^{[r]}$ satisfies:

(L1) For each $r > 0$ there exists $M(r) < \infty$ such that for all x and all locally finite point configurations $\mathcal{X} \ni x$,

$$\xi^{[r]}(x, \mathcal{X}) \leq M(r).$$

(L2) For arbitrarily small $\varepsilon > 0$ and for arbitrarily large $C > 0$ there exists $r(\varepsilon, C)$ such that for each $r > r(\varepsilon, C)$ we have for λ large enough

$$\mathbb{P}\left[||\delta_\lambda^{\xi^{[r]}}||_{TV} > \varepsilon\lambda^d\right] \leq \exp(-C\lambda^d).$$

In view of the boundedness condition **(L1)**, stabilization of $\xi^{[r]}$ yields for all $r \in (0, \infty)$ [19] a constant $\gamma^{\xi^{[r]}}$ such that

$$\gamma^{\xi^{[r]}} = \lim_{\lambda\to\infty} \frac{1}{\lambda^d} \mathbb{E} H_\lambda^{\xi^{[r]}}. \tag{1.2}$$

By condition **(L2)** there is a constant γ^ξ, referred to as the spatial constant for ξ, such that

$$\gamma^{\xi} = \lim_{\lambda\to\infty} \frac{1}{\lambda^d} \mathbb{E} H_\lambda^{\xi}. \tag{1.3}$$

The following results are Donsker-Varadhan-style *large deviation principles* and constitute the main results of this paper. As in Section 1.2 in [7], we say that a family of random elements $(\mathbb{Y}_\lambda)_{\lambda>0}$ taking values in a general topological space $\mathcal{Y}$ satisfies a large deviation principle on $\mathcal{Y}$ with a good rate function I and with speed $s(\lambda)$ iff the level sets $\{I(y) \leq M\},\ M < \infty$, are compact (thus, in particular, I is lower semicontinuous) and

$$\limsup_{\lambda\to\infty} \frac{1}{s(\lambda)} \log \mathbb{P}[\mathbb{Y}_\lambda \in F] \leq -\inf_{x\in F} I(x)$$

for all closed sets $F \subseteq \mathcal{Y}$ and

$$\liminf_{\lambda\to\infty} \frac{1}{s(\lambda)} \log \mathbb{P}[\mathbb{Y}_\lambda \in O] \geq -\inf_{x\in O} I(x)$$

for all open sets $O \subseteq \mathcal{Y}$.

Theorem 1.1 *(scalar LDP) With conditions* **(L1)** *and* **(L2)***, there exists a convex good rate function $I : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ such that the family $(\lambda^{-d} H_\lambda^\xi)_\lambda$ satisfies the full large deviation principle with speed λ^d and rate function I. Moreover, the limit*

$$\Lambda(s) := \lim_{\lambda\to\infty} \frac{1}{\lambda^d} \log \mathbb{E} \exp\left(sH_\lambda^\xi\right) \tag{1.4}$$

exists and is finite for all $s \in \mathbb{R}$ and

$$I(t) = \sup_{s\in\mathbb{R}} (ts - \Lambda(s)). \tag{1.5}$$

Furthermore,

$$\lim_{t\to+\infty} I(t)/t = +\infty \tag{1.6}$$

and $I(t) > 0$ for $t \neq \gamma^\xi$.

The next result extends Theorem 1.1 and describes the asymptotic behavior of the measures $(\lambda^{-d}\mu_\lambda^\xi)_\lambda$ instead of their total masses.

Theorem 1.2 *(measure LDP) With conditions* **(L1)** *and* **(L2)***, the family of random measures $(\lambda^{-d}\mu_\lambda^\xi)_\lambda$ satisfies on $\mathcal{M}^+([0,1]^d)$ the full large deviation principle with speed λ^d and rate function*

$$J(\theta) := \begin{cases} \int_{[0,1]^d} I\left(\frac{d\theta}{dl}\right) dl, & \text{if } \theta \ll l, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.7}$$

Remarks. (i) General laws of large numbers and central limit theorems for stabilizing functionals of point processes are established in [18, 19] and [3, 16, 17, 18], respectively; unlike Theorems 1.1 and 1.2 such results do not require existence of exponential approximations of the original process by its finite range corrections as in **(L2)**. Under conditions (**L1**) and (**L2**), Theorems 1.1 and 1.2 thus add to the existing limit theory for stabilizing functionals.

(ii) Theorems 1.1 and 1.2 extend to random functionals and random measures defined by marked Poisson point processes as follows. Let $(\mathcal{M}, \mathcal{F}, \nu)$ be a probability space of marks and $\mathcal{P}_{\tau\times\nu}$ a homogeneous Poisson point process on $\mathbb{R}^d \times \mathcal{M}$ with intensity $\tau \times \nu$. Then ξ is stabilizing at intensity τ if for each $x \in \mathbb{R}^d$ there exists an a.s. finite random variable $R(x) := R^\xi(x, \mathcal{P}_{\tau\times\nu})$ such that for all finite $\mathcal{A} \subseteq (\mathbb{R}^d \setminus B_{R(x)}(x)) \times \mathcal{M}$ we have

$$\xi(x, (\mathcal{P}_{\tau\times\nu} \cap B_{R(x)}(x)) \cup \mathcal{A}) = \xi(x, (\mathcal{P}_{\tau\times\nu} \cap B_{R(x)}(x)).$$

In this setting if $\{(X_i, \tau_i)\}_i$ is a realization of $\mathcal{P}_{\tau\times\nu}$ then put

$$\mu_\lambda^\xi := \mu_{\lambda\cdot\tau}^\xi := \sum_{x\in\{X_i\}\cap Q_\lambda} \xi(x, \{(X_i, \tau_i)\}_i)\delta_{x/\lambda}$$

and $H_\lambda^\xi := \mu_\lambda^\xi([0,1]^d)$. With these small modifications, Theorems 1.1 and 1.2 hold for the marked functionals $(H_\lambda^\xi)_\lambda$ and measures $(\mu_\lambda^\xi)_\lambda$. We shall use this modification in applications to random sequential packing (sections 2.1 and 2.2).

(iii) The non-negativity assumption on ξ is not essential to the proof of Theorem 1.1. However, the non-negativity of ξ is crucial to ensure that the empirical measures belong to the space of positive measures $\mathcal{M}^+([0,1]^d)$, which is topologically better behaved than the space of signed measures.

(iv) We are unable to establish analogues of Theorems 1.1 and 1.2 under binomial sampling, that is for measures induced by i.i.d. random variables $X_1, X_2, ..., X_{\lceil\tau\lambda\rceil}$ on Q_λ instead of by the realization of the Poisson point set $\mathcal{P}_\tau \cap Q_\lambda$.

2 Applications

We provide applications of the general LDPs given by Theorems 1.1 and 1.2 to functionals of random sequential packing models and functionals of graphs in computational geometry. The following examples have been considered in detail in the context of central limit theorems [3, 17, 20], thermodynamic limits [19], and moderate deviation principles [1].

2.1 Random sequential packing

We recall a prototypical random sequential packing model used in diverse disciplines, including physical, chemical, and biological processes. See [18] for a discussion of the many applications, the many references, and also a discussion of previous results in the scientific literature. In one dimension, this model is often referred to as the Rényi car parking model. With $\mathrm{Po}(\lambda^d\tau)$ standing for a Poisson random variable with parameter $\lambda^d\tau$, let $B_1, B_2, ..., B_{\mathrm{Po}(\lambda^d\tau)}$ be a sequence of

d-dimensional unit radius balls whose centers are i.i.d. random d-vectors $X_1, ..., X_{\mathrm{Po}(\lambda^d\tau)}$, independent of $\mathrm{Po}(\lambda^d\tau)$, with a uniform probability density function on $Q_\lambda := [0,\lambda]^d,\ d \geq 1$. Without loss of generality, assume that the balls are sequenced in the order determined by i.i.d. uniformly distributed marks (time coordinates) in $[0,\tau]$. Let the first ball B_1 be *packed*, and recursively for $i = 2, 3, \ldots$, let the i-th ball B_i be packed iff B_i does not overlap any ball in $B_1, ..., B_{i-1}$ which has already been packed. If not packed, the i-th ball is discarded. This procedure is known as random sequential adsorption (RSA).

Observe that the centers of all incoming balls form a homogeneous intensity τ Poisson point process $\mathcal{P}_\tau$ on Q_λ. When re-scaled onto $[0,1]^d$, the collection of centers of accepted balls induces a point measure on $[0,1]^d$, denoted $\mu_{\lambda;\tau}$. We call this the *random sequential packing measure* induced by unit balls with centers arising from $\mathcal{P}_\tau$.

For any finite point set $\mathcal{X} \subset \mathbb{R}^d,\ d \geq 1$, assume the points $x \in \mathcal{X}$ have time coordinates which are independent and uniformly distributed over the interval $[0,\tau]$. Assume unit balls centered at the points of $\mathcal{X}$ arrive sequentially in an order determined by the time coordinates, and assume as before that each ball is packed or discarded according to whether or not it overlaps a previously packed ball. Let $\xi(x,\mathcal{X})$ be either 1 or 0 depending on whether the ball centered at x is packed or discarded. It is known that ξ is stabilizing at intensity τ and that $R^\xi(x,\mathcal{P}_\tau)$ has exponentially decaying tails for all $\tau > 0$ [18]. For $x \in [0,1]^d$ and $\mathcal{X} \subseteq [0,1]^d$ let $\xi_\lambda(x,\mathcal{X}) := \xi(\lambda x, \lambda\mathcal{X})$, where λx denotes scalar multiplication of x and *not* the mark associated with x. The total number of balls packed with centers in $\mathcal{X}$ is given by the packing functional $H^\xi(\mathcal{X}) := \sum_{x\in\mathcal{X}} \xi(x,\mathcal{X})$. The random measure

$$\mu_\lambda^\xi := \sum_{x\in\mathcal{P}_\tau\cap Q_\lambda} \xi(x,\mathcal{P}_\tau\cap Q_\lambda)\delta_{x/\lambda}$$

coincides with the packing measure $\mu_{\lambda;\tau}$.

The following provides an LDP for the random packing functionals and measures.

Corollary 2.1 *(LDP) The family of packing functionals $(\lambda^{-d}H_\lambda^\xi)_\lambda$ and the family of packing measures $(\lambda^{-d}\mu_\lambda^\xi)_\lambda$ satisfy the LDP as in Theorems 1.1 and 1.2, respectively.*

Remarks. (i) The family of functionals $(H_\lambda^\xi)_\lambda$ and measures $(\mu_\lambda^\xi)_\lambda$ satisfy weak laws of large numbers [19], central limit theorems [2, 3, 18], and moderate deviation principles [1]. In $d = 1$ the analysis is somewhat simpler and has roots in Rényi [22] and Dvoretzky and Robbins [10], with later work by Coffman et al.[6]. Corollary 2.1 shows that the Poissonized packing functionals and measures satisfy an LDP as well.

(ii) Corollary 2.1 provides an LDP for the total number of balls accepted in the packing model with finite input, i.e., where the time coordinates are uniformly bounded. Finite input is crucial to controlling the long range interactions in packing models. In the setting of infinite input in $d \geq 2$ (i.e., where the time coordinates arise as the realization of a homogeneous Poisson point process over $[0, \infty)$), we are unable to control the long range interactions and thus are unable to establish an LDP in the infinite input setting. Central limit theorems in this context are not known either (cf. Theorem 1.2 of [18]).

Proof of Corollary 2.1. We consider the finite range correction $\xi^{[r]}$ of the packing functional ξ, given by (1.1). Clearly, only condition **(L2)** requires verification, since (**L1**) holds with $M(r) = 1$. Our argument below is based on the graphical representation of the packing process built upon a particular space-time epidemic-spreading oriented percolation model coupled with the original model, as introduced in [18].

Denote the realization of the Poisson point process $\mathcal{P}_\tau$ by $(X)_{X \in \mathcal{P}_\tau}$. Assume that each X carries a mark M_X which is uniformly distributed on $[0, \tau]$. Thus the points $(X, M_X)_{X \in \mathcal{P}_\tau}$ form a unit intensity Poisson point process $\mathcal{P}$ on $\mathbb{R}^d \times [0, \tau]$. As in [18], we make $\mathcal{P}$ into the vertex set of an oriented graph by including an edge from the point (X, T) to (Y, U) whenever (X, T) and (Y, U) are points of $\mathcal{P}$ satisfying $T \leq U$ and $|X - Y| \leq 2$.

It is useful to think of the induced graph on $\mathcal{P}$ as representing the spread of an epidemic in which new points born in the unit radius neighborhood of existing infected points are themselves instantly (and permanently) infected. A collection of points (X_i, M_{X_i}) in $\mathcal{P}$ which satisfies $|X_i - X_{i+1}| < 2$ and $M_{X_i} \leq M_{X_{i+1}}$ will be called a path of infected points or a causal chain. It is crucial to observe that the packing status of a given point $(X, M_X) \in \mathcal{P}$ cannot be affected by the points of $\mathcal{P}$ which fall outside the union of all causal chains containing (X, M_X) (section four of [18]).

To proceed, partition the large cube Q_λ into translates $Q_{L/2+r_0}[1]$, $Q_{L/2+r_0}[2], \ldots$ of $Q_{L/2+r_0}$ with L and r_0 to be specified below, but always chosen such that $L > 2r_0$ and $L/2 + r_0$ divides λ. Let $Q_{L+2r_0}[1], Q_{L+2r_0}[2], \ldots$ stand for all possible translates of Q_{L+2r_0} which can be obtained as unions of $Q_{L/2+r_0}[i]$, $i = 1, 2, \ldots$ Note that each $Q_{L+2r_0}[j]$ has non-trivial intersection with at most $3^d - 1$ other such cubes ('neighbors'). Rather than labelling these cubes with a sequence of natural numbers, it is convenient to consider $Q_{L+2r_0}[\cdot]$ indexed by a subset of the integer lattice $\mathbb{Z}^d$, namely by the subset $B[n(\lambda, L, r_0)] := \{1, \ldots, n(\lambda, L, r_0) := \lambda/(L/2 + r_0)\}^d$, with the neighborhood relation $i \sim j$ given by whether $Q_{L+2r_0}[i]$ overlaps with $Q_{L+2r_0}[j]$, which is easily seen to be equivalent to

$||i-j||_\infty := \max(|i_1 - j_1|, ..., |i_d - j_d|) = 1$. Further subdivide the cube $Q_{L+2r_0}[i]$ into a centrally located translate $Q_L[i]$ of Q_L surrounded by a corridor (moat) $C_{L,r_0}[i] := Q_{L+2r_0}[i] \setminus Q_L[i]$ of width r_0. For a fixed $\varepsilon > 0$, we choose $L := L(\varepsilon)$ so that the maximum overall volume of packed balls with their centers in $\bigcup_i C_{L,r_0}[i]$ cannot exceed $\frac{\varepsilon}{2}\lambda^d$.

We declare a cube $Q_{L+2r_0}[i]$ *bad* if it contains a causal chain (path of infected points) with a diameter larger than $r_0 - 2$. Observe that in view of the exponential decay of the diameter of a single causal chain, as established in [18], the probability of a given cube $Q_{L+2r_0}[i],\ i \in B[n(\lambda, L, r_0)]$, being bad can be made arbitrarily small uniformly in the configuration over the non-overlapping cubes $Q_{L+2r_0}[j],\ j \not\sim i$, by an appropriate choice of L and r_0. Indeed, inside the cube $Q_{L+2r_0}[i]$ we observe $\mathrm{Po}(\tau \mathrm{Vol}(Q_{L+2r_0}[i])) = \mathrm{Po}(\tau(L+2r_0)^d)$ points of the process $\mathcal{P}_\tau \cap Q_{L+2r_0}[i]$. Each of these points belongs to a causal chain whose diameter exhibits exponential tail decay. In particular, assigning to each point $x \in \mathcal{P}_\tau \cap Q_{L+2r_0}[i]$ the random variable $R[x; r_0]$ equal to the largest possible diameter R of a causal chain containing x and contained in Q_{L+2r_0} provided $R > r_0 - 2$ and 0 otherwise, we see that the expectation $\mathbb{E} \sum_{x \in \mathcal{P}_\tau \cap Q_\lambda} R[x; r_0]$ is of order $O((L+2r_0)^d \exp(-cr_0))$ for some $c > 0$. However, this expectation is an upper bound for the probability that there is at least one causal chain of diameter larger than $r_0 - 2$ within $Q_{L+2r_0}[i]$. Clearly, this argument is valid regardless of the configurations over the cubes $Q_{L+2r_0}[j],\ i \not\sim j$, i.e. those which do not overlap with $Q_{L+2r_0}[i]$.

We use now Theorem 0.0 in [12] to conclude that the random process, assigning to each site $i \in B[n(\lambda, L, r_0)]$ the value 1 if $Q_{L+2r_0}[i]$ is bad and 0 otherwise, can be stochastically dominated by an i.i.d. site percolation process Π on $B[n(\lambda, L, r_0)]$, where the probability $\pi(L, r_0) := \mathbb{P}[\Pi[i] = 1]$ can be made arbitrarily small by adjusting L and r_0. In accordance with the terminology introduced above, we shall call a site $i \in B[n(\lambda, L, r_0)]$ bad whenever $\Pi[i] = 1$ and good otherwise (note that when declaring a site in $B[n(\lambda, L, r_0)]$ bad or good we refer to the dominating i.i.d. process Π rather to the original dependent process induced on $B[n(\lambda, L, r_0)]$ by the continuum graphical construction).

It is a simple yet crucial consequence of the graphical construction that $||\delta_\lambda^{\xi^{[r]}}||_{TV}$ is bounded above by the overall number $D(\lambda, r)$ of packed (accepted) balls in $\mathcal{P}_\tau \cap Q_\lambda$ falling into causal chains of diameter larger than r, i.e.,

$$||\delta_\lambda^{\xi^{[r]}}||_{TV} \leq D(\lambda, r). \tag{2.1}$$

Now in order that a causal chain covers a Euclidean distance larger than r, observe that it is necessary that there exist a path of bad cubes of length (cardinality of the number of constituent cubes Q_{L+2r_0}) larger than Kr/L for some constant $K > 0$ depending only on the dimension d. Let $W[L+2r_0]$ stand for the maximum possible number of unit balls which can be packed in Q_{L+2r_0}. In particular, the overall number $D(\lambda, r)$ of balls belonging to causal chains having a diameter larger than r is stochastically dominated by the product of $W[L+2r_0]$ and the number $\Delta(\lambda, L, r_0; r)$ of bad sites $i \in B[n(\lambda, L, r_0)]$ for Π, falling into bad clusters of size (length) larger than Kr/L, connected with respect to the neighborhood relation $\sim$, plus the number of balls packed in the moats $\bigcup_i C_{L,r_0}[i]$. Note that we have to add the balls in the moats to take into account that even in good cubes some balls, whose distance to the boundary of the cube is smaller than r_0, may belong to causal chains passing across the boundary. Clearly, this cannot happen if the distance between a ball contained in a good cube and the boundary of the cube exceeds r_0. Indeed, by the definition of a good cube no causal chain containing this ball can reach the boundary of the cube.

To proceed, observe that since the size of the corridors was chosen so that the maximum number of balls packed there is a negligible fraction of the overall volume $\mathrm{vol}(Q_\lambda) = \lambda^d$, to complete the proof of (**L2**) it is enough to show that for each $\zeta > 0$ and $C > 0$ there exists $r(\zeta, C) > 0$ such that for all $r > r(\zeta, C)$ and all λ large

$$\mathbb{P}\left[W[L+2r_0]\Delta(\lambda, L, r_0; r) > \zeta\lambda^d\right] \leq \exp(-C\lambda^d). \tag{2.2}$$

To this end, assume that L and r_0 are chosen so that $\pi(L, r_0)$ is subcritical for the i.i.d. site percolation on $\mathbb{Z}^d$ with the neighborhood relation $\sim$. Denote by $Cl[i]$ the connected cluster of bad boxes at $i \in B[n(\lambda, L, r_0)]$. We order the points of $B[n(\lambda, L, r_0)]$ in some arbitrary way as $i_1, i_2, ...$ and define $\eta_1 = \eta_{i_1} := \mathrm{card}Cl[i_1]$ and $\eta_{k+1} = \eta_{i_{k+1}} := \mathrm{card}Cl[i_{k+1}]$ if $Cl[i_{k+1}]$ does not coincide with any of the previous clusters $Cl[i],\ i \leq k$; put $\eta_{k+1} := 0$ otherwise. It follows by the exponential decay of the cluster size in the subcritical regime (Sections 5.2 and 6.3 in [11]) that $\mathbb{P}[\mathrm{card}Cl[i] > s] \leq \exp(-sR(L, r_0))$ for some $R(L, r_0) > 0$ for all $i \in B[n(\lambda, L, r_0)]$. Consequently

$$\mathbb{P}[\eta_1 > s_1, \eta_2 > s_2, ...] \leq \exp(-R(L, r_0)[s_1 + s_2 + ...\ ]).$$

This means that the sequence $\eta_1, \eta_2, ...$ is stochastically dominated by an i.i.d. sequence $\hat{\eta}_1, \hat{\eta}_2, ...$ of exponential random variables with parameter $R(L, r_0)$. Clearly,

$$\Delta(\lambda, L, r_0; r) \leq \sum_{i \in B[n(\lambda, L, r_0)]} \eta_i \mathbf{1}_{\{\eta_i > Kr/L\}}.$$

In particular, $\Delta(\lambda, L, r_0; r)$ is stochastically bounded by the sum $\sum_{i \in B[n(\lambda, L, r_0)]} \hat{\eta}_i \mathbf{1}_{\{\hat{\eta}_i > Kr/L\}}$. As discussed above, by adjusting L and r_0 we can make $\pi(L, r_0)$ arbitrarily small, and hence $R(L, r_0)$

arbitrarily large. Using Markov's inequality we get for each $R' < R(L, r_0)$ and $\varepsilon > 0$

$$\mathbb{P}\left(\sum_{i \in B[n(\lambda,L,r_0)]} \hat{\eta}_i \mathbf{1}_{\{\hat{\eta}_i > Kr/L\}} > \varepsilon\lambda^d\right) \leq \frac{\left(\mathbb{E}\exp(R'\hat{\eta}_1 \mathbf{1}_{\{\hat{\eta}_1 > Kr/L\}})\right)^{n(\lambda,L,r_0)^d}}{\exp(R'\varepsilon\lambda^d)}.$$

Thus, choosing r large so that $\mathbb{E}\exp(R'\hat{\eta}_1 \mathbf{1}_{\{\hat{\eta}_1 > Kr/L\}}) < 1$ we can ensure that (2.2) holds. This completes the proof of (**L2**) and hence also that of Corollary 2.1. □

2.2 Related packing models

There are several variants of the basic RSA packing model which can be viewed as marked processes admitting a graphical representation similar to that of RSA packing. In this way, be following the proof of Corollary 2.1, we obtain Donsker-Varadhan LDPs for spatial birth growth models and ballistic deposition models.

(i) Spatial birth-growth models. Consider the following spatial birth-growth model in $\mathbb{R}^d$. Fix $0 < \tau < \infty$. Seeds are born at random locations $X_i \in \mathbb{R}^d$ at times T_i, $i = 1, 2, ...$ according to a unit intensity homogeneous spatial temporal Poisson point process $\Psi := \{(X_i, T_i) \in \mathbb{R}^d \times [0, \tau]\}$. When a seed is born, it forms a cell by growing radially in all directions with a constant speed $v \geq 0$. Whenever one growing cell touches another, it stops growing in that direction. Initially the seed takes the form of a ball of radius $\rho_i \geq 0$ centered at X_i. If a seed appears at X_i and if the ball centered at X_i with radius ρ_i overlaps any of the existing cells then the seed is discarded.

We assume that the ρ_i, $i = 1, 2, ...$ are i.i.d., independent of $\{(X_i, T_i)\}$, and satisfy $\rho_i \leq r$ for some $r < \infty$. In the special case when the growth rate $v = 0$ and ρ_i is constant, this model reduces to the RSA packing model. In the case where all initial radii are zero a.s., the model is known as the Johnson-Mehl model.

Define the random packing measure

$$\mu_\lambda^\xi := \sum_{X_i \in \{X_j\}_{j=1}^\infty \cap Q_\lambda} \xi(X_i, \{X_j\}_{j=1}^\infty \cap Q_\lambda)\delta_{X_i/\lambda}$$

where $\xi(X_i, \{X_j\}_{j=1}^\infty \cap Q_\lambda)$ is either 1 or 0 depending on whether the ball centered at X_i is packed or discarded. Note that the total mass of μ_λ^ξ (i.e., $H_\lambda^\xi := \mu_\lambda^\xi([0, 1]^d)$) is the number of seeds accepted by time τ. Since $\tau < \infty$ and $r < \infty$ there is a finite range of interaction between two particles and in fact no particle excludes any other particle appearing at a distance $D := 2r + 2v\tau$ from it.

The point set $\{X_i\}_{(X_i,T_i)\in\Psi}$ in $\mathbb{R}^d$ is an example of a *marked* point set where each X_i carries a random mark $M_{X_i} \in [0, \tau] \times [0, r]$ representing the arrival time and particle radius.

As in the proof of Corollary 2.1 we make $\{X_i, T_i\}$ into the vertex set of an oriented graph by including an edge from (X,T) to (Y,U) whenever (X,T) and (Y,U) are points satisfying $T \leq U$ and $|X - Y| \leq D$. Thus the graph is constructed exactly as in Section 2.1 by considering only the time component of the marks M_{X_i} (and ignoring the radius component).

By following the set-up in the proof of Corollary 2.1 we may deduce an LDP for the measures μ_λ^ξ as well as the total mass functionals $\mu_\lambda^\xi([0,1]^d)$. This adds to the central limit theorems given by [3, 5, 18].

(ii) Monolayer ballistic deposition with a rolling mechanism. Incoming particles (balls of radius r) have a downward vertical motion as in the basic packing process. Particles arrive sequentially and if a particle hits the substrate $\mathbb{R}^d$ (and not another adsorbed particle) then it is adsorbed and irreversibly fixed. If, on the other hand, a particle hits an already adsorbed particle, then it rolls, following the path of steepest descent until it reaches a stable position. The particle is discarded if it fails to reach the substrate surface. The rolling process does not modify the positions of previously deposited particles.

If the rolling process puts the particle on the substrate $\mathbb{R}^d$ then the particle is adsorbed, otherwise it is rejected from the system. The next sequenced particles are considered similarly. The result is a deposition process on $\mathbb{R}^d$ consisting of a single 'layer'.

Assuming that the rolling process maintains contact between the rolling particle and already deposited particles, it follows that there is a uniform bound D_1 on the lateral displacement of incoming balls. (This is trivial in $d = 1$ and proved in [14] for $d = 2$.) Therefore the interaction range for this process is finite.

Let $\Psi := \{(X_i, T_i) \in \mathbb{R}^d \times [0,T]\}$ be the spatial temporal Poisson point process as defined above for spatial birth growth models. Let $\xi(X_i, \{X_j\}_{j=1}^\infty \cap Q_\lambda)$ be either 1 or 0 depending on whether the particle centered at X_i is accepted or discarded according to the ballistic deposition process with rolling. Consider the random measure

$$\mu_\lambda^\xi := \sum_{X_i \in \{X_j\}_{j=1}^\infty \cap Q_\lambda} \xi(X_i, \{X_j\}_{j=1}^\infty \cap Q_\lambda) \delta_{X_i/\lambda}.$$

Exactly as in the previous examples we observe that no particle excludes any other particle appearing at a distance $D := 2r + 2D_1$ from it and we make $\{(X_i, T_i)\}$ into the vertex set of an oriented graph by including an edge from (X,T) to (Y,U) whenever (X,T) and (Y,U) are points satisfying $T \leq U$ and $|X - Y| \leq D$. Via the proof of Corollary 2.1 we deduce an LDP for $(\mu_\lambda^\xi)_\lambda$ as well as the functionals $(\mu_\lambda^\xi([0,1]^d))_\lambda$, adding to the central limit theorems of [3, 18].

(iii) Multilayer ballistic deposition. Incoming particles arrive as in monolayer ballistic deposition, but now a particle may attach itself to previously adsorbed particles instead of to the substrate. In the simplest form of continuum multilayer ballistic deposition, each particle falls vertically towards the substrate and as soon as it encounters either the substrate or another particle, it sticks (and remains in that place forever). If a particle is deposited higher than at some fixed distance from the substrate, it is rejected. The interaction range for this process is bounded by a finite number, say D_2. Expressing the total number of accepted particles (in cases where particles are not all accepted) as a sum of stabilizing functionals and by following the methods outlined above (with D_2 replacing D), yields an LDP for the total number of accepted particles. This adds to the CLTs of [3, 18].

2.3 k-nearest neighbors graphs

Let $\mathcal{X}$ be a locally finite subset of $\mathbb{R}^d$ and $G := G(\mathcal{X})$ a graph on $\mathcal{X}$. Given a vertex $x \in \mathcal{X}$, let $\mathcal{E}(x, G(\mathcal{X}))$ be the set of edges in G incident to x and let $|e|$ denote the length of an edge e.

Given $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, consider functionals of the type

$$\xi(x, \mathcal{X}) := \xi_\phi^G(x, \mathcal{X}) := \sum_{e \in \mathcal{E}(x; G(\mathcal{X}))} \phi(|e|).$$

Such functionals could represent e.g. the total length of ϕ-weighted edges in G incident to x, the number of edges in G incident to x , or the number of edges in G less than some specified length. These functionals induce functionals on $\mathcal{X}$

$$H^\xi(\mathcal{X}) := \sum_{x \in \mathcal{X}} \sum_{e \in \mathcal{E}(x; G(\mathcal{X}))} \phi(|e|).$$

As before, let $\mathcal{P}_\tau$ be the homogeneous Poisson point process in $\mathbb{R}^d$ of intensity τ. Write

$$H_\lambda^\xi := H_{\lambda;\tau}^\xi := H^\xi(\mathcal{P}_\tau \cap Q_\lambda; Q_\lambda) := \sum_{x \in \mathcal{P}_\tau \cap Q_\lambda} \sum_{e \in \mathcal{E}(x; G(\mathcal{P}_\tau \cap Q_\lambda))} \phi(|e|) \tag{2.3}$$

and

$$\mu_\lambda^\xi := \mu_{\lambda;\tau}^\xi := \mu^\xi(\mathcal{P}_\tau \cap Q_\lambda; Q_\lambda) := \sum_{x \in \mathcal{P}_\tau \cap Q_\lambda} \sum_{e \in \mathcal{E}(x; G(\mathcal{P}_\tau \cap Q_\lambda))} \phi(|e|)\delta_{x/\lambda}. \tag{2.4}$$

Let k be a positive integer. The k-nearest neighbors (undirected) graph on $\mathcal{X}$, denoted $\mathrm{NG}(\mathcal{X})$, is the graph with vertex set $\mathcal{X}$ obtained by including $\{x, y\}$ as an edge whenever y is one of the

k nearest neighbors of x and/or x is one of the k nearest neighbors of y. The k-nearest neighbors (directed) graph on $\mathcal{X}$, denoted $NG'(\mathcal{X})$, is the graph with vertex set $\mathcal{X}$ obtained by placing a directed edge between each point and its k nearest neighbors. Define the induced Poisson point measures μ_λ^ξ as in (2.4). The next result gives an LDP for the total edge length of $NG(\mathcal{P}_\tau \cap Q_\lambda)$; a similar result holds for the total edge length of $NG'(\mathcal{P}_\tau \cap Q_\lambda)$.

Corollary 2.2 *Let $\phi(x) = x$ and let G be the undirected k nearest neighbors graph. The family of functionals $(\lambda^{-d} H_\lambda^\xi)_\lambda$ and the family of measures $(\lambda^{-d}\mu_\lambda^\xi)_\lambda$ satisfy the LDP of Theorems 1.1 and 1.2, respectively.*

Remark. Corollary 2.2 adds to the existing laws of large numbers and central limit theorems for the total edge length of the undirected and directed k nearest neighbors graph [3, 17, 18, 19].

Proof. It is clear that $\xi(x, \mathcal{P}_\tau) := \sum_{e \in \mathcal{E}(x, G(\mathcal{P}_\tau))} |e|$ is stabilizing from the analysis in [17, 3]. With $\xi^{[r]}$ defined as in (1.1), condition **(L1)** is easily verified for $M(r) = kr$. It remains to show that the total variation of the disagreement process $\delta_\lambda^{\xi^{[r]}}$ satisfies condition **(L2)**. Assume that $r > 1$. It is clear that

$$||\delta_\lambda^{\xi^{[r]}}||_{TV} \leq \sum r'_i$$

where r'_i are the lengths of the k-nearest neighbors edges in $G(\mathcal{P}_\tau)$ which exceed the cut-off r. However, with probability 1

$$\sum r'_i \leq \sup[r'_1 + ... + r'_m] \tag{2.5}$$

where the *sup* ranges over all $r'_1, ..., r'_m$, $r'_i \geq r$, such that $({r'}_1^d + ... + {r'}_m^d)\omega_d \leq C(d,k)\text{vol}Q_\lambda$, where ω_d is the volume of the unit radius ball in $\mathbb{R}^d$ while $C(d,k)$ is a constant depending on k and d. Indeed, even though the balls with the radii joining points of $\mathcal{P}_\tau$ to their k-nearest neighbors are in general not disjoint, it is clear that only a finite number $C(d,k)$ of such balls can meet at a given point in $\mathbb{R}^d$. Now, it is easily seen that the right hand side in (2.5) is maximized under the imposed constraint when all r'_i are identical and equal to some $L \geq r$ (with $L = r$ iff $\omega_d r^d$ divides $\text{Vol}(Q_\lambda) = \lambda^d$). Consequently, the right hand side in (2.5) is bounded by $\frac{C(d,k)L\lambda^d}{\omega_d L^d} \leq \frac{C(d,k)\lambda^d}{\omega_d L^{d-1}} < \varepsilon\lambda^d$ if r is large, thus showing **(L2)** as desired. □

Further applications of Theorems 1.1 and 1.2 go as follows. Fix $t > 0$. Let $\phi_t(|e|)$ be either 1 or 0 depending on whether the length $|e|$ of the edge e in the graph NG($\mathcal{X}$) is bounded by t or not. With the finite range corrections $\xi^{[r]}$ given as in (1.1) the conditions (**L1**) and (**L2**) are easily verified since ξ corresponding to such ϕ_t stabilizes within radius t and hence coincides with

$\xi^{[r]},\ r > t$. Then Theorem 1.1 gives an LDP for the empirical distribution function of the re-scaled lengths of the edges in the k-nearest neighbors (undirected) graph on $\mathcal{P}_\tau \cap Q_\lambda$ When $k = 1$, this gives an LDP for the number of pairs of re-scaled points distant at most t from each other. We summarize the discussion as follows. A similar result holds for the k-nearest neighbors (directed) graph on $\mathcal{P}_\tau \cap Q_\lambda$.

Corollary 2.3 *Let $\phi := \phi_t$ be as above, $t \geq 0$, and let G be the undirected nearest neighbors graph. The family of functionals $(\lambda^{-d} H_\lambda^\xi)_\lambda$ and the family of measures $(\lambda^{-d}\mu_\lambda^\xi)_\lambda$ satisfy the LDP of Theorems 1.1 and 1.2, respectively.*

2.4 Lattice graph models

Below we consider versions of some classical graph models restricted to the discretized lattice setting. In this subsection $\mathcal{P}_\tau$ will stand for the point process in $\mathbb{Z}^d$, with each lattice site $x \in \mathbb{Z}^d$ *empty* with probability $\exp(-\tau)$ and *occupied* (i.e. containing a single point) with probability $1 - \exp(-\tau)$, independently of each other. In spite of minor formal differences, it can be shown from our method of proof that the general theory and results developed in Section 1 are valid for this lattice setting.

2.4.1 Voronoi and Delaunay graphs

We assume that $d = 2$ in this example. Given a locally finite set $\mathcal{X} \subset \mathbb{Z}^2$ and $x \in \mathcal{X}$, the locus of points in $\mathbb{R}^2$ closer to x than to any other point in $\mathcal{X}$ is called the *Voronoi cell* centered at x. The graph on the vertex set $\mathcal{X}$ in which each pair of adjacent cell centers is connected by an edge is called the *Delaunay graph* on $\mathcal{X}$ while the planar dual graph consisting of all boundaries of Voronoi cells is called the *Voronoi graph* generated by $\mathcal{X}$. Edges of the Voronoi graph can be finite or infinite. Let $\mathrm{VOR}(x, \mathcal{X})$ denote the Voronoi cell of x generated by $\mathcal{X}$ and let $\mathcal{E}(x, \mathrm{VOR}(\mathcal{X}))$ be the collection of the *finite* edges of $\mathrm{VOR}(x, \mathcal{X})$. Consider a real-valued function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{r\to\infty} \phi(r)/r = 0, \ \ \phi(r) \leq Kr \text{ for some } K > 0, \tag{2.6}$$

and put

$$\xi(x, \mathcal{X}) := \sum_{e \in \mathcal{E}(x, \mathrm{VOR}(\mathcal{X}))} \phi(|e|).$$

A representative example of such ϕ is $\phi(r) = r^p,\ p < 1$. When $p = 0$, that is when $\phi \equiv 1$, then $\xi(x, \cdot)$ counts edges of the Voronoi cells centered at x, which amounts to counting the Delaunay edges incident to x. We have

Corollary 2.4 *With the notation of this paragraph, the family of functionals* $(\lambda^{-d} H^{\xi}_{\lambda})_{\lambda}$ *and the family of measures* $(\lambda^{-d} \mu^{\xi}_{\lambda})_{\lambda}$ *satisfy the LDP of Theorems 1.1 and 1.2, respectively.*

Remark. Corollary 2.4 adds to existing laws of large numbers [19] and central limit theorems [17, 3] for functionals of Voronoi graphs in the continuum.

Proof. Corollary 2.4 will follow as a direct conclusion of Theorems 1.1 and 1.2 as soon as we verify the conditions **(L1)** and **(L2)**. We define $\xi^{[r]}$ as in (1.1). Clearly, if the radius of stabilization $R^{\xi}(x, \mathcal{P}_{\tau}) < r$, the Voronoi cell centered at x is contained in $B_{r/2}(x)$ and hence **(L1)** holds with $M(r) := K\pi r$ with K as in (2.6). To check that **(L2)** holds as well, observe that only centers $x \in \mathbb{Z}^2$ of Voronoi cells with $\mathrm{diam}(\mathrm{VOR}(x, \mathcal{P}_{\tau})) \geq r/2$ may contribute to $\delta^{\xi^{[r]}}_{\lambda}$. It is easily seen that the area $\mathrm{Vol}(\mathrm{VOR}(x, \mathcal{X}))$ of each such cell exceeds $r/2$, an estimate which may not necessarily hold in the continuum setting. In view of (2.6), we have $\xi(x, \mathcal{X}) = o(\mathrm{Vol}(\mathrm{VOR}(x, \mathcal{X})))$ as $r \to \infty$, and so each x with $R^{\xi}(x, \mathcal{P}_{\tau}) \geq r$, can contribute at most $o(\mathrm{Vol}(\mathrm{VOR}(x, \mathcal{X})))$ to $\delta^{\xi^{[r]}}_{\lambda}$. Since the Voronoi cells centered at different points are disjoint, combining the above conclusions yields a.s.

$$||\delta^{\xi^{[r]}}_{\lambda}||_{TV} = o(\mathrm{Vol}(Q_{\lambda})) = o(\lambda^2) \text{ as } r \to \infty,$$

which completes the verification of **(L2)** and hence also the proof of Corollary 2.4. □

2.4.2 Sphere of influence graphs

Given a locally finite set $\mathcal{X} \subset \mathbb{Z}^d$, the sphere of influence graph $\mathrm{SIG}(\mathcal{X})$ is a graph with vertex set $\mathcal{X}$, constructed as follows: for each $x \in \mathcal{X}$ let $B(x)$ be a ball around x with radius equal to $\min_{y \in \mathcal{X} \setminus \{x\}} \{|y - x|\}$. Then $B(x)$ is called the sphere of influence of x. Draw an edge between x and y iff the balls $B(x)$ and $B(y)$ overlap. The collection of such edges is the sphere of influence graph (SIG) on $\mathcal{X}$ and is denoted by $\mathrm{SIG}(\mathcal{X})$. We also write $\mathrm{SIG}(x, \mathcal{X})$ for the collection of all edges in $\mathrm{SIG}(x, \mathcal{X})$ incident to x. Consider a real-valued function ϕ as in (2.6) above and put

$$\xi(x, \mathcal{X}) := \sum_{e \in \mathrm{SIG}(x, \mathcal{X})} \phi(|e|). \tag{2.7}$$

Once again, a representative example is $\phi \equiv 1$, in which case $\xi(x, \cdot)$ counts the edges incident to x. We obtain

Corollary 2.5 *With the notation of this paragraph, the family of functionals* $(\lambda^{-d} H^{\xi}_{\lambda})_{\lambda}$ *and the family of measures* $(\lambda^{-d} \mu^{\xi}_{\lambda})_{\lambda}$ *satisfy the LDP of Theorems 1.1 and 1.2, respectively.*

Remark. When $\phi \equiv 1$, laws of large numbers and central limit theorems for $(H^{\xi}_{\lambda})_{\lambda}$ are established in Theorem 2.6 of [19] and Theorem 7.1 of [17], respectively.

Proof. For the purpose of this proof, rather than using (1.1) we introduce a particular finite range correction of the functional ξ. To this end, for each sample point $x \in \mathcal{X}$ set $B^{[r/2]}(x)$ to be the sphere of influence $B(x)$ if the radius of $B(x)$ does not exceed $r/2$ and let $B^{[r/2]}(x)$ be the ball of radius $r/2$ around x otherwise. Construct the r-corrected graph $\mathrm{SIG}_r(\mathcal{X})$ by drawing an edge between x and y iff $B^{[r/2]}(x)$ and $B^{[r/2]}(y)$ overlap. Note that this construction is monotone in that $\mathrm{SIG}_r(\mathcal{X}) \subseteq \mathrm{SIG}_{r'}(\mathcal{X}) \subseteq \mathrm{SIG}(\mathcal{X})$ for $r' > r$. Write $\mathrm{SIG}_r(x, \mathcal{X})$ for the collection of edges in $\mathrm{SIG}_r(\mathcal{X})$ incident with x and put

$$\xi^{[r]}(x, \mathcal{X}) := \sum_{e \in \mathrm{SIG}_r(x,\mathcal{X})} \phi(|e|). \tag{2.8}$$

It is clear that so defined $\xi^{[r]}$ satisfies **(C1)** and **(C2)**. Moreover, by the above monotonicity the difference measure $\delta^{\xi^{[r]}}$ is always non-negative.

To conclude Corollary 2.5 from Theorems 1.1 and 1.2 it suffices to verify the conditions **(L1)** and **(L2)**. Observe that for $x \in \mathcal{P}_\tau \subseteq \mathbb{Z}^d$ the edges in $\mathrm{SIG}_r(x, \mathcal{P}_\tau \cap Q_\lambda)$ have their lengths bounded by r. Since we place ourselves in the lattice setting, we conclude that the total number of such edges is of the surface order $O(r^{d-1})$ and, consequently, in view of (2.6) the value of $\xi^{[r]}$ as given by (2.8) is of order $o(r^d)$, which yields **(L1)**. Note that we would not attain this conclusion in the continuum setting. To proceed with the proof of **(L2)** note that

$$||\delta^{\xi^{[r]}}_{\lambda}||_{TV} = 2 \sum_{e \in \mathrm{SIG}(\mathcal{P}_\tau \cap Q_\lambda) \setminus \mathrm{SIG}_r(\mathcal{P}_\tau \cap Q_\lambda)} \phi(|e|). \tag{2.9}$$

Say that x is the *principal endpoint* of an edge $e := \{x, y\} \in \mathrm{SIG}(\mathcal{P}_\tau \cap Q_\lambda)$ iff the radius of the sphere of influence $B(x)$ is larger than that of $B(y)$. If the radii of $B(x)$ and $B(y)$ coincide, we break the tie arbitrarily so that e has exactly one principal endpoint. As easily seen, our lattice setting guarantees that a given point x can be the principal endpoint of at most $O(r_x^{d-1})$ edges, each of length at most r_x, where r_x is the radius of the sphere of influence $B(x)$. Thus, with $Pr[x]$ standing for the collection of edges in $\mathrm{SIG}(\mathcal{P}_\tau \cap Q_\lambda)$ whose principal endpoint is x, we easily conclude from (2.6) that

$$\sum_{e \in Pr[x]} \phi(|e|) = o(r_x^d) = o(\mathrm{Vol}(B(x))). \tag{2.10}$$

Note also that

$$\sum_{e \in \mathrm{SIG}(\mathcal{P}_\tau \cap Q_\lambda) \setminus \mathrm{SIG}_r(\mathcal{P}_\tau \cap Q_\lambda)} \phi(|e|) \leq \sum_{\substack{x \in \mathcal{P}_\tau \cap Q_\lambda \\ r_x > r/2}} \sum_{e \in Pr[x]} \phi(|e|).$$

Consequently, using (2.9) and (2.10) we obtain

$$||\delta_\lambda^{\xi^{[r]}}||_{TV} = o\left(\sum_{\substack{x\in\mathcal{P}_\tau\cap Q_\lambda\\ r_x>r/2}} B(x)\right). \tag{2.11}$$

Even though the balls $B(x)$ for $x \in \mathcal{P}_\tau \cap Q_\lambda,\ r_x > r/2$, showing up in (2.11) are in general not disjoint, it is clear that a given lattice point can belong to at most a certain dimension-dependent number b_d of them. Putting together the above considerations and (2.11) gives us a.s.

$$||\delta_\lambda^{\xi^{[r]}}||_{TV} = o(\mathrm{Vol}(Q_\lambda)) = o(\lambda^d) \text{ as } r \to \infty,$$

which immediately yields the required condition **(L2)**. □

3 Proof of Theorem 1.1

Using general results in [24] for functionals having a 'near additivity property', we establish in Lemma 3.2 below an LDP for the total masses of the r-corrected functionals $H_\lambda^{\xi^{[r]}} := \mu_\lambda^{\xi^{[r]}}([0,1]^d)$. We then conclude the LDP for $H_\lambda^\xi := \mu_\lambda^\xi([0,1]^d)$. This is done using the condition **(L2)** and applying the Inverse Bryc Lemma, see e.g. Theorems 4.4.2 and 4.4.10 in [7]. Even though we do not assume explicitly the finiteness of the Laplace transforms of H_λ^ξ, we get for all $s \in \mathbb{R}$

$$\limsup_{\lambda>0} \frac{1}{\lambda^d} \log \mathbb{E}[\exp(sH_\lambda^\xi)] < \infty. \tag{3.1}$$

Indeed, this follows directly from **(L1)** and **(L2)** in view of the inequality ([25], p. 485)

$$\mathbb{P}[\mathrm{Po}(\alpha) > t] \le \exp\left(-\frac{t}{4}\log\left(\frac{t}{2\alpha}\right)\right),\ t \ge 16\alpha, \tag{3.2}$$

with $\mathrm{Po}(\alpha)$ standing for a mean α Poisson random variable.

3.1 Auxiliary lemmas

We first recall the general results, definitions, and terminology of [24]. Let $X = \{X(n)\}$ be a real-valued process indexed by the positive integers.

Definition 3.1 *(near additivity) We say that the process $\{X(n)\}$ is nearly additive if on $(\Omega, \mathcal{F}, \mathbb{P})$ there are i.i.d. processes $\{X_i(n)\}_{i=1}^\infty$ and random variables $R(n,m)$, for all positive integers n and m, with these properties: $X_i(n) \stackrel{\mathcal{D}}{=} X(n)$ for all i and n, the inequalities*

$$\left| X(nm^d) - \sum_{i=1}^{m^d} X_i(n) \right| \le R(n,m) \tag{3.3}$$

hold for all m and n, and the error $R(n,m)$ is such that for all $\varepsilon > 0$ and $C > 0$ there exists a finite $n_0 = n_0(C, \varepsilon)$ such that

$$\mathbb{P}[R(n,m) \geq \varepsilon n m^d] \leq \exp(-Cnm^d)$$

for all $n \geq n_0$ and for all positive integers m.

Definition 3.2 *(regularity) We say that $\{X(n)\}$ is regular if for each fixed k, the following property holds: If $m = m(n)$ is defined by the requirement $km^d \leq n < k(m+1)^d$ for all n, then for all $\varepsilon > 0$ and $C > 0$ there exists a finite $n_1 = n_1(C, \varepsilon, k)$ such that*

$$\mathbb{P}[|X(n) - X(km^d)| \geq n\varepsilon) \leq \exp(-Cn) \tag{3.4}$$

for all $n \geq n_1$.

It is straightforward to check that if $\{X(n)\}$ is a regular nearly additive process and if $X(n)$ is integrable for all $n \geq 1$, then such a process satisfies a strong law of large numbers, i.e.,

$$\lim_{n\to\infty} \frac{X(n)}{n} = \lim_{n\to\infty} \frac{E[X(n)]}{n} = \gamma^X \ a.s.,$$

where γ^X is a constant depending upon the process $\{X(n)\}$, referred to as the *spatial constant* of $\{X(n)\}$ in the sequel.

In order to obtain a general LDP for $\{X(n)\}$, we make a uniform boundedness assumption on its logarithmic moment generating function:

$$A(t) := \sup_n \{n^{-1} \log E[\exp(tX(n))]\} < \infty \quad \text{for all } t \in \mathbb{R}. \tag{3.5}$$

The following is Theorem 2.1 of [24].

Lemma 3.1 *Let $X := \{X(n)\}$ be a regular nearly additive process satisfying (3.5). Then the limit*

$$\Lambda^X(s) = \lim_{n\to\infty} n^{-1} \log \mathbb{E} \exp(sX(n))$$

exists for all $s \in \mathbb{R}$, the Legendre-Fenchel transform of Λ^X

$$[\Lambda^X]^*(t) := \sup_{s\in\mathbb{R}} (ts - \Lambda^X(s))$$

has compact level sets, $[\Lambda^X]^*(x) = 0$ iff $x = \gamma^X$ and $\{X(n)\}$ satisfies a full large deviation principle with rate function $[\Lambda^X]^*$ and speed n.

We will use Lemma 3.1 to establish a LDP for the total mass of the r-corrected version $H_\lambda^{\xi^{[r]}}$.

Lemma 3.2 *The family* $(\lambda^{-d} H_\lambda^{\xi^{[r]}})_{\lambda>0}$ *satisfies on* $\mathbb{R}$ *the full large deviation principle with speed* λ^d *and a good, convex rate function* $I^{[r]}$. *Moreover,* $I^{[r]}(t) \neq 0$ *unless* $t = \gamma^{\xi^{[r]}}$.

Proof of Lemma 3.2. Fix $r > 0$ and choose some large $L > 0$ to be specified below. We partition the cube Q_λ into translates $Q_{L+2r}[1], Q_{L+2r}[2], ..., Q_{L+2r}[k(\lambda, L, r)]$ of Q_{L+2r}. To avoid unnecessary technicalities we assume without loss of generality that, for given $\lambda > 0$, $L := L(\lambda)$ is chosen so as to make Q_λ split into an integer number of such subcubes and, moreover, that $(L+r)^d$ is an integer. Each $Q_{L+2r}[i]$ can be further subdivided into the central cube $Q_L[i]$ separated from the boundary $\partial Q_{L+2r}[i]$ by moats of constant width r. Write

$$H_i^{[r]} := H^{\xi^{[r]}}(\mathcal{P}_\tau; Q_L[i])$$

and observe that, a.s.

$$H_i^{[r]} = H^{\xi^{[r]}}(\mathcal{P}_\tau \cap Q_{L+2r}[i]; Q_L[i]) \tag{3.6}$$

by the definition of the corrected functional $\xi^{[r]}$. Consequently, since the subcubes $Q_{L+2r}[\cdot]$ are disjoint by (3.6), the random variables $H_i^{[r]}, 1 \leq i \leq k(\lambda, L, r)$ are i.i.d. By **(L1)** we can assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carries i.i.d. copies $\hat{H}_i^{[r]}$ of $H^{\xi^{[r]}}(\mathcal{P}_\tau \cap Q_{L+2r}[i]; Q_{L+2r}[i])$ such that for all $1 \leq i \leq k(\lambda, L, r)$

$$|H_i^{[r]} - \hat{H}_i^{[r]}| \leq M(r)\eta_i, \tag{3.7}$$

where η_i are i.i.d. Poisson random variables with mean $(L+2r)^d - L^d$.

We will use (3.7) and Lemma 3.1 to show near additivity of $H^{\xi^{[r]}}(\mathcal{P}_\tau \cap Q_\lambda; Q_\lambda)$. To this end, define $X := \{X(n)\}$ a real-valued process indexed by the positive integers. Put $n := (L+2r)^d$ and $m := \lambda/(L+2r)$ so that $m^d = k(\lambda, L, r)$ coincides with the number of partitioning cubes $Q_{L+2r}[i]$. Further, set $X(nm^d) := H^{\xi^{[r]}}(\mathcal{P}_\tau \cap Q_\lambda; Q_\lambda)$ and $X_i(n) := \hat{H}_i^{[r]}$. By construction, $X_i(n)$ coincides in distribution with $X(n) = H^{\xi^{[r]}}(\mathcal{P}_\tau \cap Q_{L+2r}; Q_{L+2r})$. With (3.7) we easily obtain a.s.

$$\left| X(nm^d) - \sum_{j=1}^{m^d} X_i(n) \right| \leq M(r) \sum_{i=1}^{m^d} \eta_i = M(r)\hat{\eta}, \tag{3.8}$$

where $\hat{\eta}$ is a Poisson random variable with mean $\lambda^d[((L+2r)^d - L^d)/(L+2r)^d]$. Near additivity (3.3) now follows. Indeed, notice by inequality (3.2) that

$$\mathbb{P}[M(r)\hat{\eta} \geq \varepsilon nm^d] = \mathbb{P}\left[\hat{\eta} \geq \frac{\varepsilon nm^d}{M(r)}\right]$$

$$\leq \exp\left(-\frac{\varepsilon n m^d}{4M(r)} \log\left(\frac{-\frac{\varepsilon n m^d}{2M(r)}}{\frac{(L+2r)^d - L^d}{(L+2r)^d}}\right)\right) \leq \exp(-Cnm^d)$$

where the last inequality follows because we can make the ratio r/L arbitrarily small by an appropriate choice of L.

The required regularity assumption (3.4) follows from **(L1)** and (3.2) in an analogous way. Finally, the condition (3.5) ensuring appropriate behaviour of the Laplace transforms of $X(n)$, coincides with (3.1). Consequently, we can apply Lemma 3.1 to obtain the required LDP, thus completing the proof of Lemma 3.2. □

3.2 Conclusion of the proof of Theorem 1.1

The proof of Theorem 1.1 goes much along the same lines as the proof of Lemma 3.1 (Theorem 2.1 in [24]). We therefore shorten the technical details, providing only the necessary steps. The first step roughly corresponds to Proposition 3.1 in [24] and involves showing that for all bounded Lipschitz functions $F : \mathbb{R} \to \mathbb{R}$ the limit

$$\Gamma(F) := \lim_{\lambda \to \infty} \Gamma_\lambda(F) \tag{3.9}$$

exists with

$$\Gamma_\lambda(F) := \frac{1}{\lambda^d} \log \mathbb{E} \exp\left(\lambda^d F\left(\lambda^{-d} H_\lambda^\xi\right)\right).$$

In view of exponential tightness of $\lambda^{-d} H_\lambda^\xi$, as implied by (3.1), the relation (3.9) combined with the Bryc Inverse Varadhan Lemma (Theorem 4.4.10 in [7]) will imply that $\Gamma(F)$ exists for all $F \in C_b(\mathbb{R})$, where $C_b(\mathbb{R})$ denotes the bounded continuous functions on $\mathbb{R}$, and that $\lambda^{-d} H_\lambda^\xi$ satisfies the LDP with a good rate function I given by

$$I(x) = \sup_{F \in C_b(\mathbb{R})} (F(x) - \Gamma(F)), \tag{3.10}$$

see also (3.3) in [24]. However, some further steps will be needed to establish the required convexity and non-triviality properties for this I, as well as its variational representation as the Legendre-Fenchel transform of the limit of log-Laplace transforms.

Step 1. We establish (3.9) as follows. For all $r > 0$ and $\lambda > 0$, put

$$D_\lambda^{[r]} := D_\lambda^{\xi^{[r]}} := \delta_\lambda^{\xi^{[r]}}([0,1]^d).$$

Fix a bounded Lipschitz function F and let L_F denote its Lipschitz constant. Write

$$\Gamma_\lambda^{[r]}(F) := \frac{1}{\lambda^d} \log \mathbb{E} \exp\left(\lambda^d F\left(\lambda^{-d} H_\lambda^{\xi^{[r]}}\right)\right)$$

and let

$$\Gamma^{[r]}(F) := \lim_{\lambda\to\infty} \Gamma^{[r]}_\lambda(F),$$

with the existence of the above limit guaranteed by Lemma 3.2 and by the Varadhan Integral Lemma (Theorem 4.3.1 in [7]). We show that for each $\varepsilon > 0$

$$\liminf_{\lambda\to\infty} \Gamma_\lambda(F) \geq \limsup_{r\to\infty} \Gamma^{[r]}(F) - \varepsilon. \tag{3.11}$$

To this end, choose some large $C > 0$ and small $\delta > 0$ to be specified below, and let $r > r(\delta, C)$ as defined in **(L2)**. Then, in view of **(L2)**, we conclude that for λ large enough

$$\Gamma_\lambda(F) =$$

$$\frac{1}{\lambda^d}\log\left[\mathbb{E}\exp\left(\lambda^d F\left(\lambda^{-d}H^\xi_\lambda\right)\right)\mathbf{1}_{\{|D^{[r]}_\lambda|\leq\delta\lambda^d\}} + \mathbb{E}\exp\left(\lambda^d F\left(\lambda^{-d}H^\xi_\lambda\right)\right)\mathbf{1}_{\{|D^{[r]}_\lambda|>\delta\lambda^d\}}\right] \tag{3.12}$$
$$\geq \frac{1}{\lambda^d}\log\left[\mathbb{E}\exp\left(\lambda^d F\left(\lambda^{-d}H^\xi_\lambda\right)\right)\mathbf{1}_{\{|D^{[r]}_\lambda|\leq\delta\lambda^d\}}\right].$$

Since $\log[\exp^{A(\lambda)} - \exp^{B(\lambda)}] \geq A(\lambda) + o(1)$ if $B(\lambda) < A(\lambda)$, the above is bounded below by

$$\frac{1}{\lambda^d}\log\left[\exp\left(\lambda^d\left[\Gamma^{[r]}_\lambda(F) - L_F\delta\right]\right) - \exp(\lambda^d||F||_\infty)\mathbb{P}[|D^{[r]}_\lambda| > \delta\lambda^d]\right] \geq$$
$$\Gamma^{[r]}_\lambda(F) - L_F\delta - o(1)$$

provided $||F||_\infty - C < \Gamma^{[r]}_\lambda(F) - L_F\delta$ (note that such a choice of C can always be made independent of r since $\Gamma^{[r]}_\lambda(F) > -||F||_\infty$ for all r and λ). Choosing C as specified above, taking $\delta < \varepsilon/L_F$ and letting λ and r tend to infinity yields the required relation (3.11).

Likewise, to establish the corresponding upper bound

$$\limsup_{\lambda\to\infty} \Gamma_\lambda(F) \leq \liminf_{r\to\infty} \Gamma^{[r]}(F) + \varepsilon \tag{3.13}$$

write, using (3.12) and **(L2)**,

$$\Gamma_\lambda(F) \leq \frac{1}{\lambda^d}\log\left[\exp\left(\lambda^d\left[\Gamma^{[r]}_\lambda + L_F\delta\right]\right) + \exp(\lambda^d||F||_\infty)\mathbb{P}[|D^{[r]}_\lambda| > \delta\lambda^d]\right] \leq$$
$$\max\left(\Gamma^{[r]}_\lambda + L_F\delta, ||F||_\infty - C\right) + o(1)$$

for λ large enough, which yields (3.13) by choosing C so that $||F||_\infty - C < \Gamma^{[r]}_\lambda + L_F\delta$ (see the discussion above), setting $\delta < \varepsilon/L_F$ and letting λ and r tend to infinity as above. Combining (3.11) and (3.13) shows that the limit in (3.9) exists and is given by

$$\Gamma(F) = \lim_{r\to\infty} \Gamma^{[r]}(F). \tag{3.14}$$

This completes Step 1.

Step 2. As a consequence of Step 1, the LDP for $(\lambda^{-d} H^{\xi}_{\lambda})_{\lambda}$ holds with the good rate function $I(\cdot)$ given by (3.10) and the log-Laplace transform $\Lambda(\cdot)$ given by the limit (1.4) is well defined in view of the Varadhan Integral Lemma (see Theorem 4.3.1 in [7]) combined with (3.1).

Further, we claim that I in (3.10) is convex. In full analogy with the proof of Proposition 3.2 in [24] this is concluded from the following two auxiliary results:

(a) For all $x \in \mathbb{R}$, $I(x) \leq \liminf_{r \to \infty} I^{[r]}(x)$,

(b) The set $\{I < \infty\}$ is an interval, and $\lim_{r\to\infty} I^{[r]}(x) = I(x)$ for interior points of this interval.

Both these statements are proven exactly along the same lines as the corresponding assertions of Proposition 3.2 in [24], the only difference being to use the LDPs for the r-corrected approximations, as obtained in Lemma 3.2, rather than the Cramér LDPs for approximating sums of i.i.d. random variables, as in [24]. We demonstrate this substitution by rewriting from [24] the proof of (a); the statement (b) and the resulting convexity of $I(\cdot)$ follow by a similar rewriting from [24]. To this end, fix x and assume that $\liminf_{r\to\infty} I^{[r]}(x) < \infty$, for otherwise part (a) holds trivially. Let $c < I(x)$. By lower semicontinuity there exists $\varepsilon > 0$ such that $|y - x| \leq \varepsilon$ implies $I(y) \geq c$. Let $C > \liminf_{r\to\infty} I^{[r]}(x) + 1$, and let r be large enough so that $r \geq r(\varepsilon/2, C)$ (in the notation of the condition **(L2)**) and so that $I^{[r]}(x) + \varepsilon < C$. Then we have by the LDP of Lemma 3.2 for $\lambda^{-d} H^{\xi^{[r]}}_{\lambda}$,

$$-c \geq -\inf_{y\in[x-\varepsilon,x+\varepsilon]} I(y) \geq \limsup_{\lambda\to\infty} \lambda^{-d} \log \mathbb{P}\left(\lambda^{-d} H^{\xi}_{\lambda} \in [x-\varepsilon, x+\varepsilon]\right) \geq$$

$$\limsup_{\lambda\to\infty} \lambda^{-d} \log\left[\mathbb{P}\left(\lambda^{-d} H^{\xi^{[r]}}_{\lambda} \in [x-\varepsilon/2, x+\varepsilon/2]\right) - \mathbb{P}\left(||\delta^{\xi^{[r]}}_{\lambda}||_{TV} \geq \lambda^{d}\varepsilon/2\right)\right] \geq$$

$$\limsup_{\lambda\to\infty} \lambda^{-d} \log\left[\exp\left\{-\lambda^{d} I^{\xi^{[r]}}(x) - \lambda^{d}\varepsilon\right\} - \exp(-C\lambda^{d})\right] = -I^{\xi^{[r]}}(x) - \varepsilon.$$

This is true for all large enough λ, so we may let $r \to \infty$ along suitable subsequence to get $c \leq \liminf_{\lambda\to\infty} I^{\xi^{[r]}}(x) + \varepsilon$. We then let $c \uparrow I(x)$ and $\varepsilon \to 0$ to prove the assertion (a).

The convexity of I, combined with (3.1), allows us now to apply Theorem 4.5.10 in [7] to conclude that I is the Legendre-Fenchel transform of Λ i.e.

$$I(x) = \sup_{s\in\mathbb{R}} (xs - \Lambda(s)),$$

as required for (1.5). Consequently, since Λ is everywhere finite by (3.1), the relation (1.6) follows now by Lemma 2.2.20 in [7]. Step 2 is complete.

Step 3. In this final step we show that the rate function I is non-zero away from the spatial constant γ^ξ. Assume that there exists $t_0 \neq \gamma^\xi$ with $I(t_0) = 0$. Fix $C > 0$, choose arbitrarily small $\varepsilon > 0$ and let $r(\varepsilon, C)$ be as in **(L2)**. Recalling that $I^{[r]}(t) > 0$ unless $t = \gamma^{\xi^{[r]}}$ as shown in Lemma 3.2, we conclude that we must have $|t_0 - \gamma^{\xi^{[r]}}| < \varepsilon$ for all $r > r(\varepsilon, C)$. However, for the same reasons we must have $|\gamma^\xi - \gamma^{\xi^{[r]}}| < \varepsilon$ since $I(\gamma^\xi) = 0$. Consequently, we see that $|t_0 - \gamma^\xi| < 2\varepsilon$. Since ε was arbitrary, this contradicts our initial assumption that $\gamma^\xi \neq t_0$. This completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

Next, in Lemma 4.1 we extend the LDP for the total masses of the r-corrected empirical measures to an LDP for the joint behaviour of their masses on finite partitions of the cube $[0,1]^d$ into sub-cubes. The same is then done for the original functional ξ in Lemma 4.2. The proof of Theorem 1.2 is then completed by borrowing the argument from the proof of Theorem 1 in Schreiber [23], which is close in spirit to classical projective limit techniques, see Section 4.6 in [7].

Lemma 4.1 *Let $\mathcal{C} := \{C_1, ..., C_k\}$ be a finite partition of $[0,1]^d$ into sub-cubes. Then, for each $r > 0$, the family of random vectors $\langle \mu_\lambda^{\xi^{[r]}}(C_1), ..., \mu_\lambda^{\xi^{[r]}}(C_k) \rangle$ satisfies on $\mathbb{R}^k$ the full large deviation principle with speed λ^d and rate function*

$$I_{\mathcal{C}}^{[r]}(t_1, ..., t_k) := \sum_{i=1}^{k} \mathrm{vol}(C_i) I^{[r]}(t_i / \mathrm{vol}(C_i)). \tag{4.1}$$

Proof. The proof is a direct extension of the argument in Lemma 3.2. It relies on Theorem 2.1 in [24] combined with the observation that the values of the r-corrected stabilizing functionals are independent over disjoint rectangular solids in $[0, \lambda]^d$ provided that the solids are separated by moats of width $2r$ and whose total volume is negligible in the limit $\lambda \to \infty$. □

The next result follows from Lemma 4.1 by an argument completely analogous to that used to deduce Theorem 1.1 from Lemma 3.2.

Lemma 4.2 *Let $\mathcal{C} := \{C_1, ..., C_k\}$ be as in Lemma 4.1. Then the family of random vectors $\langle \mu_\lambda^{\xi}(C_1), ..., \mu_\lambda^{\xi}(C_k) \rangle$ satisfies on $\mathbb{R}^k$ the full large deviation principle with speed λ^d and rate function*

$$I_{\mathcal{C}}(t_1, ..., t_k) := \sum_{i=1}^{k} \mathrm{vol}(C_i) I(t_i / \mathrm{vol}(C_i)). \tag{4.2}$$

Proof of Theorem 1.2. We observe first that the family of random measures $(\lambda^{-d}\mu_\lambda^\xi)_\lambda$ is exponentially tight in $\mathcal{M}^+([0,1]^d)$. Indeed, this follows directly from conditions **(L1)** and **(L2)** combined with the inequality (3.2). We are thus in a position to apply Theorem 1.3.7 in [9], originally due to O'Brien, Verwaat [13] and Pukhalskii [21], stating that from an exponentially tight sequence of measures on a Polish space one can extract a subsequence which satisfies the large deviation principle with a certain good rate function. Fix an arbitrary sequence $(\lambda_n)_{n=1}^\infty$ and extract from it a subsequence $(\lambda_n')_{n=1}^\infty$ such that $(\mu_{\lambda_n'}^\xi)_{n=1}^\infty$ satisfies on $\mathcal{M}^+([0,1]^d)$ the large deviation principle with a good rate function $\hat{J}$. We will show that

$$I_{\mathcal{C}}(t_1,...,t_k) = \inf\{\hat{J}(\theta) \mid \theta \in \mathcal{M}^+([0,1]^d),\ \theta(C_i) = t_i,\ i = 1,...,k\} \tag{4.3}$$

with $\mathcal{C} = \{C_1,...,C_k\}$ and $I_{\mathcal{C}}$ as in Lemma 4.2. This will imply that $\hat{J}$ coincides with J given in (1.7), and hence complete the proof of Theorem 1.2, in exactly the same way that Theorem 1 in [23] is concluded from Lemma 3 there, by the use of Lemmas 4, 5 and 6 in [23]. Note that the argument in [23] makes essential use of the convexity of I, which is available by Theorem 1.1.

To establish (4.3) we use the Contraction Principle (Lemma 2.1.4 in [8]). Since the mapping $\mathcal{M}^+([0,1]^d) \ni \theta \mapsto \langle\theta(C_1),...,\theta(C_k)\rangle$ is not continuous in the weak topology on $\mathcal{M}^+([0,1]^d)$, some work is needed to use this lemma. To this end, consider for each $1 \le i \le k$ the sequence of continuous functions $\phi_m^{(i)} : [0,1]^d \to [0,1]$ given by

$$\phi_m^{(i)} := \begin{cases} 1, & x \in C_i, \\ 0, & \text{dist}(x, C_i) > 1/m, \\ 1 - m\text{dist}(x, C_i), & \text{otherwise}, \end{cases}$$

so that $\phi_m^{(i)}$ approximates the indicator function $\mathbf{1}_{C_i}$. We show first that

$$\lim_{m\to\infty} \sup\{\sum_{i=1}^k |\langle\phi_m^{(i)}, \theta\rangle - \theta(C^{(i)})|\ ,\ \theta \in \mathcal{M}^+([0,1]^d),\ \hat{J}(\theta) \le L\} = 0 \tag{4.4}$$

for all $L \in (0,\infty)$. Assume the contrary so that there exists L_0, ε_0 and a sequence $\theta_{m'},\ m' \to \infty$ of measures in $\mathcal{M}^+([0,1]^d)$ with $\hat{J}(\theta_{m'}) \le L_0$ and such that

$$\sum_{i=1}^k |\langle\phi_{m'}^{(i)}, \theta_{m'}\rangle - \theta_{m'}(C^{(i)})| \ge \varepsilon_0$$

for all m' so that in particular

$$\sum_{i=1}^k \theta_{m'}(\partial^{1/m'} C^{(i)}) \ge \varepsilon_0 \tag{4.5}$$

for all m' with $\partial^{1/m'} C^{(i)} := \partial C^{(i)} \cup \{x \in [0,1]^d \mid 0 < \mathrm{dist}(x, C^{(i)} \leq 1/m'\}$. Since $\hat{J}$ is a good rate function, the level set $\{\theta \mid \hat{J}(\theta) \leq L_0\}$ is compact in $\mathcal{M}^+([0,1]^d)$ and hence we can assume without loss of generality that $\theta_{m'}$ converges in $\mathcal{M}^+([0,1]^d)$ to some θ_∞ as $m' \to \infty$ and, moreover, $\hat{J}(\theta_\infty) \leq L_0$. In view of (4.5) this means that

$$\sum_{i=1}^{k} \theta_\infty(\partial C^{(i)}) \geq \varepsilon_0. \tag{4.6}$$

However, recalling that $\hat{J}$ is the rate function governing the large deviation behavior of the sequence $(\mu^\xi_{\lambda'_n})_{n=1}^\infty$, we use (4.6) combined with Lemma 4.2 to conclude by (1.6) that $\hat{J}(\theta_\infty) = +\infty$. Indeed, (4.6) allows us to find a sequence of rectangular boxes $(K_j)_{j=1}^\infty$ and some $\delta_0 > 0$ with $\theta_\infty(\mathrm{Int} K_j) > \delta_0 > 0$ for all $j = 1, 2, \ldots$ and with $\mathrm{vol}(K_j) \to 0$, we observe that $\hat{J}(\theta_\infty) \geq \mathrm{vol}(K_j) I(\delta_0/\mathrm{vol}(K_j))$ by Lemma 4.2, and we use (1.6) to show that $\mathrm{vol}(K_j) I(\delta_0/\mathrm{vol}(K_j)) \to \infty$. This contradicts our above observation that $\hat{J}(\theta_\infty) \leq L_0$, which now yields (4.4). Our next step is to show that

$$\lim_{m\to\infty} \limsup_{\lambda\to\infty} \frac{1}{\lambda^d} \log \mathbb{P}\left[\sum_{i=1}^{k} |\langle \phi_m^{(i)}, \lambda^{-d}\mu^\xi_\lambda\rangle - \lambda^{-d}\mu^\xi_\lambda(C^{(i)})| > \delta\right] = -\infty \tag{4.7}$$

for all $\delta > 0$. Of course, since $|\phi_m^{(i)}| \leq 1$, it is enough to check that

$$\lim_{m\to\infty} \limsup_{\lambda\to\infty} \frac{1}{\lambda^d} \log \mathbb{P}\left[\sum_{i=1}^{k} \lambda^{-d}\mu^\xi_\lambda(\partial^{1/m} C^{(i)}) > \delta\right] = -\infty.$$

This comes, however, as a direct consequence of the conditions **(L1)** and **(L2)** combined with the use of the inequality (3.2), whence (4.7) is established. Combining (4.4) and (4.7) we can now apply Lemma 2.1.4 in [8] to obtain (4.3). As argued above, this completes the proof of Theorem 1.2. □

References

[1] Yu. Baryshnikov, P. Eichelsbacher, T. Schreiber, and J. E. Yukich (2004), Moderate deviations and cluster measures in geometric probability, preprint, Electronically available via `http://www.lehigh.edu/~jey0/publications.html`.

[2] Yu. Baryshnikov and J. E. Yukich (2003), Gaussian fields and random packing, *Journal of Statistical Physics*, **111**, 443-463.

[3] Yu. Baryshnikov and J. E. Yukich (2005), Gaussian limits for random measures in geometric probability, *Annals Appl. Prob.*, **15**, 1A, 213-253, Electronically available via `http://www.lehigh.edu/~jey0/publications.html`.

[4] Yu. Baryshnikov and J. E. Yukich (2005), Gaussian fields and maximal points, preprint.

[5] S. N. Chiu and M. P. Quine (1997), Central limit theory for the number of seeds in a growth model in $\mathbb{R}^d$ with inhomogeneous Poisson arrivals, *Annals Appl. Prob.*, **7**, 802-814.

[6] E. G. Coffman, L. Flatto, P. Jelenković (2000), Interval packing: the vacant interval distribution *Annals of Appl. Prob.*, **10**, 240-257.

[7] A. Dembo and O. Zeitouni (1998), Large Deviations Techniques and Applications, Springer. Second Ed.

[8] J.-D. Deuschel and D. Stroock (1989), Large Deviations, Academic Press.

[9] P. Dupuis, R.S. Ellis (1997), A Weak Convergence Approach to the Theory of Large Deviations, John Wiley & Sons.

[10] A. Dvoretzky and H. Robbins (1964), On the "parking" problem,*MTA Mat Kut. Int. Kžl.*, (Publications of the Math. Res. Inst. of the Hungarian Academy of Sciences), **9**, 209-225.

[11] G. Grimmett (1999), Percolation, Grundlehren der mathematischen Wissenschaften 321, Springer.

[12] T.M. Liggett, R.H. Schonmann, A.M. Stacey (1997), Domination by product measures, *Ann. Probab.* **25**, 71-95.

[13] G.L. O'Brien, W. Verwaat (1991), Capacities, large deviations and loglog laws, in: *Stable processes and Related Topics*, **25** of *Progress in Probability*, 43-83, Birkhäuser, Boston.

[14] M. D. Penrose (2001) Limit theorems for monolayer ballistic deposition in the continuum, *J. Stat. Phys.*, **105**, 561-583.

[15] M.D. Penrose (2003), Random Geometric Graphs, Clarendon Press, Oxford.

[16] M.D. Penrose (2004), Multivariate spatial central limit theorems with applications to percolation and spatial graphs, Preprint, University of Bath. M.D. Penrose. Electronically available via `http://www.bath.ac.uk/math-sci/`

[17] M.D. Penrose and J.E. Yukich (2001), Central limit theorems for some graphs in computational geometry, *Ann. Appl. Probab.* **11**, 1005-1041.

[18] M.D. Penrose and J.E. Yukich (2002), Limit theory for random sequential packing and deposition, *Ann. Appl. Probab.* **12**, 272-301.

[19] M.D. Penrose and J.E. Yukich (2003), Weak laws of large numbers in geometric probability, *Ann. Appl. Probab.*, **13**, pp. 277-303.

[20] M.D. Penrose and J.E. Yukich (2004), Normal approximation in geometric probability, Isaac Newton Institute for Mathematical Sciences, preprint series NI04005-IGS, Electronically available via `http://www.lehigh.edu/~jey0/publications.html`.

[21] A.A. Pukhalskii (1991), On functional principle of large deviations, in: V. Saonov, T. Shevashidze eds., *New trends in probability and statistics*, pp. 198-218, VSP-Mokslas, Zeist, The Netherlands.

[22] A. Rényi (1958) On a one-dimensional random space-filling problem, *MTA Mat Kut. Int. Kžl.*, (Publications of the Math. Res. Inst. of the Hungarian Academy of Sciences) **3**, 109-127.

[23] T. Schreiber (2002), Large deviation principle for spatial density of local observables from Gibbs distributions, *Reports on Mathematical Physics*, **50**, 379-393.

[24] T. Seppäläinen and J. E. Yukich (2001), Large deviation principles for Euclidean functionals and other nearly additive processes, *Probability Theory and Related Fields* **120**, 309-345.

[25] G. R. Shorack and J. A. Wellner (1986), Emprirical Processes with Applications to Statistics, Wiley, New York.

Tomasz Schreiber, Faculty of Mathematics and Computer Science, Nicholas Copernicus University, Toruń, Poland: tomeks@mat.uni.torun.pl

J. E. Yukich, Department of Mathematics, Lehigh University, Bethlehem PA 18015, USA: joseph.yukich@lehigh.edu