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BROWNIAN LIMITS, LOCAL LIMITS AND VARIANCE ASYMPTOTICS FOR CONVEX HULLS IN THE BALL

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Dedicated to the memory of Tomasz Schreiber

Schreiber and Yukich [*Ann. Probab.* **36** (2008) 363–396] establish an asymptotic representation for random convex polytope geometry in the unit ball $\mathbb{B}^d, d \geq 2$, in terms of the general theory of stabilizing functionals of Poisson point processes as well as in terms of *generalized paraboloid growth processes*. This paper further exploits this connection, introducing also a dual object termed the *paraboloid hull process*. Via these growth processes we establish local functional limit theorems for the properly scaled radius-vector and support functions of convex polytopes generated by high-density Poisson samples. We show that direct methods lead to explicit asymptotic expressions for the fidis of the properly scaled radius-vector and support functions. Generalized paraboloid growth processes, coupled with general techniques of stabilization theory, yield Brownian sheet limits for the defect volume and mean width functionals. Finally we provide explicit variance asymptotics and central limit theorems for the k-face and intrinsic volume functionals.

1. Introduction. Let K be a smooth convex set in $\mathbb{R}^d$ of unit volume. Letting $\mathcal{P}_\lambda$ be a Poisson point process in $\mathbb{R}^d$ of intensity λ, we let K_λ be the convex hull of $K \cap \mathcal{P}_\lambda$. The random polytope K_λ, together with the analogous polytope K_n, obtained by considering n i.i.d. uniformly distributed points in K, are well-studied objects in stochastic geometry.

The study of the asymptotic behavior of the polytopes K_λ and K_n, as $\lambda \to \infty$ and $n \to \infty$, respectively, has a long history originating with the work of Rényi and Sulanke [23]. Letting $\mathbb{S}^{d-1}$ denote the unit sphere, the following functionals of K_λ have featured prominently:

- the volume $\mathrm{Vol}(K_\lambda)$ of K_λ, abbreviated as $V(K_\lambda)$;
- the number of k-dimensional faces of K_λ, denoted $f_k(K_\lambda), k \in \{0, 1, \dots, d-1\}$; in particular $f_0(K_\lambda)$ is the number of vertices of K_λ;

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- the mean width $W(K_\lambda)$ of K_λ;
- the distance between ∂K_λ and ∂K in the direction $u \in \mathbb{S}^{d-1}$, here denoted $r_\lambda(u), u \neq 0$;
- the distance between the boundary of the Voronoi flower, defined by $\mathcal{P}_\lambda$ and ∂K, in the direction $u \in \mathbb{S}^{d-1}$, here denoted $s_\lambda(u)$;
- the kth intrinsic volumes of K_λ, here denoted $V_k(K_\lambda), k \in \{1, \dots, d-1\}$.

The mean values of these functionals on general convex polytopes, as well as their counterparts for K_n, have been widely studied, and for a complete account we refer to the surveys of Affentranger [1], Buchta [6], Gruber [11], Reitzner [22], Schneider [25, 27] and Weil and Wieacker [34], together with Chapter 8.2 in Schneider and Weil [28]. There has been recent progress in establishing higher order and asymptotic normality results for these functionals, for various choices of K. We signal the important breakthroughs by Reitzner [21], Bárány and Reitzner [3], Bárány et al. [2], Pardon [14] and Vu [32, 33]. These results, together with those of Schreiber and Yukich [30], are difficult and technical, with proofs relying upon tools from convex geometry and probability, including martingales, concentration inequalities and Stein's method. When K is the unit radius d-dimensional ball $\mathbb{B}^d$ centered at the origin, Schreiber and Yukich [30] establish variance asymptotics for $f_0(K_\lambda)$ as $\lambda \to \infty$, but up to now little is known regarding explicit variance asymptotics for other functionals of K_λ.

This paper has the following goals. We first study two processes in *formal space–time* $\mathbb{R}^{d-1} \times \mathbb{R}_+$, one termed the *paraboloid growth process* and denoted by Ψ, and a second termed the *paraboloid hull process*, denoted by Φ. While the first process was introduced in [30], the second has apparently not been considered before. When $K = \mathbb{B}^d$, an embedding of convex sets into the space of continuous functions on $\mathbb{S}^{d-1}$, together with a re-scaling, show that these processes are naturally suited to the study of K_λ. Their spatial localization can be exploited to describe first and second order asymptotics of functionals of K_λ. Many of our main results, described as follows, are obtained via geometric properties of the processes Ψ and Φ. Our goals are as follows:

• Show that the distance between K_λ and $\partial\mathbb{B}^d$, upon re-scaling in a local regime, converges in law as $\lambda \to \infty$, to a continuous path stochastic process defined in terms of Φ, adding to Molchanov [13]; similarly, we show that the distance between $\partial\mathbb{B}^d$ and the Voronoi flower defined by $\mathcal{P}_\lambda$ converges in law to a continuous path stochastic process defined in terms of Ψ. In the two-dimensional case the fidis (finite-dimensional distributions) of these distances, when re-scaled, are shown to converge to the fidis of Ψ and Φ, whose description is given explicitly, adding to work of Hsing [12].

• Show, upon re-scaling in a global regime, that the suitably integrated local defect width and defect volume functionals, when considered as processes indexed by points in $\mathbb{R}^{d-1}$ mapped on $\partial\mathbb{B}^d$ via the exponential map, satisfy a functional central limit theorem, that is, converge in the space of continuous functions on

$\mathbb{R}^{d-1}$ to a Brownian sheet on the injectivity region of the exponential map, whose respective variance coefficients σ_W^2 and σ_V^2 are expressed in closed form in terms of Ψ and Φ. To the best of our knowledge, this connection between the geometry of random polytopes and Brownian sheets is new. In particular we show

$$\lim_{\lambda\to\infty} \lambda^{(d+3)/(d+1)} \operatorname{Var}[W(K_\lambda)] = \sigma_W^2 \tag{1.1}$$

and

$$\lim_{\lambda\to\infty} \lambda^{(d+3)/(d+1)} \operatorname{Var}[V(K_\lambda)] = \sigma_V^2. \tag{1.2}$$

This adds to Reitzner's central limit theorem (Theorem 1 of [21]) and his variance approximation $\operatorname{Var}[V(K_\lambda)] \approx \lambda^{-(d+3)/(d+1)}$ (Theorem 3 and Lemma 1 of [21]), both valid when K is an arbitrary smooth convex set. It also adds to Hsing [12], which is confined to the case $K = \mathbb{B}^2$.

• Establish central limit theorems and variance asymptotics for the number of k-dimensional faces of K_λ, showing for all $k \in \{0, 1, \dots, d-1\}$,

$$\lim_{\lambda\to\infty} \lambda^{-(d-1)/(d+1)} \operatorname{Var}[f_k(K_\lambda)] = \sigma_{f_k}^2, \tag{1.3}$$

where $\sigma_{f_k}^2$ is described in terms of the processes Ψ and Φ. This improves upon Reitzner (Lemma 2 of [21]), whose breakthrough paper showed $\operatorname{Var}[f_k(K_\lambda)] \approx \lambda^{(d-1)/(d+1)}$, and builds upon [30], which establishes (1.3) when $k = 0$.

• Establish central limit theorems and variance asymptotics for the intrinsic volumes $V_k(K_\lambda)$, establishing for all $k \in \{1, \dots, d-1\}$ that

$$\lim_{\lambda\to\infty} \lambda^{(d+3)/(d+1)} \operatorname{Var}[V_k(K_\lambda)] = \sigma_{V_k}^2, \tag{1.4}$$

where again $\sigma_{V_k}^2$ is described in terms of the processes Ψ and Φ. This adds to Bárány et al. (Theorem 1 of [2]), which shows $\operatorname{Var}[V_k(K_n)] \approx n^{-(d+3)/(d+1)}$.

Limits (1.1)–(1.4) resolve the issue of finding variance asymptotics for face functionals and intrinsic volumes, a long-standing problem put forth this way in the 1993 survey of Weil and Wieacker (page 1431 of [34]): "We finally emphasize that the results described so far give mean values hence first-order information on random sets and point processes… There are also some less geometric methods to obtain higher-order informations or distributions, but generally the determination of the variance, for example, is a major open problem."

These goals are stated in relatively simple terms, and yet they and the methods behind them suggest further objectives involving additional explanation. One of our chief objectives is to carefully define the growth processes Ψ and Φ and exhibit their geometric properties making them relevant to K_λ, including their localization in space, known as *stabilization*. The latter property is central to establishing variance asymptotics and the limit theory of functionals of K_λ. A second objective is to describe two natural scaling regimes, one suited for locally defined functionals of K_λ, and the other suited for the integrated characteristics of K_λ, namely the width and volume functionals. A third objective is to extend the afore-mentioned results to ones holding on the level of measures. In other words, functionals considered here are naturally associated with random measures, and we shall show variance asymptotics for such measures and also convergence of their fidis to those of a Gaussian process under suitable global scaling. We originally intended to restrict attention to convex hulls generated from Poisson points with intensity density λ, but realized that the methods easily extend to treat intensity densities decaying as a power of the distance to the boundary of the unit ball as given by (2.1) below, and so we shall include this more general case without further complication. These major objectives are discussed further in the next section.

The extension of the variance asymptotics (1.2) and (1.3) to smooth compact convex sets with a C^3 boundary of positive Gaussian curvature is nontrivial and is addressed in [7]. We expect that much of the limit theory described here can be "de-Poissonized," that is to say, extends to functionals of the polytope K_n. This extension involves challenging technical questions which we do not address here.

2. Basic functionals and their scaled versions. Given a locally finite subset $\mathcal{X}$ of $\mathbb{R}^d$, we denote by $\operatorname{conv}(\mathcal{X})$ the *convex hull* generated by $\mathcal{X}$. For a given compact convex set $K \subset \mathbb{R}^d$ containing the origin, we let $h_K : \mathbb{S}^{d-1} \to \mathbb{R}$ be the support function of K, that is to say, for all $u \in \mathbb{S}^{d-1}$, we let $h_K(u) := \sup\{\langle x, u\rangle, x \in K\}$. It is easily seen for $\mathcal{X} \subset \mathbb{R}^d$ and $u \in \mathbb{S}^{d-1}$ that

$$h_{\operatorname{conv}(\mathcal{X})}(u) = \sup\{h_{\{x\}}(u), x \in \mathcal{X}\} = \sup\{\langle x, u\rangle, x \in \mathcal{X}\}.$$

For $u \in \mathbb{S}^{d-1}$, the *radius-vector function* of K in the direction of u is given by

$$r_K(u) := \sup\{\varrho > 0, \varrho u \in K\}.$$

For $\lambda > 0$ and $\delta > 0$ we abuse notation and *henceforth denote by* $\mathcal{P}_\lambda := \mathcal{P}_\lambda(\delta)$ *the Poisson point process in* $\mathbb{B}^d$ *of intensity*

$$\lambda(1 - |x|)^\delta \, dx, \qquad x \in \mathbb{B}^d. \tag{2.1}$$

The parameter δ shall remain fixed throughout, and therefore we suppress mention of it. Further, abusing notation we put

$$K_\lambda := \operatorname{conv}(\mathcal{P}_\lambda).$$

The principal characteristics of K_λ studied here are the following functionals, the first two of which represent K_λ in terms of continuous functions on $\mathbb{S}^{d-1}$:

- The *defect support function.* For all $u \in \mathbb{S}^{d-1}$, we define

$$s_\lambda(u) := s(u, \mathcal{P}_\lambda), \tag{2.2}$$

where for $\mathcal{X} \subseteq \mathbb{B}^d$ we define $s(u, \mathcal{X}) := 1 - h_{\operatorname{conv}(\mathcal{X})}(u)$. In other words, $s_\lambda(u)$ is the defect support function of K_λ in the direction u. It is easily verified that

$s(u, \mathcal{X})$ is the distance in the direction u between the sphere $\mathbb{S}^{d-1}$ and the *Voronoi flower*

$$F(\mathcal{X}) := \bigcup_{x \in \mathcal{X}} B_d\left(\frac{x}{2}, \frac{|x|}{2}\right), \tag{2.3}$$

where for $x \in \mathbb{R}^d$ and $r > 0$ we let $B_d(x, r)$ denote the d-dimensional radius r ball centered at x.

• The *defect radius-vector function.* For all $u \in \mathbb{S}^{d-1}$, we define

$$r_\lambda(u) := r(u, \mathcal{P}_\lambda), \tag{2.4}$$

where for $\mathcal{X} \subseteq \mathbb{B}^d$ and $u \in \mathbb{S}^{d-1}$ we put $r(u, \mathcal{X}) := 1 - r_{\mathrm{conv}(\mathcal{X})}(u)$. Thus, $r_\lambda(u)$ is the distance in the direction u between $\mathbb{S}^{d-1}$ and K_λ. The convex hull K_λ contains the origin, except on a set of exponentially small probability as $\lambda \to \infty$, and thus for asymptotic purposes *we assume without loss of generality that K_λ always contains the origin*, and therefore the radius vector function $r_\lambda(\cdot)$ is well defined.

• The *numbers of k-faces*. Let $f_{k;\lambda} := f_k(K_\lambda)$, $k \in \{0, 1, \dots, d-1\}$, be the number of k-dimensional faces of K_λ. In particular, $f_{0;\lambda}$ and $f_{1;\lambda}$ are the number of vertices and edges, respectively. The spatial distribution of k-faces is captured by the k-face empirical measure (point process) $\mu_\lambda^{f_k}$ on $\mathbb{B}^d$ given by

$$\mu_\lambda^{f_k} := \sum_{f \in \mathcal{F}_k(K_\lambda)} \delta_{\mathrm{Top}(f)}. \tag{2.5}$$

Here $\mathcal{F}_k(K_\lambda)$ *is the collection of all k-faces of* K_λ and $\mathrm{Top}(f)$, $f \in \mathcal{F}_k(K_\lambda)$, is the point of f which is closest to $\mathbb{S}^{d-1}$, with ties ignored as they occur with probability zero (there are other conceivable choices for $\mathrm{Top}(f)$, but we find this one to be as good as any). The total mass $\mu_\lambda^{f_k}(\mathbb{B}^d)$ coincides with $f_{k;\lambda}$.

• *Projection avoidance functionals*. Representing intrinsic volumes of K_λ as the total masses of the corresponding curvature measures, while suitable in the local scaling regime, turns out to be less useful in the global scaling regime, as it leads to an asymptotically vanishing add-one cost for related stabilizing functionals, thus precluding normal use of stabilization theory. To overcome this problem, we shall use the following consequence of Crofton's general formula, usually going under the name of Kubota's formula; see (5.8) and (6.11) in [28]. We write

$$V_k(K_\lambda) = \frac{d!\kappa_d}{k!\kappa_k(d-k)!\kappa_{d-k}} \int_{G(d,k)} \mathrm{Vol}_k(K_\lambda | L)\, d\nu_k(L), \tag{2.6}$$

where $G(d, k)$ is the kth Grassmannian of $\mathbb{R}^d$, ν_k is the normalized Haar measure on $G(d, k)$ and $K_\lambda | L$ is the orthogonal projection of K_λ onto the k-dimensional linear space $L \in G(d, k)$. We shall only focus on the case $k \geq 1$ because for $k = 0$, we have $V_0(K_\lambda) \equiv 1$ for all nonempty, compact convex K_λ; see page 601 in [28]. Write

$$\int_{G(d,k)} \mathrm{Vol}_k(K_\lambda | L)\, d\nu_k(L) = \int_{G(d,k)} \int_L [1 - \vartheta_L(x, \mathcal{P}_\lambda)]\, dx\, d\nu_k(L),$$

where $\vartheta_L(x,\mathcal{X}) := \mathbf{1}(\{x \notin \operatorname{conv}(\mathcal{X})|L\})$. Putting $x = ru$, $u \in \mathbb{S}^{d-1}$, $r \in [0,1]$, this yields

$$\begin{aligned}
&\int_{G(d,k)} \operatorname{Vol}_k(K_\lambda|L)\, d\nu_k(L) \\
&\quad = \int_{G(d,k)} \int_{\mathbb{S}^{d-1}\cap L} \int_0^1 [1-\vartheta_L(ru,\mathcal{P}_\lambda)] r^{k-1}\, dr\, d\sigma_{k-1}(u)\, d\nu_k(L) \\
&\quad = \int_{G(d,k)} \int_{\mathbb{S}^{d-1}\cap L} \int_0^1 \frac{1}{r^{d-k}} [1-\vartheta_L(ru,\mathcal{P}_\lambda)] r^{d-1}\, dr\, d\sigma_{k-1}(u)\, d\nu_k(L).
\end{aligned}$$

Noting that $dx = r^{d-1}\, dr\, d\sigma_{d-1}(u)$ and interchanging the order of integration, we conclude, in view of the discussion on pages 590–591 of [28], that the considered expression equals

$$\frac{k\kappa_k}{d\kappa_d} \int_{\mathbb{B}^d} \frac{1}{|x|^{d-k}} \int_{G(\operatorname{lin}[x],k)} [1-\vartheta_L(ru,\mathcal{P}_\lambda)]\, d\nu_k^{\operatorname{lin}[x]}(L)\, dx,$$

where $\operatorname{lin}[x]$ is the 1-dimensional linear space spanned by x, $G(\operatorname{lin}[x],k)$ is the set of k-dimensional linear subspaces of $\mathbb{R}^d$ containing $\operatorname{lin}[x]$ and $\nu_k^{\operatorname{lin}[x]}$ is the corresponding normalized Haar measure; see [28]. Thus, putting

$$\vartheta_k^{\mathcal{X}}(x) := \int_{G(\operatorname{lin}[x],k)} \vartheta_L(x,\mathcal{X})\, d\nu_k^{\operatorname{lin}[x]}(L), \qquad x \in \mathbb{B}^d, x \neq 0, \tag{2.7}$$

and using (2.6), we are led to

$$\begin{aligned}
V_k(\mathbb{B}^d) - V_k(K_\lambda) &= \frac{\binom{d-1}{k-1}}{\kappa_{d-k}} \int_{\mathbb{B}^d} \frac{1}{|x|^{d-k}} \vartheta_k^{\mathcal{P}_\lambda}(x)\, dx \\
&= \frac{\binom{d-1}{k-1}}{\kappa_{d-k}} \int_{\mathbb{B}^d\setminus K_\lambda} \frac{1}{|x|^{d-k}} \vartheta_k^{\mathcal{P}_\lambda}(x)\, dx.
\end{aligned} \tag{2.8}$$

We will refer to $\vartheta_k^{\mathcal{P}_\lambda}$ as the *projection avoidance function* for K_λ.

The large λ asymptotics of the above characteristics of K_λ are studied in two natural scaling regimes, the *local* and the *global* one, as discussed below.

Local scaling regime and locally re-scaled functionals. The first scaling we consider is referred to as the *local scaling* in the sequel. It stems from the following observation, which, while considered before in [3], shall be discussed here in the context of stabilization of growth processes. If we consider the local behavior of functionals of K_λ in the vicinity of two fixed boundary points $u, u' \in \mathbb{S}^{d-1}$, with $\lambda \to \infty$, then these behaviors become asymptotically independent. Moreover, if $u' := u'(\lambda)$ approaches u slowly enough as $\lambda \to \infty$, the asymptotic independence is preserved. On the other hand, if the distance between u and $u' := u'(\lambda)$ decays rapidly enough, then both behaviors coincide for large λ, and the resulting picture is rather uninteresting. As in [30], it is therefore natural to ask for the frontier of these two asymptotic regimes and to expect that this corresponds to the natural *characteristic scale* between the observation directions u and u' where the crucial features of the local behavior of K_λ are revealed.

To render the characteristic scale as transparent as possible, we start with some simple yet important observations, which shall eventually lead to asymptotic independence of local convex hull geometries and which shall also suggest the proper scaling limits of convex hull statistics. For arbitrary points $x_1, \dots, x_k \in \mathbb{B}^d$, the support function of the convex hull $\operatorname{conv}(x_1, \dots, x_k)$ satisfies for all $u \in \mathbb{S}^{d-1}$, the relation

$$h_{\operatorname{conv}(x_1,\dots,x_k)}(u) = \max_{1 \le i \le k} h_{x_i}(u).$$

We make the fundamental observation that the epigraph of $s(u, \{x_i\}_{i=1}^k) := 1 - h_{\operatorname{conv}(x_1,\dots,x_k)}(u)$ *is thus the union of epigraphs which, locally near the apices, are of parabolic structure.* Any scaling transformation for K_λ on the characteristic scale must preserve this structure, as should the scaling limit for K_λ.

To determine the proper local scaling for our model, we consider the following intuitive argument. To obtain a nontrivial limit behavior we should re-scale K_λ in a neighborhood of $\mathbb{S}^{d-1}$, both in the $d-1$ surfacial (tangential) directions with factor λ^β and radial direction with factor λ^γ with suitable scaling exponents β and γ so that:

• The re-scaling compensates the intensity of $\mathcal{P}_\lambda$ with growth factor λ. In other words, a subset of $\mathbb{B}^d$ in the vicinity of $\mathbb{S}^{d-1}$, having a unit volume scaling image, should host on average $\Theta(1)$ points of the point process $\mathcal{P}_\lambda$. Since the integral of the intensity density (2.1) scales as $\lambda^{\beta(d-1)}$, with respect to the $d-1$ tangential directions, and since it scales as $\lambda^{\gamma(1+\delta)}$ with respect to the radial direction, where we take into account the integration over the radial coordinate, we are led to $\lambda^{\beta(d-1)+\gamma(1+\delta)} = \lambda$ and thus

$$\beta(d-1) + \gamma(1+\delta) = 1. \tag{2.9}$$

• The local behavior of the convex hull close to the boundary of $\mathbb{S}^{d-1}$, as described by the locally parabolic structure of s_λ, should preserve parabolic epigraphs, implying for $u \in \mathbb{S}^{d-1}$ that $(\lambda^\beta |u|)^2 = \lambda^\gamma |u|^2$, and thus

$$\gamma = 2\beta. \tag{2.10}$$

Solving the system (2.9), (2.10) we end up with the following *scaling exponents:*

$$\beta = \frac{1}{d+1+2\delta}, \qquad \gamma = 2\beta. \tag{2.11}$$

We next describe scaling transformations for K_λ. Fix $u_0 \in \mathbb{S}^{d-1}$, and let $T_{u_0} := T_{u_0}\mathbb{S}^{d-1}$ denote the tangent space to $\mathbb{S}^{d-1}$ at u_0. The *exponential map* $\exp_{u_0} : T_{u_0}\mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ maps a vector v of the tangent space T_{u_0} to the point $u \in \mathbb{S}^{d-1}$, such that u lies at the end of the geodesic of length $|v|$ starting at u_0 and having direction v. Note that $\mathbb{S}^{d-1}$ is geodesically complete in that the exponential map $\exp_{u_0}$ is well defined on the whole tangent space $\mathbb{R}^{d-1} \simeq T_{u_0}\mathbb{S}^{d-1}$, although it is injective only on $\{v \in T_{u_0}\mathbb{S}^{d-1}, |v| < \pi\}$. Instead of $\exp_{u_0}$, we shall write $\exp_{d-1}$ or simply exp, and we make the default choice $u_0 := (0, 0, \dots, 1)$. We use the isomorphism $T_{u_0}\mathbb{S}^{d-1} \simeq \mathbb{R}^{d-1}$ without further mention, and *we shall denote the closure of the injectivity region* $\{v \in T_{u_0}\mathbb{S}^{d-1}, |v| < \pi\}$ *of the exponential map simply by* $\mathbb{B}_{d-1}(\pi)$. Thus we have $\exp(\mathbb{B}_{d-1}(\pi)) = \mathbb{S}^{d-1}$.

Further, consider the following scaling transform T^λ mapping $\mathbb{B}^d$ into $\mathbb{R}^{d-1} \times \mathbb{R}_+$

$$T^\lambda(x) := \left(\lambda^\beta \exp_{d-1}^{-1}\left(\frac{x}{|x|}\right), \lambda^\gamma(1 - |x|)\right), \qquad x \in \mathbb{B}^d \setminus \{\mathbf{0}\}. \tag{2.12}$$

Here $\exp^{-1}(\cdot)$ is the inverse exponential map, which is well defined on $\mathbb{S}^{d-1} \setminus \{-u_0\}$ and which takes values in the injectivity region $\mathbb{B}_{d-1}(\pi)$. For formal completeness, on the "missing" point $-u_0$, we let $\exp^{-1}$ admit an arbitrary value, say $(0, 0, \dots, \pi)$, and likewise we put $T^\lambda(\mathbf{0}) := (\mathbf{0}, \lambda^\gamma)$, where $\mathbf{0}$ denotes either the origin of $\mathbb{R}^{d-1}$ or $\mathbb{R}^d$, according to the context. It is easily seen that T^λ is a.e. (with respect to Lebesgue measure on $\mathbb{B}^d$) a bijection from $\mathbb{B}^d$ onto the d-dimensional solid cylinders

$$\mathcal{R}_\lambda := \lambda^\beta \mathbb{B}_{d-1}(\pi) \times [0, \lambda^\gamma). \tag{2.13}$$

Throughout points in $\mathbb{B}^d$ are written as $x := (r, u)$, and we represent generic points in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ by (v, h), whereas we write (v', h') to represent points in the scaled region $\mathcal{R}_\lambda$. We assert that the transformation T^λ, defined at (2.12), maps the Poisson point process $\mathcal{P}_\lambda$ to $\mathcal{P}^{(\lambda)}$, where $\mathcal{P}^{(\lambda)}$ is the dilated Poisson point process in the region $\mathcal{R}_\lambda$ having intensity

$$(v', h') \mapsto \frac{\sin^{d-2}(\lambda^{-\beta}|v'|)}{|\lambda^{-\beta}v'|^{d-2}}(1 - \lambda^{-\gamma}h')^{d-1}h'^\delta \, dv' \, dh' \tag{2.14}$$

at $(v', h') \in \mathcal{R}_\lambda$. Indeed, this intensity measure is the image by the transformation T^λ of the measure on $\mathbb{B}^d$ given by

$$\lambda(1 - |x|)^\delta \, dx = \lambda(1 - r)^\delta r^{d-1} \, dr \, d\sigma_{d-1}(u) \tag{2.15}$$

introduced in (2.1), where we put $x = (r, u)$. To obtain (2.14), we first make a change of variables,

$$h' := \lambda^\gamma(1 - r) \quad \text{and} \quad v' := \lambda^\beta \exp_{d-1}^{-1}(u) := \lambda^\beta v.$$

Next, notice that the exponential map $\exp_{d-1} : T_{u_0}\mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ has the following expression:

$$\exp_{d-1}(v') = \cos(|v'|)(0, \dots, 0, 1) + \sin(|v'|)\left(\frac{v'}{|v'|}, 0\right), \tag{2.16}$$

with $v' \in \mathbb{R}^{d-1} \setminus \{\mathbf{0}\}$. Therefore, since $v := \exp_{d-1}^{-1}(u)$, we have

$$d\sigma_{d-1}(u) = \sin^{d-2}(|v|)d(|v|)\, d\sigma_{d-2}\left(\frac{v}{|v|}\right) = \frac{\sin^{d-2}(|v|)\, dv}{|v|^{d-2}}.$$

Since $v' = \lambda^{\beta} v$, this gives

$$d\sigma_{d-1}(u) = \frac{\sin^{d-2}(\lambda^{-\beta}|v'|)}{|\lambda^{-\beta} v'|^{d-2}} \lambda^{-\beta(d-1)}\, dv'. \tag{2.17}$$

We also have that

$$(1-r)^{\delta} r^{d-1}\, dr = \lambda^{-\gamma\delta} h'^{\delta} (1 - \lambda^{-\gamma} h')^{d-1} \lambda^{-\gamma}\, dh'. \tag{2.18}$$

Inserting (2.17) and (2.18) in (2.15) and using (2.9) to obtain $\lambda\lambda^{-\beta(d-1)}\lambda^{-\gamma(1+\delta)} = 1$, we obtain (2.14).

In Section 4, following [30], we shall embed $T^{\lambda}(K_{\lambda})$ into a space of paraboloid growth processes on $\mathcal{R}_{\lambda}$. One such process, denoted by $\Psi^{(\lambda)}$ and defined at (4.2), is a *generalized growth process with overlap* whereas the second, a dual process denoted by $\Phi^{(\lambda)}$ and defined at (4.8) is termed the *paraboloid hull process*. Infinite volume counterparts to $\Psi^{(\lambda)}$ and $\Phi^{(\lambda)}$, described fully in Section 3 and denoted by Ψ and Φ, respectively, play a natural role in describing the asymptotic behavior of our basic functionals of interest, re-scaled as follows:

• The *re-scaled versions of the defect support function (2.2) and the radius support function (2.4)*, defined, respectively, by

$$\hat{s}_{\lambda}(v) := \lambda^{\gamma} s_{\lambda}(\exp_{d-1}(\lambda^{-\beta} v)), \qquad v \in \mathbb{R}^{d-1}, \tag{2.19}$$

$$\hat{r}_{\lambda}(v) := \lambda^{\gamma} r_{\lambda}(\exp_{d-1}(\lambda^{-\beta} v)), \qquad v \in \mathbb{R}^{d-1}. \tag{2.20}$$

• The *re-scaled version of the projection avoidance function (2.7)* defined by

$$\hat{\vartheta}_k^{\mathcal{P}_{\lambda}}(x) := \vartheta_k^{\mathcal{P}_{\lambda}}([T^{\lambda}]^{-1}(x)), \qquad x \in \mathcal{R}_{\lambda}. \tag{2.21}$$

Global scaling regime and globally re-scaled functionals. The asymptotic independence of local convex hull geometries at distinct points of $\mathbb{S}^{d-1}$, as discussed above, suggests that the global behavior of both s_{λ} and r_{λ} is, in large λ asymptotics, that of the *white noise*. Therefore it is natural to consider the corresponding integral characteristics of K_{λ} and to ask whether, under proper scaling, they converge in law to a Brownian sheet. Define the processes

$$W_{\lambda}(v) := \int_{\exp([\mathbf{0},v])} s_{\lambda}(u)\, d\sigma_{d-1}(u), \qquad v \in \mathbb{R}^{d-1}, \tag{2.22}$$

and

$$V_{\lambda}(v) := \int_{\exp([\mathbf{0},v])} r_{\lambda}(u)\, d\sigma_{d-1}(u), \qquad v \in \mathbb{R}^{d-1}, \tag{2.23}$$

where the "segment" $[\mathbf{0}, v]$ for $v \in \mathbb{R}^{d-1}$ is the rectangular solid in $\mathbb{R}^{d-1}$ with vertices $\mathbf{0}$ and v, that is to say, $[\mathbf{0}, v] := \prod_{i=1}^{d-1}[\min(0, v^{(i)}), \max(0, v^{(i)})]$, with $v^{(i)}$ standing for the ith coordinate of v. We shall also consider the cumulative values

$$W_\lambda := W_\lambda(\infty) := \int_{\mathbb{S}^{d-1}} s_\lambda(u)\, d\sigma_{d-1}(u);$$
$$V_\lambda := V_\lambda(\infty) := \int_{\mathbb{S}^{d-1}} r_\lambda(u)\, d\sigma_{d-1}(u). \tag{2.24}$$

Notice that the radius-vector function of the Voronoi flower $F(\mathcal{P}_\lambda)$ coincides with the support function of K_λ. In particular, the volume outside $F(\mathcal{P}_\lambda)$ is equal to

$$\int_{\mathbb{S}^{d-1}} \left[\int_{1-s_\lambda(u)}^{1} \rho^{d-1}\, d\rho\right] d\sigma_{d-1}(u) = \int_{\mathbb{S}^{d-1}} \frac{1-(1-s_\lambda(u))^d}{d}\, d\sigma_{d-1}(u). \tag{2.25}$$

Since s_λ goes to 0 uniformly, the volume outside $F(\mathcal{P}_\lambda)$ is asymptotically equivalent to the integral of the defect support function, which in turn is proportional to the defect mean width by Cauchy's formula. Moreover, in two dimensions the mean width is the ratio of the perimeter to π (see page 210 of [26]), and so $W_\lambda(\infty)/\pi$ *coincides with* 2 *minus the mean width of* K_λ*, and consequently* $W_\lambda(\infty)$ *itself equals* 2π *minus the perimeter of* K_λ *for* $d = 2$. On the other hand, $V_\lambda(\infty)$ is asymptotic to the volume of $\mathbb{B}^d \setminus K_\lambda$, *whence the notation* W *for (asymptotic) width and* V *for (asymptotic) volume*.

To get the desired convergence to a Brownian sheet, we put

$$\zeta := \beta(d-1) + 2\gamma = \frac{d+3}{d+1+2\delta}; \tag{2.26}$$

we show in Section 8 that it is natural to re-scale the processes $(W_\lambda(v) - \mathbb{E}W_\lambda(v))$ and $(V_\lambda(v) - \mathbb{E}V_\lambda(v))$ by $\lambda^{\zeta/2}$ and that the resulting re-scaled processes

$$\hat{W}_\lambda(v) := \lambda^{\zeta/2}\big(W_\lambda(v) - \mathbb{E}W_\lambda(v)\big) \quad \text{and}$$
$$\hat{V}_\lambda(v) := \lambda^{\zeta/2}\big(V_\lambda(v) - \mathbb{E}V_\lambda(v)\big), \qquad v \in \mathbb{R}^{d-1}, \tag{2.27}$$

converge in law to a Brownian sheet with an explicit variance coefficient.

Putting the picture together. The remainder of this paper is organized as follows.

Section 3. Though the formulation of our results might suggest otherwise, there are crucial connections between the local and global scaling regimes. These regimes are linked by stabilization and the objective method, which together show that the behavior of locally defined processes on the finite volume rectangular solids $\mathcal{R}_\lambda$, defined at (2.13), can be well approximated by the local behavior of a related "candidate object," either *a generalized growth process* Ψ or a *dual paraboloid hull process* Φ, on an *infinite volume half-space*. While generalized growth processes were developed in [30] in a larger context, our limit theory depends heavily on a new object, the dual paraboloid hull process. The purpose of

Section 3 is to carefully define these processes and to establish properties relevant to their asymptotic analysis.

Section 4. We show that as $\lambda \to \infty$, both $\hat{s}_\lambda$ and $\hat{r}_\lambda$, defined, respectively, at (2.19) and (2.20), converge in law to continuous path stochastic processes explicitly constructed in terms of the paraboloid generalized growth process Ψ and the paraboloid hull process Φ, respectively. This adds to Molchanov [13], who considers the "epiconvergence" in the space $\mathbb{S}^{d-1} \times \mathbb{R}$ of the random process, arising as the binomial counterpart of λr_λ. Molchanov's results [13] are not framed in terms of the rescaled function $\hat{r}_\lambda$, and thus they do not involve the paraboloid growth processes described in this paper.

Section 5. When $d = 2$, after re-scaling in space by a factor of $\lambda^{1/3}$ and in time (height coordinate) by $\lambda^{2/3}$, we use nonasymptotic direct considerations to provide explicit asymptotic expressions for the fidis of $\hat{s}_\lambda$ and $\hat{r}_\lambda$ as $\lambda \to \infty$. These distributions coincide with the fidis of the parabolic growth process Ψ and the parabolic hull process Φ, respectively.

Section 6. Both the paraboloid growth process Ψ and its dual paraboloid hull process Φ are shown to enjoy a localization property, which expresses, in geometric terms, a type of spatial mixing. This provides a direct route toward establishing first and second order asymptotics for the convex hull functionals of interest.

Section 7. This section establishes closed form variance asymptotics for the total number of k-faces as well as the intrinsic volumes for the random polytope K_λ. We also establish variance asymptotics and a central limit theorem for the properly scaled integrals of continuous test functions against the empirical measures associated with the functionals under proper scaling.

Section 8. Using the stabilization properties established in Section 6, we establish a functional central limit theorem for $\hat{W}_\lambda$ and $\hat{V}_\lambda$, showing that these processes converge, as $\lambda \to \infty$ in the space of continuous functions on $\mathbb{R}^{d-1}$, to Brownian sheets with variance coefficients given in terms of the processes Ψ and Φ, respectively.

3. Paraboloid growth and hull processes. In this section we introduce the paraboloid *growth* and *hull* processes in the upper half-space $\mathbb{R}^{d-1} \times \mathbb{R}_+$ often interpreted as formal *space–time* below, with $\mathbb{R}^{d-1}$ standing for the spatial dimension and $\mathbb{R}_+$ standing for the time dimension. Although this interpretation is purely formal in the convex hull set-up, it provides a link to a well-established theory of growth processes studied by means of stabilization theory; see below for further details. These processes turn out to be infinite volume counterparts to finite volume paraboloid growth processes, which are defined in the next section, and which are used to describe the behavior of our basic re-scaled functionals and measures.

Poisson point process on half-spaces. Fix $\delta > 0$, and let $\mathcal{P}(\delta)$ be a Poisson point process in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ with intensity density

$$h^\delta \, dh \, dv \qquad \text{at } (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+. \tag{3.1}$$

In the sequel we shall show that the scaled Poisson point process $\mathcal{P}^{(\lambda)} := T^{\lambda}(\mathcal{P}_{\lambda})$ with intensity defined at (2.14) converges to $\mathcal{P}(\delta)$ on compacts, but for now we use the process $\mathcal{P}$ to define growth processes on half-spaces. As with $\mathcal{P}_{\lambda}$ and $\mathcal{P}^{(\lambda)}$, we suppress δ and simply write $\mathcal{P}$ for $\mathcal{P}(\delta)$.

Paraboloid growth processes on half-spaces. We introduce the *paraboloid generalized growth process with overlap* (paraboloid growth process for short), specializing to our present set-up the corresponding general concept defined in Section 1.1 of [30] and designed to constitute the asymptotic counterpart of the Voronoi flower $F(K_{\lambda})$. Let $\Pi^{\uparrow}$ be the epigraph of the standard paraboloid $v \mapsto |v|^2/2$, that is,

$$\Pi^{\uparrow} := \left\{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_{+}, h \geq \frac{|v|^2}{2}\right\}.$$

We introduce one of the fundamental objects of this paper.

DEFINITION 3.1. Given a locally finite point set $\mathcal{X}$ in $\mathbb{R}^{d-1} \times \mathbb{R}_{+}$, the paraboloid growth model $\Psi(\mathcal{X})$ is defined as the Boolean model with paraboloid grain $\Pi^{\uparrow}$ and with germ collection $\mathcal{X}$, namely

$$\Psi(\mathcal{X}) := \mathcal{X} \oplus \Pi^{\uparrow} = \bigcup_{x \in \mathcal{X}} x \oplus \Pi^{\uparrow}, \tag{3.2}$$

where $\oplus$ stands for Minkowski addition. In particular, we define the paraboloid growth process $\Psi := \Psi(\mathcal{P})$, where $\mathcal{P}$ is the Poisson point process defined at (3.1).

The model $\Psi(\mathcal{X})$ arises as the union of upwards paraboloids with apices at the points of $\mathcal{X}$ (see Figure 1), in close analogy to the Voronoi flower $F(\mathcal{X})$, where to each $x \in \mathcal{X}$ we attach a ball $B_d(x/2, |x|/2)$ (which asymptotically scales to an upwards paraboloid as we shall see in the sequel) and take the union thereof.

The name *generalized growth process with overlap* comes from the original interpretation of this construction [30], where $\mathbb{R}^{d-1} \times \mathbb{R}_{+}$ stands for *space–time* with $\mathbb{R}^{d-1}$ corresponding to the *spatial* coordinates and the semi-axis $\mathbb{R}_{+}$ corresponding to the *time* (*or height*) coordinate, and where the grain $\Pi^{\uparrow}$, possibly admitting

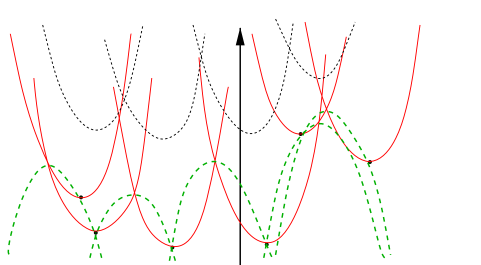

FIG. 1. *Example of paraboloid and growth processes for $d = 2$.*

more general shapes as well, arises as the graph of the growth of a germ born at the apex of $\Pi^{\uparrow}$ and growing thereupon in time with properly varying speed. We say that the process *admits overlaps* because the growth does not stop when two grains overlap, unlike in traditional growth schemes. We shall often use this space–time interpretation and refer to the respective coordinate axes as to the spatial and time (height) axis.

The boundary $\partial\Psi$ of the random closed set $\Psi := \Psi(\mathcal{P})$ constitutes a graph of a continuous function from $\mathbb{R}^{d-1}$ (space) to $\mathbb{R}_+$ (time), also denoted by $\partial\Psi$ in the sequel. In what follows we interpret $\hat{s}_\lambda$, defined at (2.19), as the boundary of a growth process $\Psi^{(\lambda)}$, defined at (4.2) below, on the finite region $\mathcal{R}_\lambda$ at (2.13); we shall see in Section 4 that $\partial\Psi$ is the scaling limit for the boundary of $\Psi^{(\lambda)}$.

A germ point $x \in \mathcal{P}$ is called *extreme* in the paraboloid growth process Ψ if and only if its associated epigraph $x \oplus \Pi^{\uparrow}$ is *not* contained in the union of the paraboloid epigraphs generated by other germ points in $\mathcal{P}$, that is to say,

$$x \oplus \Pi^{\uparrow} \not\subseteq \bigcup_{y\in\mathcal{P}, y\neq x} (y \oplus \Pi^{\uparrow}). \tag{3.3}$$

For x to be extreme, it is sufficient, but not necessary, that x fails to be contained in paraboloid epigraphs of other germs. Write $\mathrm{ext}(\Psi)$ for the set of all extreme points.

Paraboloid hull process on half-spaces. The *paraboloid hull process* Φ can be regarded as the dual to the paraboloid growth process. At the same time, the paraboloid hull process is designed to exhibit geometric properties analogous to those of convex polytopes with *paraboloids* playing the role of *hyperplanes*, with the *spatial coordinates* playing the role of *spherical coordinates* and with the *height/time coordinate* playing the role of the *radial coordinate*. The motivation of this construction is to mimic the convex geometry on second order paraboloid structures in order to describe the local second order geometry of convex polytopes, which dominates their limit behavior in smooth convex bodies. As we shall see, this intuition is indeed correct and results in a detailed description of the limit behavior of K_λ.

To proceed with our definitions, we let $\Pi^{\downarrow}$ be the downwards space–time paraboloid hypograph

$$\Pi^{\downarrow} := \left\{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R}, h \leq -\frac{|v|^2}{2}\right\}. \tag{3.4}$$

The idea behind our interpretation of the paraboloid process is that the shifts of $\Pi^{\downarrow}$ correspond to half-spaces not containing $\mathbf{0}$ in the Euclidean space $\mathbb{R}^d$. We shall argue the *paraboloid convex sets* have properties strongly analogous to those related to the usual concept of convexity. The corresponding proofs are not difficult and will be presented in enough detail to make our presentation self-contained, but it should be emphasized that alternatively the entire argument of this paragraph could be re-written in terms of the following *trick*. Considering the transform $(v, h) \mapsto (v, h + |v|^2/2)$, we see that it maps translates of $\Pi^{\downarrow}$ to half-spaces and thus whenever we make a statement below in terms of paraboloids and claim it is analogous to a standard statement of convex geometry, we can alternatively apply the above auxiliary transform, use the classical result and then transform back to our set-up. We do not choose this option here, finding it more aesthetic to work directly in the paraboloid set-up, but we indicate at this point the availability of this alternative.

The next definitions are central to the description of the paraboloid hull process. Recall that the affine hull $\operatorname{aff}[v_1, \dots, v_k]$ is the set of all affine combinations $\alpha_1 v_1 + \dots + \alpha_k v_k, \alpha_1 + \dots + \alpha_k = 1, \alpha_i \in \mathbb{R}$.

DEFINITION 3.2. For any collection $x_1 := (v_1, h_1), \dots, x_k := (v_k, h_k), k \le d$, of points in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ with affinely independent spatial coordinates v_i, we define $\Pi^{\downarrow}[x_1, \dots, x_k]$ to be the hypograph in $\operatorname{aff}[v_1, \dots, v_k] \times \mathbb{R}$ of the unique space–time paraboloid in the affine space $\operatorname{aff}[v_1, \dots, v_k] \times \mathbb{R}$ with quadratic coefficient $-1/2$ and passing through $x_1, \dots, x_k$.

In other words $\Pi^{\downarrow}[x_1, \dots, x_k]$ is the intersection of $\operatorname{aff}[v_1, \dots, v_k] \times \mathbb{R}$ and a translate of $\Pi^{\downarrow}$ having all $x_1, \dots, x_k$ on its boundary; while such translates are nonunique for $k < d$, their intersections with $\operatorname{aff}[v_1, \dots, v_k]$ all coincide.

DEFINITION 3.3. For $x_1 := (v_1, h_1) \neq x_2 := (v_2, h_2) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$, the *parabolic segment* $\Pi^{[\cdot]}[x_1, x_2]$ is the unique parabolic segment with quadratic coefficient $-1/2$ joining x_1 to x_2 in $\operatorname{aff}[v_1, v_2] \times \mathbb{R}$. More generally, for any collection $x_1 := (v_1, h_1), \dots, x_k := (v_k, h_k), k \le d$, of points in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ with affinely independent spatial coordinates, we define the *paraboloid face* $\Pi^{[\cdot]}[x_1, \dots, x_k]$ by

$$\Pi^{[\cdot]}[x_1, \dots, x_k] := \partial \Pi^{\downarrow}[x_1, \dots, x_k] \cap [\operatorname{conv}(v_1, \dots, v_k) \times \mathbb{R}]. \tag{3.5}$$

Clearly, $\Pi^{[\cdot]}[x_1, \dots, x_k]$ is the smallest set containing $x_1, \dots, x_k$ and with the *paraboloid convexity* property: For any two y_1, y_2 it contains, it also contains $\Pi^{[\cdot]}[y_1, y_2]$. In these terms, $\Pi^{[\cdot]}[x_1, \dots, x_k]$ is the *paraboloid convex hull* p-hull$(\{x_1, \dots, x_k\})$. In particular, we readily derive the property

$$\begin{aligned} &\Pi^{[\cdot]}[x_1, \dots, x_i, \dots, x_k] \cap \Pi^{[\cdot]}[x_i, \dots, x_k, \dots, x_m] \\ &\quad = \Pi^{[\cdot]}[x_i, \dots, x_k], \qquad 1 < i < k. \end{aligned} \tag{3.6}$$

Next, we say that $A \subseteq \mathbb{R}^{d-1} \times \mathbb{R}_+$ is *upwards paraboloid convex* (up-convex for short) if and only if:

- for each two $x_1, x_2 \in A$ we have $\Pi^{[\cdot]}[x_1, x_2] \subseteq A$;
- and for each $x = (v, h) \in A$ we have $x^{\uparrow} := \{(v, h'), h' \ge h\} \subseteq A$.

Whereas the first condition in the definition above is quite intuitive, the second will be seen to correspond to our requirement that $\mathbf{0} \in K_\lambda$ as $\mathbf{0}$ gets transformed to *upper infinity* in the limit of our re-scalings. Indeed, though T^λ is not defined at $x = \mathbf{0}$, the last coordinate of $T^\lambda(x)$ goes to λ^γ when $x \to \mathbf{0}$, and λ^γ goes to ∞ when $\lambda \to \infty$.

With the notation introduced above, we now define the second fundamental object of this paper.

DEFINITION 3.4. Given $A \subseteq \mathbb{R}^{d-1} \times \mathbb{R}_+$, by the paraboloid hull (up-hull for short) of A, we mean the smallest up-convex set containing A. Given a locally finite point set $\mathcal{X} \in \mathbb{R}^{d-1} \times \mathbb{R}_+$, we define the paraboloid hull $\Phi(\mathcal{X})$ to be the up-hull of $\mathcal{X}$, that is,

$$\Phi(\mathcal{X}) := \text{up-hull}(\mathcal{X}).$$

In particular, we define the paraboloid hull process Φ in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ as the up-hull of $\mathcal{P}$, that is to say,

$$\Phi := \Phi(\mathcal{P}) := \text{up-hull}(\mathcal{P}). \tag{3.7}$$

For $A \subseteq \mathbb{R}^{d-1} \times \mathbb{R}_+$ we put $A^\uparrow := \{(v, h'), (v, h) \in A \text{ for some } h \leq h'\}$ and observe that if $x_1' \in x_1^\uparrow, x_2' \in x_2^\uparrow$, then

$$\Pi^{[\cdot]}[x_1', x_2'] \subset [\Pi^{[\cdot]}[x_1, x_2]]^\uparrow \tag{3.8}$$

and, more generally, by definition of $\Pi^{[\cdot]}[x_1, \dots, x_k]$ and by induction in k, $\Pi^{[\cdot]}[x_1', \dots, x_k'] \subset [\Pi^{[\cdot]}[x_1, \dots, x_k]]^\uparrow$. Consequently, we conclude that

$$\Phi = [\text{p-hull}(\mathcal{P})]^\uparrow, \tag{3.9}$$

which, in terms of our analogy between convex polytopes and paraboloid hulls processes, reduces to the trivial statement that a convex polytope containing $\mathbf{0}$ arises as the union of radial segments joining $\mathbf{0}$ to convex combinations of its vertices. This statement is somewhat more interesting in the present set-up where $\mathbf{0}$ *disappears* at infinity, and we formulate it here for further use.

LEMMA 3.1. *With probability* 1 *we have*

$$\Phi = \bigcup_{\{x_1, \dots, x_d\} \subset \mathcal{P}} [\Pi^{[\cdot]}[x_1, \dots, x_d]]^\uparrow. \tag{3.10}$$

This statement corresponds to the property of d-dimensional polytopes containing $\mathbf{0}$, stating that the convex hull of a collection of points containing $\mathbf{0}$ is the union of all d-dimensional simplices with vertex sets running over all cardinality $(d + 1)$ sub-collections of the generating collection which contain $\mathbf{0}$. Subsets

$\{x_1, \dots, x_d\} \subset \mathcal{P}$ have their spatial coordinates affinely independent with probability 1 and thus the right-hand side in (3.10) is a.s. well defined; in the sequel we shall say that points of $\mathcal{P}$ are a.s. *in general position*.

PROOF. Observe that, in view of (3.9) and the fact that

$$\bigcup_{\{x_1,\dots,x_d\}\subset\mathcal{P}} \Pi^{[\cdot]}[x_1, \dots, x_d] \subset \text{p-hull}(\mathcal{P}),$$

(3.10) will follow as soon as we show that

$$\text{p-hull}(\mathcal{P}) \subset \bigcup_{\{x_1,\dots,x_d\}\subset\mathcal{P}} [\Pi^{[\cdot]}[x_1, \dots, x_d]]^{\uparrow}. \tag{3.11}$$

To establish (3.11) it suffices to show that adding an extra point x_{d+1} in general position to a set $\bar{x} = \{x_1, \dots, x_d\}$ results in having

$$\text{p-hull}(\bar{x}^+ := \bar{x} \cup \{x_{d+1}\}) \subset \bigcup_{i=1}^{d+1} [\Pi^{[\cdot]}[\bar{x}^+ \setminus \{x_i\}]]^{\uparrow}, \tag{3.12}$$

and inductive use of this fact readily yields the required relation (3.11). To verify (3.12) choose $y := (v, h) \in \text{p-hull}(\bar{x}^+)$. Then there exists $y' = (v', h') \in \Pi^{[\cdot]}[x_1, \dots, x_d]$ such that $y \in \Pi^{[\cdot]}[y', x_{d+1}]$. Consider the section of $\Pi^{[\cdot]}[x_1, \dots, x_d]$ by the plane $\text{aff}[v', v_{d+1}] \times \mathbb{R}$ and y'' be its point with the lowest height coordinate. Clearly then there exists x_i, $i \in \{1, \dots, d\}$ such that $y'' \in \Pi^{[\cdot]}[\bar{x} \setminus \{x_i\}]$. On the other hand, by the choice of y'' and by (3.8), $y \in \Pi^{[\cdot]}[y', x_{d+1}] \subset [\Pi^{[\cdot]}[y', y'']]^{\uparrow} \cup [\Pi^{[\cdot]}[y'', x_{d+1}]]^{\uparrow}$. Consequently, $y \in [\Pi^{[\cdot]}[\bar{x}]]^{\uparrow} \cup [\Pi^{[\cdot]}[\bar{x}^+ \setminus \{x_i\}]]^{\uparrow}$, which completes the proof of (3.12) and thus also of (3.11) and (3.10). This completes the proof of Lemma 3.1. □

To formulate our next statement, we say that a collection $\{x_1, \dots, x_d\}$ is *extreme* in $\mathcal{P}$ if and only if $\Pi^{[\cdot]}[x_1, \dots, x_d] \subset \partial\Phi$. Note that, by (3.8) and Lemma 3.1, this is equivalent to having

$$\Phi \cap \Pi^{\downarrow}[x_1, \dots, x_d] = \Pi^{[\cdot]}[x_1, \dots, x_d]. \tag{3.13}$$

Each such $\Pi^{[\cdot]}[x_1, \dots, x_d]$ is referred to as a *paraboloid sub-face*. Further, say that two extreme collections $\{x_1, \dots, x_d\}$ and $\{x'_1, \dots, x'_d\}$ in $\mathcal{P}$ are co-paraboloid if and only if $\Pi^{\downarrow}[x_1, \dots, x_d] = \Pi^{\downarrow}[x'_1, \dots, x'_d]$. By a $(d-1)$-*dimensional paraboloid face* of Φ, we shall understand the union of each maximal collection of co-paraboloid sub-faces. Clearly, these correspond to $(d-1)$-dimensional faces of convex polytopes. It is not difficult to check that $(d-1)$-dimensional paraboloid faces of Φ are p-convex, and their union is $\partial\Phi$. In fact, since $\mathcal{P}$ is a Poisson process, with probability one all $(d-1)$-dimensional faces of Φ consist of precisely one sub-face; in particular all $(d-1)$-dimensional faces of Φ are bounded. By (3.13) we have for each $(d-1)$-dimensional face f,

$$\Phi \cap \Pi^{\downarrow}[f] = f, \tag{3.14}$$

which corresponds to the standard fact of the theory of convex polytopes, stating that the intersection of a d-dimensional polytope containing $\mathbf{0}$ with a half-space determined by a $(d-1)$-dimensional face, and looking away from $\mathbf{0}$, is precisely the face itself. Further, pairs of adjacent $(d-1)$-dimensional paraboloid faces intersect yielding $(d-2)$-dimensional paraboloid manifolds, called $(d-2)$-dimensional paraboloid faces. More generally, $(d-k)$-dimensional paraboloid faces arise as $(d-k)$-dimensional paraboloid manifolds obtained by intersecting suitable k-tuples of adjacent $(d-1)$-dimensional faces. Finally, we end up with zero dimensional faces, which are the *vertices* of Φ, and which are easily seen to belong to $\mathcal{P}$. The set of vertices of Φ is denoted by $\mathrm{Vertices}(\Phi)$. In other words, we obtain a full analogy with the geometry of faces of d-dimensional polytopes. Clearly, $\partial\Phi$ is the graph of a continuous piecewise paraboloid function from $\mathbb{R}^{d-1}$ to $\mathbb{R}$.

As a consequence of the above description of the geometry of Φ in terms of its faces, particularly (3.14), we conclude that

$$\Phi = \mathrm{cl}\biggl(\biggl[\bigcup_{f\in\mathcal{F}_{d-1}(\Phi)} \Pi^{\downarrow}[f]\biggr]^c\biggr) = \bigcap_{f\in\mathcal{F}_{d-1}(\Phi)} \mathrm{cl}([\Pi^{\downarrow}[f]]^c), \tag{3.15}$$

with $\mathrm{cl}(\cdot)$ standing for the topological closure, and with $(\cdot)^c$ denoting the complement in $\mathbb{R}^{d-1}\times\mathbb{R}_+$. This is the parabolic counterpart to the standard fact that a convex polytope is the intersection of closed half-spaces determined by its $(d-1)$-dimensional faces and containing $\mathbf{0}$. From (3.15) it follows that for each point $x \notin \Phi$, there exists a translate of $\Pi^{\downarrow}$ containing x, but not intersecting Φ, hence in particular not intersecting $\mathcal{P}$, which is the paraboloid version of the standard separation lemma of convex geometry. On the other hand, if x is contained in a translate of $\Pi^{\downarrow}$ not hitting $\mathcal{P}$, then $x \notin \Phi$. Consequently

$$\begin{aligned}\Phi &= \biggl[\bigcup_{x\in\mathbb{R}^{d-1}\times\mathbb{R}_+,[x\oplus\Pi^{\downarrow}]\cap\mathcal{P}=\varnothing} x\oplus\Pi^{\downarrow}\biggr]^c\\ &= \bigcap_{x\in\mathbb{R}^{d-1}\times\mathbb{R}_+,[x\oplus\Pi^{\downarrow}]\cap\mathcal{P}=\varnothing} [x\oplus\Pi^{\downarrow}]^c.\end{aligned} \tag{3.16}$$

Alternatively, Φ arises as the complement of the morphological opening of $\mathbb{R}^{d-1}\times\mathbb{R}_+\setminus\mathcal{P}$ with downwards paraboloid structuring element $\Pi^{\downarrow}$, that is to say,

$$\Phi^c = [\mathcal{P}^c \ominus \Pi^{\downarrow}] \oplus \Pi^{\downarrow}$$

with $\ominus$ standing for Minkowski erosion. In intuitive terms this means that the complement of Φ is obtained by trying to *fill* $\mathbb{R}^{d-1}\times\mathbb{R}_+$ with downwards paraboloids $\Pi^{\downarrow}$ forbidden to hit any of the Poisson points in $\mathcal{P}$—the random open set obtained as the union of such paraboloids is precisely Φ^c.

To link the paraboloid hull and growth processes, note that a point $x\in\mathcal{P}$ is a vertex of Φ if and only if $x\notin \text{up-hull}(\mathcal{P}\setminus\{x\})$. By (3.16) this means that $x\in \mathrm{Vertices}(\Phi)$ if and only if there exists y such that $[y\oplus\Pi^{\downarrow}]\cap\mathcal{P}=\{x\}$ and, since

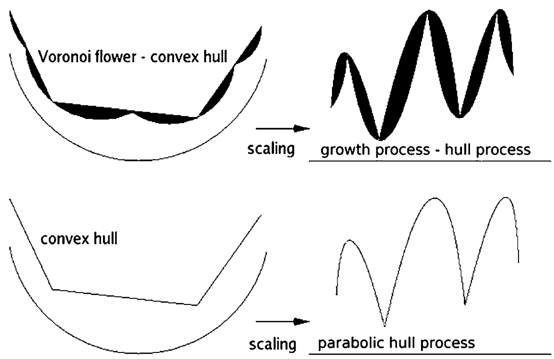

FIG. 2. *Convex hull, Voronoi flower and their scaling limits.*

the set of y such that $y \oplus \Pi^{\downarrow} \ni x$ is simply $x \oplus \Pi^{\uparrow}$, this condition is equivalent to having $x \oplus \Pi^{\uparrow}$ not entirely contained in $[\mathcal{P} \setminus \{x\}] \oplus \Pi^{\uparrow}$. In view of (3.3) this means that

$$\mathrm{ext}(\Psi) = \mathrm{Vertices}(\Phi). \tag{3.17}$$

The theory developed in this section admits a particularly simple form when $d = 2$. To see it, say that two points $x, y \in \mathrm{ext}(\Psi)$ are neighbors in Ψ, with notation $x \sim_{\Psi} y$ or simply $x \sim y$, if and only if there is no point in $\mathrm{ext}(\Psi)$ with its spatial coordinate between those of x and y. Then $\mathrm{Vertices}(\Phi) = \mathrm{ext}(\Psi)$ as in the general case, and $\mathcal{F}_1(\Phi) = \{\Pi^{[\cdot]}[x, y], x \sim y \in \mathrm{ext}(\Psi)\}$. In this context it is also particularly easy to display the relationships between the parabolic growth process Ψ and the parabolic hull process Φ in terms of the analogous relations between the convex hull K_λ and the Voronoi flower $F(\mathcal{P}_\lambda)$ upon the transformation T^λ at (2.12) in large λ asymptotics. To this end, see Figure 2 and note that, in large λ asymptotics, we have:

- The extreme points in Ψ, coinciding with $\mathrm{Vertices}(\Phi)$, correspond to the vertices of K_λ.
- Two points $x, y \in \mathrm{ext}(\Psi)$ are neighbors $x \sim y$ if and only if the corresponding vertices of K_λ are adjacent, that is to say, connected by an edge of ∂K_λ.
- The circles $S^1(x/2, |x|/2)$ and $S^1(y/2, |y|/2)$ of two adjacent vertices x, y of K_λ, whose pieces mark the boundary of the Voronoi flower $F(\mathcal{P}_\lambda)$, are easily seen to have their unique nonzero intersection point z collinear with x and y. Moreover, z minimizes the distance to 0 among the points on the $\overline{xy}$-line and $\overline{xy} \perp \overline{0z}$. For the parabolic processes this is reflected by the fact that the intersection point of two upwards parabolae with apices at two neighboring points x and y of $\mathrm{Vertices}(\Phi) = \mathrm{ext}(\Psi)$ coincides with the apex of the downwards parabola $\Pi^{\downarrow}[x, y]$ as readily verified by a direct calculation.
- Finally, relation (3.15) becomes here $\Phi = \bigcap_{x \sim y \in \mathrm{ext}(\Psi)} \mathrm{cl}([\Pi^{\downarrow}[x, y]]^c)$ which is reflected by the fact that K_λ coincides with the intersection of all closed half-spaces containing $\mathbf{0}$ determined by segments of the convex hull boundary ∂K_λ.

We conclude this paragraph by defining the paraboloid *avoidance function* $\hat{\vartheta}_k^{\infty}, k \in \{1, 2, \dots, d\}$. To this end, for each $x := (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ let $x^{\updownarrow} := \{(v, h'), h' \in \mathbb{R}\}$ be the infinite vertical ray (line) determined by x, and let $A(x^{\updownarrow}, k)$ be the collection of all k-dimensional affine spaces in $\mathbb{R}^d$ containing $x^{\updownarrow}$, regarded as the asymptotic equivalent of the restricted Grassmannian $G(\mathrm{lin}[x], k)$ considered in the definition (2.7) of the nonrescaled function $\vartheta_k^{\mathcal{P}_\lambda}$. Next, for $L \in A(x^{\updownarrow}, k)$ we define the orthogonal paraboloid surface $\Pi^{\perp}[x; L]$ to L at x given by

$$\Pi^{\perp}[x; L] := \left\{x' = (v', h') \in \mathbb{R}^{d-1} \times \mathbb{R}, (x - x') \perp L, h' = h - \frac{d(x, x')^2}{2}\right\}. \tag{3.18}$$

Note that this is an analog of the usual orthogonal affine space $L^{\perp} + x$ to L at x, with the second order parabolic correction typical in our asymptotic setting—recall that nonradial hyperplanes get asymptotically transformed onto downwards paraboloids. Further, for $L \in A(x^{\updownarrow}, k)$, we put

$$\vartheta_L^{\infty}(x) := \mathbf{1}(\{\Pi^{\perp}[x; L] \cap \Phi = \varnothing\}).$$

Observe that this is a direct analog of $\vartheta_L(x, \mathcal{P}_\lambda)$, assuming the value 1 precisely when $x \notin K_\lambda | L \Leftrightarrow [L^{\perp} + x] \cap K_\lambda = \varnothing$. Finally, in full analogy to (2.7) set

$$\vartheta_k^{\infty}(x) := \int_{A(x^{\updownarrow}, k)} \vartheta_L^{\infty}(x) \, d\mu_k^{x^{\updownarrow}}(L) \tag{3.19}$$

with $\mu_k^{x^{\updownarrow}}$ standing for the normalized Haar measure on $A(x^{\updownarrow}, k)$; see page 591 in [28].

Duality relations between growth and hull processes. As already signaled, there are close relationships between the paraboloid growth and hull processes, which we refer to as *duality*. Here we discuss these connections in more detail. The first observation is that

$$\Psi = \Phi \oplus \Pi^{\uparrow} = \mathrm{Vertices}(\Phi) \oplus \Pi^{\uparrow}. \tag{3.20}$$

This is verified either directly by the construction of Φ and Ψ, or, less directly but more instructively, by using the fact, established in detail in Section 4 below, that Φ arises as the scaling limit of K_λ, whereas Ψ is the scaling limit of the Voronoi flower

$$F(\mathcal{P}_\lambda) = \bigcup_{x \in \mathcal{P}_\lambda} B_d\left(\frac{x}{2}, \frac{|x|}{2}\right) = \bigcup_{x \in \mathrm{Vertices}(K_\lambda)} B_d\left(\frac{x}{2}, \frac{|x|}{2}\right),$$

defined at (2.3) and then by noting that the balls $B_d(x/2, |x|/2)$ asymptotically either scale into upward paraboloids or they "disappear at infinity"; see the proof of Theorem 4.1 below, and recall that the support function of K_λ coincides with the radius-vector function of $F(K_\lambda)$ as soon as $\mathbf{0} \in K_\lambda$ (which, recall, happens with overwhelming probability). Thus, it is straightforward to transform Φ into Ψ. To construct the dual transform, say that $v \in \mathbb{R}^{d-1}$ is an *extreme direction* for Ψ if $\partial\Psi$ admits a local maximum at v. Further, say that $x \in \partial\Psi$ is an extreme directional point for Ψ, written $x \in \text{ext-dir}(\Psi)$, if and only if $x = (v, \partial\Psi(v))$ for some extreme direction v. Then we have

$$\Phi^c = \Psi^c \oplus \Pi^{\downarrow} \quad \text{and} \quad \text{cl}(\Phi^c) = \text{ext-dir}(\Psi) \oplus \Pi^{\downarrow}. \tag{3.21}$$

Again, this can be directly proved, yet it is more appealing to observe that this statement is simply an asymptotic counterpart of the usual procedure of restoring the convex polytope K_λ given its support function. Indeed, the complement of the polytope arises as the union of all half-spaces of the form $H_x := \{y \in \mathbb{R}^d, \langle y - x, x\rangle \geq 0\}$ (asymptotically transformed onto suitable translates of $\Pi^{\downarrow}$ under the action of $T^\lambda, \lambda \to \infty$) with x ranging through $x = ru, r > h_{K_\lambda}(u), r \in \mathbb{R}, u \in \mathbb{S}^{d-1}$ which corresponds to taking x in the epigraph of h_{K_λ} (transformed onto Ψ^c in our asymptotics). This explains the first equality in (3.21). The second one comes from the fact that it is enough in the above procedure to consider half-spaces H_x for x in extreme directions only, corresponding to directions orthogonal to $(d-1)$-dimensional faces of K_λ and marked by local minima of the support function h_{K_λ} (asymptotically mapped onto local maxima of $\partial\Psi$). It is worth noting that all extreme directional points of Ψ arise as d-fold intersections of boundaries of upwards paraboloids $\partial[x \oplus \Pi^{\uparrow}], x \in \text{ext}(\Psi)$, although not all such intersections give rise to extreme directional points [they do so precisely when the apices of d intersecting upwards paraboloids are vertices of the same $(d-1)$-dimensional face of Φ, which is not difficult to prove but which is not needed here].

4. Local scaling limits. The re-scaled processes $\hat{s}_\lambda$ and $\hat{r}_\lambda$, defined at (2.19) and (2.20), respectively, are locally parabolic, and here we show that their graphs have scaling limits given by the boundaries of the paraboloid growth processes Ψ and Φ, respectively. Recall from Definition 3.1 that both Ψ and Φ are defined in terms of $\mathcal{P}$, the Poisson point process in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ with intensity density $h^\delta\, dh\, dv$. Recall that $B_d(x, r)$ stands for the d-dimensional radius r ball centered at x.

THEOREM 4.1. *For any $R > 0$, the random functions $\hat{s}_\lambda$ and $\hat{r}_\lambda$ converge in law as $\lambda \to \infty$ to $\partial\Psi$ and $\partial\Phi$, respectively, in the space $\mathcal{C}(B_{d-1}(\mathbf{0}, R))$ of continuous functions on $B_{d-1}(\mathbf{0}, R)$ endowed with the supremum norm.*

REMARK. Theorem 4.1 adds to Molchanov [13], who establishes convergence of the nonrescaled process $nr(\cdot, \{X_i\}_{i=1}^n)$ in $\mathbb{S}^{d-1} \times \mathbb{R}$, where X_i are i.i.d. uniform in $\mathbb{B}^d$. It also adds to Eddy [10], who considers convergence of the properly scaled defect support function for i.i.d. random variables with a circularly symmetric standard Gaussian distribution.

PROOF OF THEOREM 4.1. The convergence in law for $\hat{s}_\lambda$ may be shown to follow from the more general theory of generalized growth processes developed in [30], but we provide here an argument specialized to our present set-up. Recall that we place ourselves on the event that $\mathbf{0} \in K_\lambda$ which is exponentially unlikely to fail as $\lambda \to \infty$ and thus, for our purposes, may be assumed to hold without loss of generality. Further, the support function $h_{\{x\}} : \mathbb{S}^{d-1} \to \mathbb{R}$ of a point $x \in \mathbb{B}^d$ is given for all $u \in \mathbb{S}^{d-1}$ by $h_{\{x\}}(u) = |x| \cos(d_{\mathbb{S}^{d-1}}(u, x/|x|))$ with $d_{\mathbb{S}^{d-1}}$ standing for the geodesic distance in $\mathbb{S}^{d-1}$.

Recall that $\mathcal{P}^{(\lambda)} := T^\lambda(\mathcal{P}_\lambda)$, where T^λ is defined at (2.12) and where $\mathcal{P}^{(\lambda)}$ has density given by (2.14). Write $x := (v_x, h_x)$ for the points in $\mathcal{P}^{(\lambda)}$. Under T^λ we may write $\hat{s}_\lambda(v)$, $v \in \lambda^\beta \mathbb{B}_{d-1}(\pi)$, as

$$\begin{aligned}
\hat{s}_\lambda(v) &= \lambda^\gamma \Big(1 - \max_{x=(v_x,h_x)\in\mathcal{P}^{(\lambda)}} [1 - \lambda^{-\gamma} h_x] \\
&\qquad\qquad \times [\cos[d_{\mathbb{S}^{d-1}}(\exp_{d-1}(\lambda^{-\beta} v), \exp_{d-1}(\lambda^{-\beta} v_x))]]\Big) \\
&= \lambda^\gamma \min_{x\in\mathcal{P}^{(\lambda)}} \big[1 - (1 - \lambda^{-\gamma} h_x) \\
&\qquad\qquad \times \big(1 - \big(1 - \cos[d_{\mathbb{S}^{d-1}}(\exp_{d-1}(\lambda^{-\beta} v), \exp_{d-1}(\lambda^{-\beta} v_x))]\big)\big)\big] \\
&= \min_{x\in\mathcal{P}^{(\lambda)}} \big[h_x + \lambda^\gamma \big(1 - \cos(d_{\mathbb{S}^{d-1}}(\exp_{d-1}(\lambda^{-\beta} v), \exp_{d-1}(\lambda^{-\beta} v_x)))\big) \\
&\qquad\qquad - h_x\big(1 - \cos[d_{\mathbb{S}^{d-1}}(\exp_{d-1}(\lambda^{-\beta} v), \exp_{d-1}(\lambda^{-\beta} v_x))]\big)\big].
\end{aligned} \tag{4.1}$$

Thus, by (2.2) and (2.19), the graph of $\hat{s}_\lambda$ coincides with the lower boundary of the following *generalized growth process*

$$\Psi^{(\lambda)} := \bigcup_{x\in\mathcal{P}^{(\lambda)}} [\Pi^\uparrow]_x^{(\lambda)}, \tag{4.2}$$

where for $x := (v_x, h_x)$ we have

$$\begin{aligned}
[\Pi^\uparrow]_x^{(\lambda)} = \big\{&(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+, h \geq h_x \\
&+ \lambda^\gamma\big(1 - \cos[e_\lambda(v, v_x)]\big) - h_x\big(1 - \cos[e_\lambda(v, v_x)]\big)\big\},
\end{aligned} \tag{4.3}$$

with

$$e_\lambda(v, v_x) := d_{\mathbb{S}^{d-1}}(\exp_{d-1}(\lambda^{-\beta} v), \exp_{d-1}(\lambda^{-\beta} v_x)). \tag{4.4}$$

We now show for fixed $R \in (0, \infty)$ that the lower boundary of the process $\Psi^{(\lambda)}$ converges in law to $\partial\Psi$ in the space $\mathcal{C}(B_{d-1}(\mathbf{0}, R))$. This goes as follows. With R fixed, for all $H \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}^+$, let $E_1(R, H, \lambda)$ be the event that the heights of the lower boundaries of Ψ and $\Psi^{(\lambda)}$ are at most H over the spatial region $B_{d-1}(\mathbf{0}, R)$. Interpreting the boundary $\partial\Psi^{(\lambda)}$ as the graph of a function from $\mathbb{R}^{d-1}$

to $\mathbb{R}_+$, it follows from straightforward modifications of Lemma 3.2 in [30] that there is a $\lambda_0 \in (0, \infty)$ such that, uniformly for $\lambda \geq \lambda_0$, we have

$$P\Big[\sup_{v \in B_{d-1}(\mathbf{0}, R)} \partial \Psi^{(\lambda)}(v) \geq H\Big] \leq C(R) \exp\big(-c\big[H^{(d+1)/2} \wedge R^{d-1} H^{1+\delta}\big]\big) \tag{4.5}$$

with $c > 0$ and $C(R) < \infty$ (note that the extra term $R^{d-1}H^{1+\delta}$ in the exponent corresponds to the probability of having $B_{d-1}(\mathbf{0}, R) \times [0, H]$ devoid of points of $\mathcal{P}$ and $\mathcal{P}^{(\lambda)}$). Lemma 3.2 in [30] likewise gives a similar bound for $P[\sup_{v \in B_{d-1}(\mathbf{0}, R)} \partial \Psi(v) \geq H]$. Thus $P[E_1(R, H, \lambda)^c]$ decays exponentially fast in H, uniformly in λ and it is enough to show, conditional on $E_1(R, H, \lambda)$, that $\hat{s}_\lambda(\cdot)$ converges to $\partial \Psi$ in the space $\mathcal{C}(B_{d-1}(\mathbf{0}, R))$.

Next, with H fixed, observe that for each R there exists a constant $R' := R'(R, H)$ such that for all λ large enough, the behavior of $\Psi^{(\lambda)}$ and Ψ restricted to $B_{d-1}(\mathbf{0}, R) \times [0, H]$ only depends on the restriction to $B_{d-1}(\mathbf{0}, R') \times [0, H]$ of the processes $\mathcal{P}^{(\lambda)}$ and $\mathcal{P}$, respectively. For instance in the case of Ψ it is enough that the region $B_{d-1}(\mathbf{0}, R') \times [0, H]$ contain the apices of all translates of $\Pi^\uparrow$ which hit $B_{d-1}(\mathbf{0}, R) \times [0, H]$, that is to say, the choice $R' := R + \sqrt{2H}$ will suffice.

We also assert for these fixed R' and H that $\mathcal{P}$ and $\mathcal{P}^{(\lambda)}$ can be coupled on a common probability space so that on a set $E_2(R', H, \lambda)$, with $P[E_2(R', H, \lambda)] \to 1$ as $\lambda \to \infty$, their restrictions agree on $B_{d-1}(\mathbf{0}, R') \times [0, H]$. This assertion, referred to as "total variation convergence on compact sets," follows by combining Theorem 3.2.2 in [20], which upper bounds total variation distance between Poisson measures by a multiple of the L^1 norm of the difference of their densities, with the observation that the intensity density of $\mathcal{P}^{(\lambda)}$, as given by (2.14), converges in $L^1(B_{d-1}(\mathbf{0}, R') \times [0, H])$ to the intensity density of $\mathcal{P}$, as given by (3.1).

Let $E(R, H, \lambda) := E_1(R, H, \lambda) \cap E_2(R', H, \lambda)$ and note that $P[E(R, H, \lambda)] \to 1$ as $\lambda \to \infty$. It is enough to show, conditional on the event $E(R, H, \lambda)$, that $\hat{s}_\lambda(\cdot)$ converges to $\partial \Psi$ in the space $\mathcal{C}(B_{d-1}(\mathbf{0}, R))$.

Now we examine the lower boundary of $\mathcal{P}^{(\lambda)}$ given the event $E(R, H, \lambda)$. On this event we have

$$\Psi^{(\lambda)} := \bigcup_{x \in \mathcal{P}} [\Pi^\uparrow]_x^{(\lambda)}$$

with $[\Pi^\uparrow]_x^{(\lambda)}$ given by (4.3). Recalling the definition of $e_\lambda(v, v_x)$ at (4.4) and recalling $\gamma = 2\beta$ from (2.10) we have (using that the ratio of the Euclidean norm and geodesic norm converges to 1)

$$\lambda^\gamma (e_\lambda(v, v_x))^2 = \lambda^\gamma \left(\frac{e_\lambda(v, v_x)}{|\lambda^{-\beta} v - \lambda^{-\beta} v_x|}\right)^2 |\lambda^{-\beta} v - \lambda^{-\beta} v_x|^2 \to |v - v_x|^2.$$

Using the Taylor expansion of the cosine function up to second order in (4.3), it follows that on $E(R, H, \lambda)$ the graph of the lower boundary of $[\Pi^\uparrow]_x^{(\lambda)}, x \in \mathcal{P}$, converges with respect to the sup norm distance on $B_{d-1}(\mathbf{0}, R') \times [0, H]$) to the graph of the lower boundary of the paraboloid $v \mapsto h_x + |v - v_x|^2/2$, that is to say, the lower boundary of $x \oplus \Pi^{\uparrow}$. In the space $\mathcal{C}(B_{d-1}(\mathbf{0}, R))$ the lower boundary of $\Psi^{(\lambda)}$ is with probability one determined by a finite number of $[\Pi^{\uparrow}]_x^{(\lambda)}$ and thus as $\lambda \to \infty$, $\hat{s}_\lambda$ converges in law to $\partial\Psi$, as claimed. This shows Theorem 4.1 for $\hat{s}_\lambda$.

To prove Theorem 4.1 for $\hat{r}_\lambda$, consider the spherical cap

$$\operatorname{cap}_\lambda[v^*, h^*] := \{x \in \mathbb{B}^d, \langle x, \exp_{d-1}(\lambda^{-\beta} v^*)\rangle \geq 1 - \lambda^{-\gamma} h^*\}, \qquad (v^*, h^*) \in \mathbb{R}^{d-1} \times \mathbb{R}_+, \tag{4.6}$$

and note that with $x := (|x|, u) \in \mathbb{B}^d$, we equivalently have

$$\operatorname{cap}_\lambda[v^*, h^*] := \left\{x \in \mathbb{B}^d, (1 - |x|) \leq \max\left(0, 1 - \frac{(1 - \lambda^{-\gamma} h^*)}{\cos\theta}\right)\right\},$$

where θ denotes the angle between x and $\exp_{d-1}(\lambda^{-\beta} v^*)$. Under the transformation T^λ the cap transforms into

$$\begin{aligned}
&\operatorname{cap}^{(\lambda)}[v^*, h^*]\\
&\quad := \Biggl\{(v, h) \in \mathcal{R}_\lambda,\\
&\qquad\quad h \leq \lambda^\gamma \max\left(0, 1 - \frac{1 - \lambda^{-\gamma} h^*}{\cos(d_{\mathbb{S}^{d-1}}(\exp_{d-1}(\lambda^{-\beta} v), \exp_{d-1}(\lambda^{-\beta} v^*)))}\right)\Biggr\}\\
&\quad = \left\{(v, h) \in \mathcal{R}_\lambda, h \leq \lambda^\gamma \max\left(0, 1 - \frac{1 - \lambda^{-\gamma} h^*}{\cos(e_\lambda(v, v^*))}\right)\right\},
\end{aligned}$$

where $e_\lambda(v, v^*)$ is as in (4.4).

Using that $\mathbb{B}^d \setminus K_\lambda$ is the union of all spherical caps not hitting any of the points in $\mathcal{P}_\lambda$, we conclude that under the mapping $T^\lambda : \mathcal{P}_\lambda \to \mathcal{P}^{(\lambda)}$, the complement of K_λ in $\mathbb{B}^d$ gets transformed into the union

$$\bigcup\{\operatorname{cap}^{(\lambda)}[v^*, h^*], (v^*, h^*) \in \mathcal{R}_\lambda, \operatorname{cap}^{(\lambda)}[v^*, h^*] \cap \mathcal{P}^{(\lambda)} = \varnothing\}. \tag{4.7}$$

Let the *paraboloid hull process* $\Phi^{(\lambda)}$ be the complement of this union in $\mathbb{R}^{d-1} \times \mathbb{R}_+$, that is,

$$\Phi^{(\lambda)} := \left(\bigcup\{\operatorname{cap}^{(\lambda)}[v^*, h^*], (v^*, h^*) \in \mathcal{R}_\lambda, \operatorname{cap}^{(\lambda)}[v^*, h^*] \cap \mathcal{P}^{(\lambda)} = \varnothing\}\right)^c. \tag{4.8}$$

To prove the asserted convergence of $\hat{r}_\lambda$, we modify the approach given for the convergence of $\hat{s}_\lambda$. Let $F_1(R, H, \lambda)$ be the event that the heights of the lower boundaries of Φ and $\Phi^{(\lambda)}$ are at most H over the spatial region $B_{d-1}(\mathbf{0}, R)$. As in (4.5) we get that $P[F_1(R, H, \lambda)^c]$ decays exponentially fast in H, uniformly in λ, implying that it is enough to show, conditional on $F_1(R, H, \lambda)$, that $\hat{r}_\lambda$ converges to $\partial\Phi$ in $C(B_{d-1}(\mathbf{0}, R))$.

Both Φ and $\Phi^{(\lambda)}$ are locally determined in the sense that for any $R, H, \varepsilon > 0$ there exist $R'', H'' > 0$, such that, with probability at least $1 - \varepsilon$, the restrictions of Φ and $\Phi^{(\lambda)}$ to $B_{d-1}(\mathbf{0}, R) \times [0, H]$ are determined by the restrictions to $B_{d-1}(\mathbf{0}, R'') \times [0, H'']$ of $\mathcal{P}^{(\lambda)}$ and $\mathcal{P}$, respectively. Indeed if the geometry of Φ within $B_{d-1}(\mathbf{0}, R) \times [0, H]$ were affected by the status of a point $x \in \mathbb{R}^{d-1} \times \mathbb{R}_+$, there would exist a translate of $\Pi^{\downarrow}$ such that the translate: (i) hits $B_{d-1}(\mathbf{0}, R) \times [0, H]$; (ii) contains x on its boundary; (iii) is devoid of other points of $\mathcal{P}$. Thus the probability of such an influence being exerted by a faraway point x tends to 0 with the distance of x from $B_{d-1}(\mathbf{0}, R) \times [0, H]$. The argument for $\Phi^{(\lambda)}$ and $\mathcal{P}^{(\lambda)}$ is analogous. Statements of this kind, going under the general name of stabilization, shall be discussed in more detail in Lemma 6.1 below.

As above, we may couple $\mathcal{P}$ and $\mathcal{P}^{(\lambda)}$ on a common probability space so that their restrictions to $B_{d-1}(\mathbf{0}, R'') \times [0, H'']$ agree on a set $F_2(R'', H'', \lambda)$, with $P[F_2^c(R'', H'', \lambda)] \to 0$ as $\lambda \to \infty$. Put $F(R, H, \lambda) := F_1(R, H, \lambda) \cap F_2(R'', H'', \lambda)$, and note that $P[F(R, H, \lambda)] \to 1$ as $\lambda \to \infty$. We now show on the event $F(R, H, \lambda)$ that $\hat{r}_\lambda(\cdot)$ converges to $\partial\Psi$ as $\lambda \to \infty$.

We Taylor-expand the cosine function up to second order to get that

$$\operatorname{cap}^{(\lambda)}[v^*, h^*] = \left\{(v, h) \in \mathcal{R}_\lambda, h \le \max\left(0, \lambda^\gamma - \frac{\lambda^\gamma - h^*}{1 - e_\lambda(v, v^*)^2/2 + \cdots}\right)\right\}.$$

Using the convergence $\lambda^\gamma e_\lambda^2(v, v^*) \to |v - v^*|^2$ and the expansion $1/(1 - r) = 1 + r + r^2 + \cdots$ for r small, we see that the upper boundary of $\operatorname{cap}^{(\lambda)}[v^*, h^*]$ converges as $\lambda \to \infty$ with respect to the sup norm distance on $B_{d-1}(\mathbf{0}, R) \times [0, H]$ to the graph of the upper boundary of the paraboloid

$$\left\{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+, h \le h^* - \frac{|v - v^*|^2}{2}\right\},$$

that is, the graph of the upper boundary of $(v^*, h^*) \oplus \Pi^{\downarrow}$. In the space $\mathcal{C}(B_{d-1}(\mathbf{0}, R))$ the upper boundary of $\Phi^{(\lambda)}$ is with probability one determined by a finite number of $[\Pi^{\downarrow}]_x^{(\lambda)}$.

This observation, the definition of $\hat{r}_\lambda$, and the relation (4.7), show that $\hat{r}_\lambda$ converges in law in the space $\mathcal{C}(B_{d-1}(\mathbf{0}, R))$ equipped with the supremum norm to the continuous function determined by the upper boundary of the process

$$\bigcup_{x \in \mathbb{R}^{d-1} \times \mathbb{R}_+, [x \oplus \Pi^{\downarrow}] \cap \mathcal{P} = \varnothing} x \oplus \Pi^{\downarrow},$$

which coincides with $\partial\Phi$ in view of (3.16). This completes the proof of Theorem 4.1. $\square$

5. Exact distributional results for scaling limits. This section is restricted to dimension $d = 2$ and to the homogeneous Poisson point process in the unit-disk. Here we provide explicit formulae for the fidis of the processes $\hat{s}_\lambda$ and $\hat{r}_\lambda$ and give their explicit asymptotics, confirming a posteriori the existence of the limiting parabolic growth and hull processes of Section 3.

5.1. *The process* $\hat{s}_\lambda$. This subsection calculates the distribution of $s(\theta_0, \mathcal{P}_\lambda)$ and establishes the convergence of the fidis of both the process and its re-scaled version. Throughout this section we identify the unit sphere $\mathbb{S}^1$ with the segment $[0, 2\pi)$, whence the notation $s(\theta, \cdot)$, $\theta \in [0, 2\pi)$, and likewise for the radius-vector function $r(\theta, \cdot)$. A first elementary result is the following:

LEMMA 5.1. *For every $h > 0$, $u \in \mathbb{S}^1$ and $\lambda > 0$, we have*

$$P[s(u, \mathcal{P}_\lambda) \geq h] = \exp\{-\lambda(\arccos(1-h) - (1-h)\sqrt{2h-h^2})\}.$$

PROOF. Notice that $(s(u, \mathcal{P}_\lambda) \geq h)$ is equivalent to $\mathrm{cap}_1[u, h] \cap \mathcal{P}_\lambda = \varnothing$, where $\mathrm{cap}_1[u, h]$ is defined at (4.6). Since the Lebesgue measure $\ell(\mathrm{cap}_1[u, h])$ of $\mathrm{cap}_1[u, h]$ satisfies

$$\ell(\mathrm{cap}_1[u, h]) = \arccos(1-h) - (1-h)\sqrt{2h-h^2}, \tag{5.1}$$

the lemma follows by the Poisson property of the process $\mathcal{P}_\lambda$. $\square$

We focus on the asymptotic behavior of the process s when λ is large. When we scale in space, we obtain the fidis of white noise and when we scale in both time and space to get $\hat{s}$, we obtain the fidis of the parabolic growth process Ψ defined in Section 3. Let $\mathbb{N}$ denote the positive integers. In dimension two, by the representation (2.16), we notice that the exponential map obtained for the choice $u_0 = (0, 1)$ has the following basic expression:

$$\exp_1(\theta) = (\sin(\theta), \cos(\theta)), \qquad \theta \in \mathbb{R}.$$

PROPOSITION 5.1. *Let $n \in \mathbb{N}$, $0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$ and $h_i \in (0, \infty)$ for all $i = 1, \ldots, n$. Then*

$$\lim_{\lambda\to\infty} P[\lambda^{2/3} s(\exp_1(\theta_1), \mathcal{P}_\lambda) \geq h_1; \ldots; \lambda^{2/3} s(\exp_1(\theta_n), \mathcal{P}_\lambda) \geq h_n]$$
$$= \prod_{k=1}^{n} \exp\left\{-\frac{4\sqrt{2}}{3} h_k^{3/2}\right\}.$$

Moreover, for every $v_1 < v_2 < \cdots < v_n \in \mathbb{R}$, we have

$$\lim_{\lambda\to\infty} P[\lambda^{2/3} s(\exp_1(\lambda^{-1/3} v_1), \mathcal{P}_\lambda) \geq h_1; \ldots; \lambda^{2/3} s(\exp_1(\lambda^{-1/3} v_n), \mathcal{P}_\lambda) \geq h_n]$$
$$= \exp\biggl(-\int_{\inf_{1\leq i\leq n}(v_i - \sqrt{2h_i})}^{\sup_{1\leq i\leq n}(v_i + \sqrt{2h_i})} \sup_{1\leq i\leq n} \biggl[\biggl(h_i - \frac{1}{2}(u - v_i)^2\biggr) \times \mathbf{1}(|u - v_i| \leq \sqrt{2h_i})\biggr] du\biggr).$$

PROOF. The first assertion is obtained by noticing that the events $\{s(\exp_1(\theta_i), \mathcal{P}_\lambda) \geq \lambda^{-2/3} h_i\}, 1 \leq i \leq n$, are independent as soon as $h_i \in (0, \frac{\lambda^{2/3}}{2} \min_{1\leq k \leq n}(1-\cos(\theta_{k+1}-\theta_k)))$. We then apply Lemma 7.7 to estimate the probability of each of these events. Let us recall beforehand that $\arccos(1-x)$ is expanded as $\sqrt{2x} + \sqrt{2x^3}/12 + \cdots$ when $x \to 0$. For every $1 \leq i \leq n$, we have

$$\begin{aligned}
&-\log P[\lambda^{2/3} s(\exp_1(\theta_i), \mathcal{P}_\lambda) \geq h_i] \\
&\quad = \lambda\big[\arccos(1-\lambda^{-2/3}h_i) - (1-\lambda^{-2/3}h_i)\sqrt{2\lambda^{-2/3}h_i - \lambda^{-4/3}h_i^2}\big] \\
&\quad \underset{\lambda\to\infty}{=} \lambda\bigg[\sqrt{2}\lambda^{-1/3}\sqrt{h_i} + \frac{\sqrt{2}}{12}\lambda^{-1}h_i^{3/2} \\
&\qquad\qquad - (1-\lambda^{-2/3}h_i)\sqrt{2}\lambda^{-1/3}\sqrt{h_i}\Big(1 - \frac{1}{4}\lambda^{-2/3}h_i\Big) + o(\lambda^{-1})\bigg] \\
&\quad \underset{\lambda\to\infty}{=} \Big(\frac{1}{12} + 1 + \frac{1}{4}\Big)\sqrt{2}h_i^{3/2} + o(1) \\
&\quad \underset{\lambda\to\infty}{=} \frac{4}{3}\sqrt{2}h_i^{3/2} + o(1).
\end{aligned}$$

Here and elsewhere in this section, the terminology $f(\lambda) \underset{\lambda\to\infty}{\sim} g(\lambda)$ [resp., $f(\lambda) \underset{\lambda\to\infty}{=} o(g(\lambda))$] signifies that $\lim_{\lambda\to\infty} f(\lambda)/g(\lambda) = 1$ [resp., $\lim_{\lambda\to\infty} f(\lambda)/g(\lambda) = 0$]. For the second assertion, it suffices to determine the area $\ell(\mathcal{D}_n)$ of the domain

$$\mathcal{D}_n := \bigcup_{1\leq i\leq n} \mathrm{cap}_\lambda[v_i, h_i].$$

This set is contained in the angular sector between $\alpha_n := \inf_{1\leq i\leq n}[\lambda^{-1/3}v_i - \arccos(1-\lambda^{-2/3}h_i)]$ and $\beta_n := \sup_{1\leq i\leq n}[\lambda^{-1/3}v_i + \arccos(1-\lambda^{-2/3}h_i)]$. Denote by $\rho_n(\cdot)$ the radial function which associates to θ the distance between the origin and the point in $\mathcal{D}_n$ closest to the origin lying on the half-line making angle θ with the positive x-axis. Then

$$\begin{aligned}
\ell(\mathcal{D}_n) &= \int_{\alpha_n}^{\beta_n} \frac{1}{2}\big(1-\rho_n^2(\theta)\big)\,d\theta \\
&= \lambda^{-1/3}\int_{\lambda^{1/3}\alpha_n}^{\lambda^{1/3}\beta_n} \frac{1}{2}\big(1-\rho_n^2(\lambda^{-1/3}u)\big)\,du \\
&\underset{\lambda\to\infty}{\sim} \lambda^{-1/3}\int_{\inf_{1\leq i\leq n}(v_i-\sqrt{2h_i})}^{\sup_{1\leq i\leq n}(v_i+\sqrt{2h_i})} \big(1-\rho_n(\lambda^{-1/3}u)\big)\,du.
\end{aligned}$$

Each set $\mathrm{cap}_\lambda[v_i, h_i]$ is bounded by a line with the polar equation

$$\rho = \frac{1-\lambda^{-2/3}h_i}{\cos(\theta - \lambda^{-1/3}v_i)}.$$

Consequently, the function $\rho_n(\cdot)$ satisfies, for every $\theta \in (0, 2\pi)$,

$$1 - \rho_n(\theta) = \sup_{1 \le i \le n} \left[\frac{\cos(\theta - \lambda^{-1/3} v_i) - 1 + \lambda^{-2/3} h_i}{\cos(\theta - \lambda^{-1/3} v_i)} \times \mathbf{1}\big(|\theta - \lambda^{-1/3} v_i| \le \arccos(1 - \lambda^{-2/3} h_i)\big) \right].$$

It remains to determine the asymptotics of the above function. We obtain that

$$1 - \rho_n(\lambda^{-1/3} u) \underset{\lambda \to \infty}{\sim} \lambda^{-2/3} \sup_{1 \le i \le n} \left[\left(h_i - \frac{1}{2}(u - v_i)^2 \right) \mathbf{1}\big(|u - v_i| \le \sqrt{2h_i}\big) \right].$$

Considering that the required probability is equal to $\exp(-\lambda \ell(\mathcal{D}_n))$, we complete the proof. $\square$

REMARK 1. Proposition 5.1 could have been obtained through the use of the growth process Φ. Indeed, we have $\partial \Phi(v_i)$ greater than h_i for every $1 \le i \le n$ if and only if none of the points (v_i, h_i) is covered by a parabola of Φ. Equivalently, this means that there is no point of $\mathcal{P}$ in the region arising as union of translated downward parabolae $\Pi^{\downarrow}$ with peaks at (v_i, h_i). Calculating the area of this region yields Proposition 5.1.

5.2. *The process* $\hat{r}_\lambda$. This subsection, devoted to distributional results for $\hat{r}_\lambda$, follows the same lines as the previous one. The problem of determining the distribution of $r(\cdot, \mathcal{P}_\lambda)$ seems to be a bit more tricky. To proceed, we fix a direction $u \in \mathbb{S}^1$ and a point $x = (1 - h)u$ ($h \in [0, 1]$) inside the unit-disk $\mathbb{B}^2$. Consider an angular sector centered at x and opening away from the origin. Open the sector until it first meets a point of the Poisson point process at the angle $\mathcal{A}_{\lambda,h}$ (the set with dashed lines must be empty in Figure 3). Let $\mathcal{A}_{\lambda,h}$ be the minimal angle of

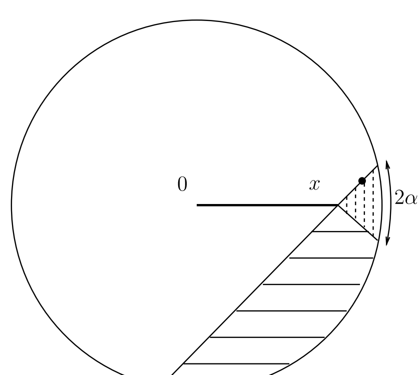

FIG. 3. *When is a point included in the convex hull?*

opening from $x = (1-h)u$ in order to meet a point of $\mathcal{P}_\lambda$ in the opposite side of the origin. In particular, when $\mathcal{A}_{\lambda,h} = \alpha$, there is no point of $\mathcal{P}_\lambda$ in

$$\mathcal{S}_{\alpha,h} := \{y \in \mathbb{B}^2, \langle y - x, u\rangle \ge \cos(\alpha)|y - x|\}.$$

Consequently, we have

$$P[\mathcal{A}_{\lambda,h} \ge \pi/2] = P[s(u, \mathcal{P}_\lambda) \ge h]. \tag{5.2}$$

The next lemma provides the distribution of $\mathcal{A}_{\lambda,h}$.

LEMMA 5.2. *For every $0 \le \alpha \le \pi/2$ and $h \in [0, 1]$, we have*

$$P[\mathcal{A}_{\lambda,h} \ge \alpha] = \exp\{-\lambda\ell(\mathcal{S}_{\alpha,h})\} \tag{5.3}$$

with

$$\begin{aligned}\ell(\mathcal{S}_{\alpha,h}) = \biggl(\alpha &+ \frac{(1-h)^2}{2}\sin(2\alpha) - (1-h)\sin(\alpha)\sqrt{1-(1-h)^2\sin^2(\alpha)} \\ &- \arcsin\bigl((1-h)\sin(\alpha)\bigr)\biggr).\end{aligned} \tag{5.4}$$

When λ goes to infinity, $\mathcal{A}_{\lambda,\lambda^{-2/3}h}$ converges in distribution to a measure with mass 0 on $[0, \pi/2)$ and mass $(1 - \exp\{-\frac{4\sqrt{2}}{3}h^{2/3}\})$ on $\{\pi/2\}$.

PROOF. A quick geometric consideration shows that the set $\mathcal{S}_{\alpha,h}$ is seen from the origin with an angle equal to

$$2\beta = 2\bigl[\alpha - \arcsin\bigl((1-h)\sin(\alpha)\bigr)\bigr]. \tag{5.5}$$

To obtain (5.4), we first integrate in polar coordinates, giving

$$\begin{aligned}\ell(\mathcal{S}_{\alpha,h}) &= 2\int_0^\beta \biggl[\int_{\sin(\alpha-\gamma)/\sin(\alpha-\theta)}^1 \rho\, d\rho\biggr] d\theta \\ &= \int_0^\beta \biggl(1 - \frac{(1-h)^2\sin^2(\alpha)}{\sin^2(\alpha-\theta)}\biggr) d\theta \\ &= \beta - (1-h)^2\sin^2(\alpha)\biggl(\frac{1}{\tan(\alpha-\beta)} - \frac{1}{\tan(\alpha)}\biggr).\end{aligned}$$

We then use (5.5) to get (5.4).

Let us show now the last assertion of Lemma 5.2. Using Proposition 5.1 and (5.2), we get that

$$\lim_{\lambda\to\infty} P[\mathcal{A}_{\lambda,\lambda^{-2/3}h} \ge \pi/2] = \exp\biggl(-\frac{4\sqrt{2}}{3}h^{2/3}\biggr).$$

It remains to remark that for every $\alpha < \pi/2$, $\lim_{\lambda\to\infty} P[\mathcal{A}_{\lambda,\lambda^{-2/3}h} \geq \alpha] = 1$. Indeed, a direct expansion in (5.4) shows that

$$\ell(\mathcal{S}_{\alpha,\lambda^{-2/3}h}) \underset{\lambda\to\infty}{\sim} \left(\sin(\alpha)\cos(\alpha) + 2\frac{\sin^3(\alpha)}{\cos(\alpha)} - \frac{\sin^3(\alpha)}{2\cos^3(\alpha)}\right)\lambda^{-4/3}h^2.$$

Inserting this estimation in (5.3) completes the proof. $\square$

The next lemma provides the explicit distribution of $r(u, \mathcal{P}_\lambda)$ in terms of $\mathcal{A}_{\lambda,h}$.

LEMMA 5.3. *For all $h \in [0, 1]$ and $u \in \mathbb{S}^1$,*

$$\begin{aligned} &P[r(u,\mathcal{P}_\lambda) \geq h] \\ &\quad = P[s(u,\mathcal{P}_\lambda) \geq h] \\ &\qquad + \lambda \int_0^{\pi/2} \frac{\partial \ell(\mathcal{S}_{\alpha,h})}{\partial\alpha} \exp\{-\lambda\ell(\mathrm{cap}_1[u, (1-(1-h)\sin(\alpha))])\}\, d\alpha, \end{aligned} \tag{5.6}$$

where $\ell(\mathrm{cap}_1[u, (1-(1-h)\sin(\alpha))])$ and $\ell(\mathcal{S}_{\alpha,h})$ are defined at (5.1) *and* (5.4), *respectively.*

PROOF. For fixed $h \in [0, 1]$ and $\alpha \in [0, \pi/2)$, we define the set (which is hatched in Figure 3)

$$\mathcal{F}_{h,\alpha} := \mathrm{cap}_1[\mathrm{rot}_{\alpha-\pi/2}(u), (1-(1-h)\sin(\alpha))] \setminus \mathcal{S}_{\alpha,h},$$

where rot_θ is the classical rotation of angle $\theta \in [0, 2\pi)$ defined on $\mathbb{S}^1$.

We remark that x is outside the convex hull if and only if either $\mathcal{A}_{\lambda,h}$ is bigger than $\pi/2$, or $\mathcal{F}_{h,\alpha}$ is empty. Consequently, we have for $u \in \mathbb{S}^1$

$$P[r(u,\mathcal{P}_\lambda) \geq h] = P[\mathcal{A}_{\lambda,h} \geq \pi/2] + \int_0^{\pi/2} \exp\{-\lambda\ell(\mathcal{F}_{h,\alpha})\}\, dP_{\mathcal{A}_{\lambda,h}}(\alpha),$$

where dP_X denotes the distribution of X. Applying Lemma 5.2 yields the result. $\square$

The next proposition provides the asymptotic behavior of the distribution of $\widehat{r}_\lambda(\cdot)$:

PROPOSITION 5.2. *We have for all $h \geq 0$ and $u \in \mathbb{S}^1$,*

$$\begin{aligned} \lim_{\lambda\to\infty} P[\lambda^{2/3} r(u,\mathcal{P}_\lambda) \geq h] = \exp\left\{-\frac{4\sqrt{2}h^{3/2}}{3}\right\} \\ + 2\int_0^\infty \exp\left\{-\frac{4\sqrt{2}}{3}\left(h + \frac{t^2}{2}\right)^{3/2}\right\} t^2\, dt - 1. \end{aligned}$$

PROOF. We focus on the asymptotic behavior of the integral in the relation (5.6) where h is replaced with $\lambda^{-2/3}h$. We proceed with the change of variable $\alpha = \frac{\pi}{2} - \lambda^{-1/3}t$, which gives

$$
\begin{aligned}
&\lambda \int_0^{\pi/2} \frac{\partial \ell(\mathcal{S}_{\alpha,h})}{\partial \alpha}(\alpha, \lambda^{-2/3}h) \exp\{-\lambda \ell(\mathrm{cap}_1[u, (1-(1-\lambda^{-2/3}h)\sin(\alpha))])\}\, d\alpha \\
(5.7)\quad &= \lambda^{2/3} \int_0^{(\pi/2)\lambda^{1/3}} \frac{\partial \ell(\mathcal{S}_{\alpha,h})}{\partial \alpha}\left(\frac{\pi}{2} - \lambda^{-1/3}t, \lambda^{-2/3}\right) \\
&\qquad \times \exp\{-\lambda \ell(\mathrm{cap}_1[u, (1-(1-\lambda^{-2/3}h)\cos(\lambda^{-1/3}t))])\}\, dt.
\end{aligned}
$$

Using (5.1), we find the exponential part of the integrand, which yields

$$
\begin{aligned}
(5.8)\quad &\lim_{\lambda\to\infty} \exp\left\{-\lambda \ell\left(\mathrm{cap}_1\left[u, \left(1-(1-\lambda^{-2/3}h)\sin\left(\frac{\pi}{2} - \lambda^{-1/3}t\right)\right)\right]\right)\right\} \\
&\quad = \exp\left\{-\frac{4\sqrt{2}}{3}\left(h + \frac{t^2}{2}\right)^{3/2}\right\}.
\end{aligned}
$$

Moreover, the derivative of the area of $\mathcal{S}_{\alpha,h}$ is

$$
\frac{\partial \ell(\mathcal{S}_{\alpha,h})}{\partial \alpha} = 1 + (1-h)^2\cos(2\alpha) - 2(1-h)\cos(\alpha)\sqrt{1-(1-h)^2\sin^2(\alpha)}.
$$

In particular, we have

$$
(5.9)\quad \frac{\partial \ell(\mathcal{S}_{\alpha,h})}{\partial \alpha}\left(\frac{\pi}{2} - \lambda^{-1/3}t, \lambda^{-2/3}h\right) \underset{\lambda\to\infty}{\sim} 2\lambda^{-2/3}\left[h + t^2 - t\sqrt{2h+t^2}\right].
$$

Inserting (5.8) and (5.9) into (5.7) and using (5.6), we obtain the required result. $\square$

REMARK 2. In connection with Section 3, the above calculation could have been alternatively based on the limiting hull process related to $\widehat{r}$. Indeed, for fixed $v \in \mathbb{R}, h \in \mathbb{R}_+$, saying that $\partial\Psi(v)$ is greater than h means that there is no translate of the standard downward parabola $\Pi^{\downarrow}$ containing two extreme points on its boundary and lying underneath the point (v, h). We define a random variable D related to the point (v, h); see Figure 4. If $\mathcal{P} \cap ((v, h) \oplus \Pi^{\downarrow})$ is empty, then we take $D = 0$. Otherwise, we consider all the translates of $\Pi^{\downarrow}$ containing on the boundary at least one point from $\mathcal{P} \cap ((v, h) \oplus \Pi^{\downarrow})$ and the point (v, h). There is almost surely precisely one among them which has the farthest peak (with respect to the first coordinate) from (v, h). The random variable D is then defined as the difference between the v-*coordinate* of the farthest peak and v. The distribution of $|D|$

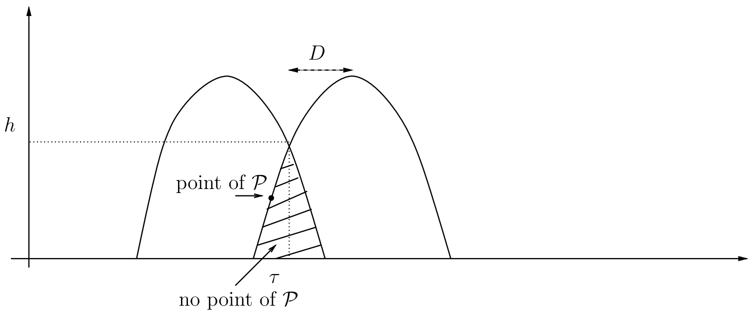

FIG. 4. *Definition of the r.v. D.*

can be made explicit:

$$P[|D| \le t] = \exp\{-\tfrac{2}{3}(2h+t^2)^{3/2} + t(2h + \tfrac{2}{3}t^2)\}, \qquad t \ge 0.$$

Conditionally on $|D|$, $\partial\Psi(v)$ is greater than h if and only if the region between the v-axis and the parabola with the farthest peak does not contain any point of $\mathcal{P}$ in its interior. Consequently, we have

$$\begin{aligned} P[\partial\Psi(v) \ge h] &= P[D=0] \\ &\quad + \int_0^\infty \exp\biggl\{\biggl(-\frac{4\sqrt{2}}{3}\biggl(h + \frac{t^2}{2}\biggr)^{3/2} \\ &\qquad\qquad - \frac{2}{3}(2h+t^2)^{3/2} - t\biggl(2h + \frac{2}{3}t^2\biggr)\biggr)\biggr\}\, dP_{|D|}(t), \end{aligned}$$

which provides the result of Proposition 5.2.

The final proposition is the analog of Proposition 5.1 where the radius-vector function of the flower is replaced by the one of the convex hull itself.

PROPOSITION 5.3. *Let $n \in \mathbb{N}$, $0 \le \theta_1 < \theta_2 \cdots < \theta_n < 2\pi$ and $h_i \in (0,\infty)$ for all $i = 1, \dots, n$. Then*

$$P\Bigl[\lambda^{2/3} r(\exp_1(\theta_1), \mathcal{P}_\lambda) \ge h_1; \dots; \lambda^{2/3} r(\exp_1(\theta_n), \mathcal{P}_\lambda) \ge h_n\Bigr]$$

$$\underset{\lambda\to\infty}{\sim} \prod_{i=1}^n P[\lambda^{2/3} r(\exp_1(\theta_i), \mathcal{P}_\lambda) \ge h_i].$$

Moreover, for every $v_1 < v_2 < \cdots < v_n \in \mathbb{R}$, we have

$$\begin{aligned} &\lim_{\lambda\to\infty} P[\lambda^{2/3} r(\exp_1(\lambda^{-1/3} v_1), \mathcal{P}_\lambda) \ge h_1; \dots; \lambda^{2/3} r(\exp_1(\lambda^{-1/3} v_n), \mathcal{P}_\lambda) \ge h_n] \\ &\quad = \int_{\mathbb{R}^n} \exp\{-F((t_i, h_i, v_i)_{1\le i\le n})\}\, dP_{(D_1,\dots,D_n)}(t_1, \dots, t_n), \end{aligned}$$

where $D_1, \dots, D_n$ are symmetric variables such that

$$
\begin{aligned}
&P[|D_1| \le t_1; \dots; |D_n| \le t_n] \\
&\quad = \exp\biggl(-\int \sup_{1 \le i \le n} \biggl[\biggl(h_i + \frac{t_i^2}{2} - \frac{(|v - v_i| + t_i)^2}{2}\biggr) \vee 0\biggr] dv\biggr),
\end{aligned} \tag{5.10}
$$

and F is the area

$$
\begin{aligned}
F((t_i, h_i, v_i)_{1 \le i \le n}) = \int_{\mathbb{R}} \biggl\{ &\sup_{1 \le i \le n} \biggl[\biggl(h_i + \frac{t_i^2}{2} - \frac{(v - v_i - t_i)^2}{2}\biggr) \vee 0\biggr] \\
&- \sup_{1 \le i \le n} \biggl[\biggl(h_i + \frac{t_i^2}{2} - \frac{(|v - v_i| + t_i)^2}{2}\biggr) \vee 0\biggr]\biggr\} dv.
\end{aligned} \tag{5.11}
$$

PROOF. We prove the first assertion and denote by $\mathcal{A}_1, \dots, \mathcal{A}_n$ the angles (as defined by Lemma 5.2) corresponding to the couples $(\theta_1, \lambda^{-2/3} h_1), \dots, (\theta_n, \lambda^{-2/3} h_n)$. Conditionally on $\{\mathcal{A}_i = \alpha_i\}$, the event $\{\lambda^{2/3} r(\exp_1(\theta_i), \mathcal{P}_\lambda) \ge h_i\}$ only involves the points of the point process $\mathcal{P}_\lambda$ included in the circular cap $\mathrm{cap}_1[\theta_i - \frac{\pi}{2} + \alpha_i, (1 - (1 - \lambda^{-2/3} h_i)\sin(\alpha_i))]$; see the proof of Lemma 5.3. Moreover there exists $\delta \in (0, \pi/2)$ such that for λ large enough and $\alpha_i \in (\delta, \frac{\pi}{2})$ for every i, these circular caps are all disjoint. Consequently, we obtain that, conditionally on $\{\mathcal{A}_i > \delta\ \forall i\}$, the events $\{\lambda^{2/3} r(\exp_1(\theta_i), \mathcal{P}_\lambda) \ge h_i\}$ are independent. It remains to remark that Lemma 5.2 implies

$$\lim_{\lambda \to \infty} P[\exists 1 \le i \le n; \mathcal{A}_i \le \delta] = 0.$$

Let us consider now the second assertion, which could be obtained by a direct estimation of the joint distribution of the angles $\mathcal{A}_i$ [corresponding to the points $(\lambda^{-1/3} v_i, \lambda^{-2/3} h_i)$]. But it is easier to prove it with the use of the boundary $\partial\Psi$ of the hull process. As in Remark 2, we define for each point (v_i, h_i), the random variable D_i as the difference between the v-coordinate of the farthest peak of a downward parabola arising as a translate of $\Pi^{\downarrow}$ (denoted by Par_i) containing on its boundary (v_i, h_i) and a point of $\mathcal{P}$. Then $|D_i|$ is less than t_i for every $1 \le i \le n$ if and only if there is no point of $\mathcal{P}$ inside a region delimited by the v-axis and the supremum of n functions $g_1, \dots, g_n$ defined in the following way: $g_i(v_i + \cdot)$ is an even function with a support equal to $[t_i - \sqrt{2h_i + t_i^2}, t_i + \sqrt{2h_i + t_i^2}]$ and identified with the parabola $\mathrm{Par}_i(\cdot - v_i)$ on the segment $[t_i - \sqrt{2h_i + t_i^2}, 0]$; see Figure 4. We deduce from this assertion the result (5.10). Conditionally on $\{D_1 = t_1, \dots, D_n = t_n\}$, $\partial\Psi(v_i)$ is greater than h_i for every i if and only if the region between the functions g_i and the parabolae Par_i does not contain any point of $\mathcal{P}$; see Figure 5. This implies result (5.11) and completes the proof. $\square$

REMARK 3. Convergence of the fidis of the radius-vector function of the convex hull of n uniform points in the disk has already been derived in Theorem 2.3

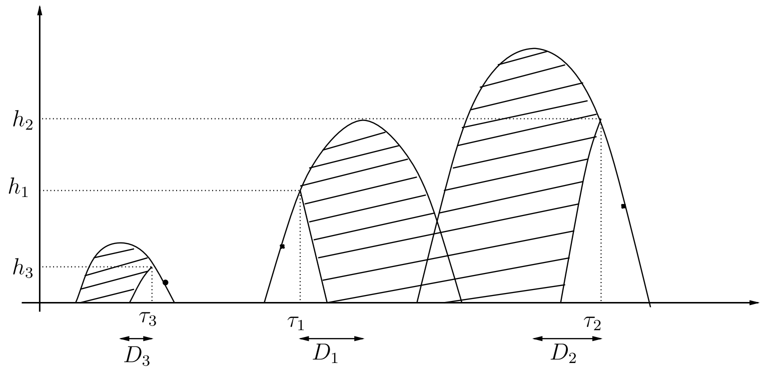

FIG. 5. *Definition of the area F (hatched region). The black points belong to $\mathcal{P}$.*

of [12]. Still we feel that the results presented in this section are obtained in a more direct and explicit way. Moreover, they are characterized by the parabolic growth and hull processes, which provides an elementary representation of the asymptotic distribution. The explicit fidis and the convergence of these fidis to those of $\partial\Psi$ and $\partial\Phi$ can be used to obtain explicit formulae for second-order characteristics of the point process of extremal points.

6. Stabilizing functional representation for convex hull characteristics. The purpose of this section is to link the convex hull characteristics considered in Section 1 with the theory of stabilizing functionals, a tool for proving limit theorems in geometric contexts; see [4, 15–19] and [29].

The collection Ξ of basic geometric functionals. We let Ξ be the collection of four basic functionals $\{\xi_s, \xi_r, \xi_{\vartheta_k}, \xi_{f_k}\}$, where each $\xi_\cdot$ is defined on pairs $(x, \mathcal{X})$, with $x \in \mathcal{X} \subset \mathbb{B}^d$, according to the following definitions. When $x \notin \mathcal{X}$, we write $\xi(x, \mathcal{X})$ instead of $\xi(x, \mathcal{X} \cup \{x\})$.

The point-configuration functional $\xi_s(x, \mathcal{X})$, $x \in \mathcal{X} \subset \mathbb{B}^d$, for finite $\mathcal{X} \subset \mathbb{B}^d$ is set to be zero if x is not a vertex of $\mathrm{conv}(\mathcal{X})$, and otherwise it is defined as follows. Let $\mathcal{F}(x, \mathcal{X})$ be the (possibly empty) collection of faces f in $\mathcal{F}_{d-1}(\mathrm{conv}(\mathcal{X}))$ such that $x = \mathrm{Top}(f)$, where we recall from (2.5) that $\mathrm{Top}(f)$ is the point in f which is closest to $\mathbb{S}^{d-1}$. Let $\mathrm{cone}(\mathcal{F}(x, \mathcal{X})) := \{ry, r > 0, y \in \mathcal{F}(x, \mathcal{X})\}$ be the corresponding cone. Recalling that $F(\cdot)$ is the Voronoi flower defined at (2.3), for $x \notin \mathrm{ext}(\mathcal{X})$, we put $\xi_s(x, \mathcal{X}) = 0$, and for $x \in \mathrm{ext}(\mathcal{X})$, we put

$$\xi_s(x, \mathcal{X}) := \mathrm{Vol}\big([\mathbb{B}^d \setminus F(\mathcal{X})] \cap \mathrm{cone}(\mathcal{F}(x, \mathcal{X}))\big).$$

Then the volume of $\mathbb{B}^d \setminus F(\mathcal{P}_\lambda)$ equals $\sum_{x \in \mathcal{P}_\lambda} \xi_s(x, \mathcal{P}_\lambda)$. Also, we know from (2.25) that W_λ is asymptotically equivalent to the volume of $\mathbb{B}^d \setminus F(\mathcal{P}_\lambda)$.

Likewise, given $x \in \mathcal{X} \subset \mathbb{B}^d$, for $x \notin \mathrm{ext}(\mathcal{X})$, we put $\xi_r(x, \mathcal{X}) = 0$, and otherwise we put

$$\xi_r(x, \mathcal{X}) = \mathrm{Vol}\big([\mathbb{B}^d \setminus \mathrm{conv}(\mathcal{X})] \cap \mathrm{cone}(\mathcal{F}(x, \mathcal{X}))\big).$$

Thus the volume of $\mathbb{B}^d \setminus K_\lambda$ equals $\sum_{x\in\mathcal{P}_\lambda} \xi_r(x,\mathcal{P}_\lambda)$, and we note that V_λ is asymptotically equivalent to the volume of $\mathbb{B}^d \setminus K_\lambda$.

The kth order projection avoidance functional $\xi_{\vartheta_k}(x,\mathcal{X})$, $x\in\mathcal{X}\subset\mathbb{B}^d$, with $k\in\{1,\dots,d\}$ is zero if $x\notin \operatorname{ext}(\mathcal{X})$, and otherwise equal to

$$\xi_{\vartheta_k}(x,\mathcal{X}) := \int_{[\mathbb{B}^d\setminus\operatorname{conv}(\mathcal{X})]\cap\operatorname{cone}(\mathcal{F}(x,\mathcal{X}))} \frac{1}{|x|^{d-k}} \vartheta_k^{\mathcal{X}}(x)\,dx;$$

see (2.7). In particular, (2.8) yields

$$V_k(\mathbb{B}^d) - V_k(K_\lambda) = \frac{\binom{d-1}{k-1}}{\kappa_{d-k}} \Bigl[\sum_{x\in\mathcal{P}_\lambda} \xi_{\vartheta_k}(x,\mathcal{P}_\lambda)\Bigr]. \tag{6.1}$$

The k-face functional $\xi_{f_k}(x;\mathcal{X})$, defined for finite $\mathcal{X}$ in $\mathbb{B}^d$, $x\in\mathcal{X}$ and $k\in\{0,1,\dots,d-1\}$, is the number of k-dimensional faces f of $\operatorname{conv}(\mathcal{X})$ with $x=\operatorname{Top}(f)$, if x belongs to Vertices[conv($\mathcal{X}$)], and zero otherwise. Thus $\sum_{x\in\mathcal{X}} \xi_{f_k}(x,\mathcal{X})$ is the total number of k-faces in $\operatorname{conv}(\mathcal{X})$. In particular, setting $\mathcal{X}:=\mathcal{P}_\lambda$, the total mass of $\mu_\lambda^{f_k}$ is

$$f_k(K_\lambda) = \sum_{x\in\mathcal{P}_\lambda} \xi_{f_k}(x,\mathcal{P}_\lambda). \tag{6.2}$$

It is readily seen by the definition of $\mu_\lambda^{f_k}$ at (2.5) that

$$\mu_\lambda^{f_k} := \mu_\lambda^{\xi_{f_k}} := \sum_{x\in\mathcal{P}_\lambda} \xi_{f_k}(x,\mathcal{P}_\lambda)\delta_x. \tag{6.3}$$

The collection $\Xi^{(\infty)}$ *of scaling counterparts to elements of* Ξ. In the spirit of the local scaling Section 4, we shall construct *scaling counterparts* to each functional $\xi\in\Xi$; we shall define these counterparts in terms of the paraboloid growth and hull processes. To reflect this correspondence we write $\xi_\cdot^{(\infty)}$ to denote the local scaling limit analog of $\xi_\cdot$ with the (∞) superscript.

The functional $\xi_s^{(\infty)}(x,\mathcal{P})$ is defined to be zero if $x\notin\operatorname{ext}(\Psi)$ and otherwise is defined as follows. Let $\mathcal{F}^\infty(x,\mathcal{P})$ stand for the collection of paraboloid faces f of Φ for which $x=\operatorname{Top}(f)$ [recall (2.5)] and let v-cone$(\mathcal{F}^\infty(x,\mathcal{P}))$ be the cylinder (vertical cone) in $\mathbb{R}^{d-1}\times\mathbb{R}_+$ generated by $\mathcal{F}^\infty(x,\mathcal{P})$, that is to say, v-cone$(\mathcal{F}^\infty(x,\mathcal{P})) := \{(v,h), \exists h' : (v,h')\in\mathcal{F}^\infty(x,\mathcal{P})\}$. Then, if $x\in\operatorname{ext}(\Psi)$, we set

$$\xi_s^{(\infty)}(x,\mathcal{P}) := \operatorname{Vol}\bigl(\text{v-cone}(\mathcal{F}^\infty(x,\mathcal{P}))\setminus\Psi\bigr).$$

Formally we should define $\xi_s^{(\infty)}(x;\mathcal{X})$ for general $\mathcal{X}$ rather than just for $\mathcal{P}$, but we bypass this formality so as to avoid extra notation. We will mainly consider $\mathcal{X}=\mathcal{P}$ anyway and the general definition can be readily recovered by formally conditioning on $\mathcal{P}=\mathcal{X}$. This simplifying convention will also be applied for the remaining local scaling functionals below.

Likewise, $\xi_r^{(\infty)}(x,\mathcal{P})$ is zero if $x \notin \operatorname{ext}(\Psi)$, and otherwise

$$\xi_r^{(\infty)}(x,\mathcal{P}) := \operatorname{Vol}\bigl(\text{v-cone}(\mathcal{F}^\infty(x,\mathcal{P})) \setminus \Phi\bigr).$$

The kth order projection avoidance functional $\xi_{\vartheta_k}^{(\infty)}(x,\mathcal{P})$ is zero if $x \notin \operatorname{ext}(\Psi)$, and otherwise

$$\xi_{\vartheta_k}^{(\infty)}(x,\mathcal{P}) := \int_{\text{v-cone}(\mathcal{F}^\infty(x,\mathcal{P}))\setminus\Phi} \vartheta_k^\infty(u)\,du \tag{6.4}$$

with $\vartheta_k^\infty(\cdot)$ defined in (3.19). Note that the extra factor $\frac{1}{|x|^{d-k}}$ in (2.8), where $x \in \mathbb{B}^d \setminus K_\lambda$, converges to one under the scaling T^λ defined at (2.12), and thus is not present in the asymptotic functional.

The k-face functional $\xi_{f_k}^{(\infty)}(x,\mathcal{P})$, defined for $x \in \mathcal{P}$, and $k \in \{0,1,\dots,d-1\}$, is the number of k-dimensional paraboloid faces f of the hull process Φ for which $x = \operatorname{Top}(f)$, if x belongs to $\operatorname{ext}(\Psi) = \operatorname{Vertices}(\Phi)$, and zero otherwise.

The collection $\Xi^{(\lambda)}$ of finite-size scaling counterparts to elements of Ξ. For each of the four basic functionals $\xi \in \Xi$, and each $\lambda \geq 1$, consider the collection $\Xi^{(\lambda)}$ of the finite-size scaling counterparts $\xi^{(\lambda)}$ given by

$$\xi^{(\lambda)}(x,\mathcal{X}) := \xi([T^\lambda]^{-1}x, [T^\lambda]^{-1}\mathcal{X}), \qquad x \in \mathcal{X} \subset \mathcal{R}_\lambda \subset \mathbb{R}^{d-1} \times \mathbb{R}_+, \tag{6.5}$$

where T^λ is the scaling transform (2.12) and $\mathcal{R}_\lambda$ its image (2.13). Again resorting to the theory developed in Section 4, we see for $\xi \in \Xi$ that $\xi^{(\lambda)}$ can be regarded as *interpolating* between ξ and $\xi^{(\infty)}$; as such it is the analog of $\xi^{(\lambda)}$ defined at (1.17) of [30]. However, due to the differing natures of the functionals considered here, different scaling pre-factors are needed to ensure nontrivial scaling behaviors. More precisely, for each $\xi^{(\lambda)} \in \Xi^{(\lambda)}$ we define its *proper scaling prefactor* $\lambda^{\eta[\xi]}$ where:

- $\eta[\xi_s] = \eta[\xi_r] = \eta[\xi_{\vartheta_k}] = \beta(d-1)+\gamma$, $k \in \{0,1,\dots,d-1\}$, since for each of these three functionals, the spatial scaling involves dilation by λ^β, whereas the time scaling involves λ^γ. [Note that the re-scaled projection avoidance function (2.21) involves no scaling prefactor.]
- $\eta[\xi_{f_k}] = 0$ because the number of k-faces does not undergo any scaling.

To proceed, for any measurable $D \subseteq \mathbb{R}^{d-1} \times \mathbb{R}_+$ and generic scaling limit functional $\xi^{(\infty)} \in \Xi^{(\infty)}$, by its *restricted version* we mean by $\xi_D^{(\infty)}(x,\mathcal{P}) := \xi^{(\infty)}(x,\mathcal{P}\cap D)$, $x \in \mathbb{R}^{d-1} \times \mathbb{R}_+$. Note that the so-defined restricted functionals in case of D bounded, or of bounded spatial extent, clearly involve growth and hull processes built on input of bounded spatial extent, in which case the definition (3.7) for $\mathcal{P}$ replaced with $\mathcal{P}\cap D$ yields infinite vertical faces at the boundary of $D's$ spatial extent. This makes some of the functionals considered in this paper infinite or even undefined for points close to these infinite faces. For such points $x, x := (v_x, h_x)$, and such sets D, we may formally put $\xi_D^{(\infty)} = \infty$. Fortunately, such pathologies do not arise in the sequel. Indeed, we will restrict to cylinder sets

D centered around the vertical axis $\{(v_x, h), h \geq 0\}$ whose radius (termed stabilization radius below) is sufficiently large so that with probability one the faces meeting x, as defined by the input $\mathcal{P} \cap D$, coincide with the faces meeting x when the input is $\mathcal{P}$. We now make these ideas precise.

Having now defined the class Ξ of four basic functionals, together with the finite-size scaling version $\Xi^{(\lambda)}, \lambda \geq 1$, and the infinite scaling version $\Xi^{(\infty)}$, we are ready to establish some crucial localization properties of the functionals in $\Xi^{(\lambda)}$ and $\Xi^{(\infty)}$. Recalling that $B_{d-1}(v, r)$ is the $(d-1)$ dimensional ball centered at $v \in \mathbb{R}^{d-1}$ with radius r, let $C_{d-1}(v, r)$ be the cylinder $B_{d-1}(v, r) \times \mathbb{R}_+$. Given a generic scaling limit functional $\xi^{(\infty)} \in \Xi^{(\infty)}$, we shall write $\xi^{(\infty)}_{[r]} := \xi^{(\infty)}_{C_{d-1}(v,r)}$. Likewise, for the finite scaling functionals $\xi^{(\lambda)} \in \Xi^{(\lambda)}$, we shall use the notation $\xi^{(\lambda)}_{[r]}$ with a fully analogous meaning.

Given $\xi^{(\infty)} \in \Xi^{(\infty)}$, a random variable $R := R^{\xi^{(\infty)}}[x]$ is called a *localization radius* for $\xi^{(\infty)}$ if and only if a.s.

$$\xi^{(\infty)}(x, \mathcal{P}) = \xi^{(\infty)}_{[r]}(x, \mathcal{P}) \qquad \text{for all } r \geq R.$$

Given $\xi^{(\lambda)} \in \Xi^{(\lambda)}$, we analogously define $R^{\xi^{(\lambda)}}[x]$ to be a localization radius for $\xi^{(\lambda)}$ if and only if a.s.

$$\xi^{(\lambda)}(x, \mathcal{P}^{(\lambda)}) = \xi^{(\lambda)}_{[r]}(x, \mathcal{P}^{(\lambda)}) \qquad \text{for all } r \geq R.$$

The notion of localization, developed in [30], is a variant of a general concept of stabilization [4, 15, 18, 19]. A crucial property of the functionals $\xi^{(\lambda)} \in \Xi^{(\lambda)}, \lambda \geq 1$ and $\xi^{(\infty)} \in \Xi^{(\infty)}$ is that they admit localization radii with tails decaying super-exponentially fast.

LEMMA 6.1. *For each $\xi \in \Xi$, the functionals $\xi^{(\infty)}$ and $\xi^{(\lambda)}, \lambda \geq 1$, admit localization radii with the property that*

$$\begin{aligned} &P[R^{\xi^{(\infty)}}[x] > L] \leq C \exp\left(-\frac{L^{d+1}}{C}\right) \quad \text{and} \\ &P[R^{\xi^{(\lambda)}}[x] > L] \leq C \exp\left(-\frac{L^{d+1}}{C}\right) \end{aligned} \tag{6.6}$$

for some finite positive constant C, uniformly in λ large enough and uniformly in x.

PROOF. The proof is given only for the scaling limit functionals $\xi^{(\infty)} \in \Xi^{(\infty)}$; the argument for the finite scaling functionals $\xi^{(\lambda)} \in \Xi^{(\lambda)}, \lambda \geq 1$, is fully analogous and is omitted.

For a point $x := (v, h) \in \mathcal{P}$, denote by $\mathcal{P}[[x]]$ the collection of all vertices of $(d-1)$-dimensional faces of Φ meeting at x if $x \in \text{Vertices}(\Phi)$ and $\mathcal{P}[[x]] := \{x\}$

otherwise. If $x \in \mathrm{Vertices}(\Phi)$, the collection $\mathcal{P}[[x]]$ uniquely determines the *local facial structure* of Φ at x, understood as the collection of all $(d-1)$-dimensional faces $f_1[x], \dots, f_m[x], m = m[x] < \infty$ meeting at x. We shall show that there exists a random variable $R' := R'[x]$ with these two properties:

• With probability one the facial structure $\mathcal{P}_{[r]}[[x]]$ at x determined upon restricting to $\mathcal{P} := \mathcal{P} \cap C_{\mathbb{R}^{d-1}}(v, r)$ coincides with $\mathcal{P}[[x]]$ for all $r \geq R'$; in the sequel we say that $\mathcal{P}[[x]]$ is fully determined within radius R' in such a case.

• We have

$$P[R' > L] \leq C \exp\left(-\frac{L^{d+1}}{C}\right). \tag{6.7}$$

Before proceeding, we note that to conclude the statement of Lemma 6.1, it is enough to establish (6.7). Indeed, this is because of these three observations:

• The values of functionals $\xi_s^{(\infty)}$, ξ_r^* and $\xi_{f_k}^{(\infty)}$, $k \in \{0, \dots, d-1\}$, at $x \in \mathcal{P}$, are uniquely determined given $\mathcal{P}[[x]]$, and thus R' can be taken as the localization radius.

• The values of functionals $\xi_{\vartheta_k}^{(\infty)}(x, \mathcal{P}), k \in \{1, \dots, d-1\}, x := (v, h)$, are determined, given the intersection of the hull process Φ with $\Theta[x] := [\text{v-cone}(\mathcal{F}^{(\infty)}(x, \mathcal{P})) \setminus \Phi] \oplus \Pi^{\downarrow}$; see (3.19) and the definition of $\xi_{\vartheta_k}^{(\infty)}$ at (6.4). It is readily seen that this intersection $\Theta[x] \cap \Phi$ is in its turn uniquely determined by $\Theta[x] \cap \mathrm{Vertices}(\Phi)$. Thus, to know it, it is enough to know the facial structure at x and at all vertices of Φ falling into $\Theta[x]$. To proceed, note that the spatial diameter of $\Theta[x]$ is certainly bounded by $R'[x]$ plus $2\sqrt{2}$ times the square root of the highest height coordinate of $\partial\Phi$ within spatial distance $R'[x]$ from v. Use (4.5) to bound this height coordinate and thus to establish a superexponential bound $\exp(-\Omega(L^{d+1}))$ for tail probabilities of the spatial diameter $R''[x]$ of $\Theta[x]$. Finally, we set the localization radius to be $\max_{y \in \mathrm{Vertices}(\Phi), y \in C_{\mathbb{R}^{d-1}}(v, R''[x])} R'[y]$ which is again easily verified to exhibit the desired tail behavior as the number of vertices within $C_{\mathbb{R}^{d-1}}(v, R''[x])$ grows polynomially in $R''[x]$ with overwhelming probability; see Lemma 3.2 in [30].

To proceed with the proof, suppose first that x is not extreme in Φ. Then, by Lemma 3.1 in [30] and its proof, there exists $R' = R'[x]$ satisfying (6.7) and such that the extremality status of x localizes within radius R'. In this particular case of x not extreme in Φ, this also implies localization for $\mathcal{P}[[x]] = \{x\}$. Assume now that x is an extreme point in $\mathcal{P}$. Enumerate the $(d-1)$-dimensional faces meeting x by $f_1, \dots, f_m$. The local facial structure $\mathcal{P}[[x]]$ is determined by the parabolic faces of the space–time region $\bigcup_{i \leq m} \Pi^{\downarrow}[f_i]$, which by (3.16) is devoid of points from $\mathcal{P}$. Note that this region contains all vertices of $f_1, \dots, f_m$ on its upper boundary. Moreover, Poisson points outside this region do not change the status of the faces $f_1, \dots, f_m$ as these faces will not be subsumed by larger faces meeting x unless Poisson points lie on the boundary of the hull process, an event of probability zero. It follows that $\mathcal{P}[[x]]$ is fully determined by the point configuration $\mathcal{P} \cap C_{d-1}(v, R')$ where R' is the smallest integer r such that

$$\bigcup_{i \le m} [\Pi^{\downarrow}[f_i] \cap (\mathbb{R}^{d-1} \times \mathbb{R}_+)] \subset C_{d-1}(v, r). \tag{6.8}$$

To establish (6.7) for R', we note that if R' exceeds L, then, by standard geometry, within distance $O(L^2)$ from x, we can find a point x' in $\mathbb{Z}^d$ with the properties that:

- the downwards parabolic solid $x' \oplus \Pi^{\downarrow}$ is contained in $\bigcup_{i \le m} \Pi^{\downarrow}[f_i]$ and thus in particular devoid of points of $\mathcal{P}$;
- the spatial diameter (the diameter of spatial projection on $\mathbb{R}^{d-1}$) of $[x' \oplus \Pi^{\downarrow}] \cap (\mathbb{R}^{d-1} \times \mathbb{R}_+)$ does exceed $L/2$.

Since the intensity measure of $\mathcal{P}$ assigns to such $[x' \oplus \Pi^{\downarrow}] \cap (\mathbb{R}^{d-1} \times \mathbb{R}_+)$ mass of order at least $\Omega(L^{d+1})$ [in fact even $\Omega(L^{d+1+2\delta})$; see the proof of Lemma 3.1 in [30] for details in a much more general set-up], the probability of having $x' \oplus \Pi^{\downarrow}$ devoid of points of $\mathcal{P}$ is $\exp(-\Omega(L^{d+1}))$. Since the cardinality of $B_d(x, L^2) \cap \mathbb{Z}^{d-1}$ is bounded by CL^{2d}, Boole's inequality gives

$$P[R' > L] \le CL^{2d} \exp\left(-\frac{L^{d+1}}{C}\right),$$

which yields the required inequality (6.7) and thus completes the proof of Lemma 6.1. $\square$

7. Variance asymptotics and Gaussian limits for empirical measures. Sections 1–5 establish the asymptotic embedding in $\mathbb{R}^{d-1} \times \mathbb{R}_+$ of convex polytope characteristics, whereas Section 6 establishes their localization properties. The present section establishes variance asymptotics and Gaussian limits of these characteristics by exploiting this embedding within the framework of general methods of stabilization theory for point processes on $\mathbb{R}^{d-1} \times \mathbb{R}_+$.

Given a generic functional $\xi \in \Xi$, recall from (6.5) its finite size scaling counterpart $\xi^{(\lambda)} \in \Xi^{(\lambda)}$, namely

$$\xi^{(\lambda)}(x, \mathcal{X}) := \xi([T^{\lambda}]^{-1}x, [T^{\lambda}]^{-1}\mathcal{X}), \qquad x \in \mathcal{X} \subset \mathcal{R}_{\lambda} \subset \mathbb{R}^{d-1} \times \mathbb{R}_+.$$

Put

$$\mu_{\lambda}^{\xi} := \sum_{x \in \mathcal{P}^{(\lambda)}} \xi^{(\lambda)}(x, \mathcal{P}^{(\lambda)}) \delta_x \tag{7.1}$$

and $\bar{\mu}_{\lambda}^{\xi} := \mu_{\lambda}^{\xi} - \mathbb{E}\mu_{\lambda}^{\xi}$.

As in Section 6, we write $\xi^{(\infty)} \in \Xi^{(\infty)}$ to denote the local scaling limit analog of ξ; $\xi^{(\infty)}$ is defined on pairs $(x, \mathcal{X})$, with $x \in \mathcal{X} \subset \mathbb{R}^{d-1} \times \mathbb{R}_+$. Recall that when $x \notin \mathcal{X}$, we write $\xi(x, \mathcal{X})$ instead of $\xi(x, \mathcal{X} \cup \{x\})$, with a similar convention for

$\xi^{(\lambda)}$. Recall from Definition 3.1 that $\mathcal{P}$ is a Poisson point process in the upper half-space $\mathbb{R}^{d-1} \times \mathbb{R}_+$ with intensity density $h^\delta\, dh\, dv$. Following [30], we define the second order correlation functions for $\xi^{(\infty)}$ given by

$$\varsigma_{\xi^{(\infty)}}(x) := \mathbb{E}[\xi^{(\infty)}(x, \mathcal{P})]^2, \qquad x \in \mathbb{R}^{d-1} \times \mathbb{R}_+, \tag{7.2}$$

whereas, for all $x, y \in \mathbb{R}^{d-1} \times \mathbb{R}_+$, we put

$$\begin{aligned}\varsigma_{\xi^{(\infty)}}(x, y) := {}& \mathbb{E}[\xi^{(\infty)}(x, \mathcal{P} \cup \{y\})\xi^{(\infty)}(y, \mathcal{P} \cup \{x\})] \\ & - \mathbb{E}[\xi^{(\infty)}(x, \mathcal{P})]\mathbb{E}[\xi^{(\infty)}(y, \mathcal{P})].\end{aligned} \tag{7.3}$$

Define also the asymptotic variance expression

$$\begin{aligned}\sigma^2(\xi^{(\infty)}) := {}& \int_0^\infty \varsigma_{\xi^{(\infty)}}((\mathbf{0}, h))h^\delta\, dh \\ & + \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{d-1}} \varsigma_{\xi^{(\infty)}}((\mathbf{0}, h), (v', h'))h^\delta h'^\delta\, dh\, dh'\, dv'.\end{aligned} \tag{7.4}$$

These expressions are the counterparts to (1.7) and (1.8) in [30]; recall that here we are working in the isotropic regime, corresponding to $\rho_0 \equiv 1$ in [30].

Given $\xi \in \Xi$, consider the sum $\sum_{x \in \mathcal{P}^{(\lambda)}} \lambda^{\eta[\xi]}\xi^{(\lambda)}(x, \mathcal{P}^{(\lambda)})$. There are roughly $\lambda^{\beta(d-1)}$ terms which do not vanish, and thus one expects growth of order λ^τ with

$$\tau = \beta(d-1) = \frac{d-1}{d+1+2\delta}. \tag{7.5}$$

Upon centering and scaling by $\lambda^{-\tau/2}$, one may also expect asymptotic normality as $\lambda \to \infty$. The following theorem, one of the main results of this paper, makes this intuition precise. It establishes a weak law of large numbers, variance asymptotics and a central limit theorem for the afore-mentioned sums as well as for $\lambda^{-\tau/2}\langle g, \lambda^{\eta[\xi]}\mu_\lambda^\xi\rangle = \lambda^{\zeta/2}\langle g, \mu_\lambda^\xi\rangle$, $g \in \mathcal{C}(\mathbb{B}^d)$, where we have $\zeta = -\tau + 2\eta$ from (2.26).

THEOREM 7.1. *For all $\xi \in \Xi$ and all $g \in \mathcal{C}(\mathbb{B}^d)$, we have*

$$\lim_{\lambda \to \infty} \lambda^{-\tau}\mathbb{E}[\langle g, \lambda^{\eta[\xi]}\mu_\lambda^\xi\rangle] = \int_0^\infty \mathbb{E}[\xi^{(\infty)}(\mathbf{0}, h)]h^\delta\, dh \int_{\mathbb{S}^{d-1}} g(u)\, d\sigma_{d-1}(u). \tag{7.6}$$

The integral in (7.4) converges, and for all $g \in \mathcal{C}(\mathbb{B}^d)$, we have

$$\lim_{\lambda \to \infty} \lambda^{-\tau} \operatorname{Var}[\langle g, \lambda^{\eta[\xi]}\overline{\mu}_\lambda^\xi\rangle] = V^{\xi^{(\infty)}}[g] := \sigma^2(\xi^{(\infty)}) \int_{\mathbb{S}^{d-1}} g^2(u)\, d\sigma_{d-1}(u). \tag{7.7}$$

Furthermore, the random variables $\lambda^{-\tau/2}\langle g, \lambda^{\eta[\xi]}\bar{\mu}_\lambda^\xi\rangle$ converge in law to $N(0, V^{\xi^{(\infty)}}[g])$ as $\lambda \to \infty$. Finally, if $\sigma^2(\xi^{(\infty)}) > 0$, then for all $g \in \mathcal{C}(\mathbb{B}^d)$ not identically zero, we have

$$\begin{aligned}& \sup_t \left| P\left[\frac{\langle g, \lambda^{\eta[\xi]}\bar{\mu}_\lambda^\xi\rangle}{\sqrt{\operatorname{Var}[\langle g, \lambda^{\eta[\xi]}\bar{\mu}_\lambda^\xi\rangle]}} \le t\right] - P[N(0,1) \le t]\right| \\ & \quad = O(\lambda^{-\tau/2}(\log \lambda)^{3d+4\delta+1}).\end{aligned} \tag{7.8}$$

REMARKS. (i) The expectation limit (7.6) and variance limit (7.7) generalize the analogous limits appearing at (2.2) and (2.3) in Theorem 2.1 of [30], which is restricted to the case that ξ is the k-face functional with $k=0$. Likewise, convergence in law of $\lambda^{-\tau/2}\langle g, \lambda^{\eta[\xi]}\bar{\mu}_\lambda^\xi\rangle$ and the rate result (7.8) extend the distributional results of Theorem 2.1 of [30].

(ii) We refer to the statements (7.7) and (7.8) as *measure-level variance asymptotics and measure level central limit theorems* for $\lambda^{\eta[\xi]}\mu_\lambda^\xi$, with scaling exponent $-\tau/2$ and with variance density $\sigma^2(\xi^{(\infty)})$. When $g\equiv 1$, we obtain the limit theory for the total mass of μ_λ^ξ, giving *scalar variance asymptotics* and central limit theorems. For all $\xi\in\Xi$, Theorem 7.1 admits a multivariate version giving a central limit theorem for the random vector $(\lambda^{-\tau/2}\langle g_1, \lambda^{\eta[\xi]}\bar{\mu}_\lambda^\xi\rangle, \dots, \lambda^{-\tau/2}\langle g_m, \lambda^{\eta[\xi]}\bar{\mu}_\lambda^\xi\rangle)$, with $g_i\in\mathcal{C}(\mathbb{B}^d)$ for all $i=1,\dots,m$, which follows from the Cramér–Wold device.

(iii) Given $\xi^{(\infty)}\in\Xi^{(\infty)}$, the question whether $\sigma^2(\xi^{(\infty)})$ is strictly positive is nontrivial, and the application of general techniques of stabilization theory designed to check this condition may be far from straightforward. These issues are discussed at the end of this section.

(iv) We have not tried for optimal rates in (7.8) and expect that the exponents on the logarithm can be improved.

The proof of Theorem 7.1 depends on the following three lemmas, which establish further properties of *the scaling limit functionals* $\xi^{(\infty)}\in\Xi^{(\infty)}$ *and local scaling functionals* $\xi^{(\lambda)}\in\Xi^{(\lambda)}$, $\lambda\ge 1$.

LEMMA 7.1. *For all $p>0$ and all $\xi\in\Xi$, we have*

$$\sup_{x\in\mathbb{R}^{d-1}}\mathbb{E}\big[\big|\xi^{(\infty)}(x,\mathcal{P})\big|^p\big]<\infty \quad \textit{and}$$
$$\sup_{\lambda\ge 1}\sup_{x\in\mathcal{R}_\lambda}\mathbb{E}\big[\big|\lambda^{\eta[\xi]}\xi^{(\lambda)}\big(x,\mathcal{P}^{(\lambda)}\big)\big|^p\big]<\infty. \tag{7.9}$$

PROOF. We only give the proof for $\xi^{(\infty)}$, the finite scaling case $\xi^{(\lambda)}$ being fully analogous. This is done separately for all functionals considered.

For $\xi_s^{(\infty)}(x,\mathcal{P})$ and $\xi_r^{(\infty)}(x,\mathcal{P})$ we only consider the case of x extreme, for otherwise both functionals are zero. With $x\in\text{Vertices}(\Phi)$ we make use of (4.5) to bound the height and of (6.7) and (6.8) to bound the spatial size of the regions whose volumes define $\xi_s^{(\infty)}$ and $\xi_r^{(\infty)}$. Since these bounds yield superexponential decay rates on each dimension separately, the volume admits uniformly controllable moments of all orders. Finally, by (6.4), $0\le\xi_{\vartheta_k}^{(\infty)}\le\xi_r^{(\infty)}$ whence (7.9) follows for ξ_{ϑ_k} as well.

For $\xi_{f_k}^{(\infty)}(x,\mathcal{P})$, we only consider the case $x\in\text{Vertices}(\Phi)$, and we let $N:=N[x]$ be the number of extreme points in $\mathcal{P}\cap C_{d-1}(v,R'[x])$ with R' as in (6.8).

Then $\xi_{f_k}^{(\infty)}(x, \mathcal{P})$ is upper bounded by $\binom{N}{k-1}$. By Lemma 3.2 of [30], the probability that a point (v_1, h_1) is extreme in Φ falls off superexponentially fast in h_1; see again (4.5). Consequently, in view of (6.7), the random variables $\binom{N}{k-1}$ and $\xi_{f_k}^{(\infty)}(x, \mathcal{P})$ admit finite moments of all orders.

The proof of Lemma 7.1 is now complete. □

For all $h \in \mathbb{R}_+$, $(v', h') \in \mathcal{R}_\lambda$, and $\xi \in \Xi$, we put

$$\begin{aligned} c^{(\lambda)}((\mathbf{0}, h), (v', h')) := \mathbb{E}\big[&\lambda^{\eta[\xi]}\xi^{(\lambda)}\big((\mathbf{0}, h), \mathcal{P}^{(\lambda)} \cup (v', h')\big) \\ &\times \lambda^{\eta[\xi]}\xi^{(\lambda)}\big((v', h'), \mathcal{P}^{(\lambda)} \cup (\mathbf{0}, h)\big)\big] \\ &- \mathbb{E}\big[\lambda^{\eta[\xi]}\xi^{(\lambda)}\big((\mathbf{0}, h), \mathcal{P}^{(\lambda)}\big)\big]\mathbb{E}\big[\lambda^{\eta[\xi]}\xi^{(\lambda)}\big((v', h'), \mathcal{P}^{(\lambda)}\big)\big]. \end{aligned}$$

The next lemma makes use of the moment bounds of Lemma 7.1 and is proved through straightforward modifications of the proofs of Lemmas 3.3 and 3.4 in [30].

LEMMA 7.2. *For all $h \in \mathbb{R}_+$, $(v', h') \in \mathcal{R}_\lambda$ and $\xi \in \Xi$ we have as $\lambda \to \infty$,*

$$\mathbb{E}\big[\lambda^{\eta[\xi]}\xi^{(\lambda)}\big((\mathbf{0}, h), \mathcal{P}^{(\lambda)}\big)\big] \to \mathbb{E}\big[\xi^{(\infty)}((\mathbf{0}, h), \mathcal{P})\big]$$

and

$$c^{(\lambda)}((\mathbf{0}, h), (v', h')) \to \varsigma_{\xi^{(\infty)}}((\mathbf{0}, h), (v', h')).$$

The next lemma is the analog of Lemma 3.5 in [30] and is proved similarly.

LEMMA 7.3. *There is a constant $C < \infty$ such that for all $h \in \mathbb{R}_+$, $(v', h') \in \mathcal{R}_\lambda$, and all $\xi \in \Xi$, we have*

$$\big|c^{(\lambda)}((\mathbf{0}, h), (v', h'))\big| \le C \exp\left(\frac{-1}{C} \max(|v'|, h, h')\right)$$

and

$$\big|\varsigma_{\xi^{(\infty)}}((\mathbf{0}, h), (v', h'))\big| \le C \exp\left(\frac{-1}{C} \max(|v'|, h, h')\right).$$

Equipped with these lemmas, we now prove Theorem 7.1. We shall give separate proofs for (7.6), (7.7) and (7.8), following closely the methods of [30].

Proof of the expectation formula (7.6). For $g \in \mathcal{C}(\mathbb{B}^d)$, we have for all $\xi \in \Xi$,

$$\mathbb{E}[\langle g, \mu_\lambda^\xi \rangle] = \lambda \int_{\mathbb{B}^d} g(x)\mathbb{E}[\xi(x, \mathcal{P}_\lambda)](1 - |x|)^\delta \, dx. \tag{7.10}$$

By rotation invariance, we have that $\xi(x, \mathcal{P}_\lambda) \stackrel{\mathcal{D}}{=} \xi(x^\theta, \mathcal{P}_\lambda^\theta)$, where x^θ is x rotated by the angle θ, and similarly for $\mathcal{P}_\lambda^\theta$. Letting $\theta := \theta_x$ be the rotation sending $x/|x|$

to u_0, gives $\mathbb{E}\xi(x,\mathcal{P}_\lambda) = \mathbb{E}\xi^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)})$, where $h := \lambda^\gamma(1-|x|)$. Thus we rewrite (7.10) as

$$\begin{aligned}\mathbb{E}[\langle g,\mu_\lambda^\xi\rangle] &= \lambda\int_{\mathbb{B}^d} g(x)\mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0},h),\mathcal{P}^{(\lambda)}\big)\big]\lambda^{-\gamma\delta}h^\delta\,dx\\ &= \lambda^{1-\gamma\delta}\int_{\mathbb{S}^{d-1}}\int_0^{\lambda^\gamma} g\big(u(1-\lambda^{-\gamma}h)\big)\mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0},h),\mathcal{P}^{(\lambda)}\big)\big]\\ &\qquad\times h^\delta(1-\lambda^{-\gamma}h)^{d-1}\lambda^{-\gamma}\,dh\,d\sigma_{d-1}(u).\end{aligned}\tag{7.11}$$

Noting that $\tau = 1-\delta\gamma-\gamma$ and multiplying through by $\lambda^{-\tau+\eta[\xi]}$, we obtain

$$\begin{aligned}\lambda^{-\tau+\eta[\xi]}\mathbb{E}[\langle g,\mu_\lambda^\xi\rangle] &= \int_{\mathbb{S}^{d-1}}\int_0^{\lambda^\gamma} g\big(u(1-\lambda^{-\gamma}h)\big)\mathbb{E}\big[\lambda^{\eta[\xi]}\xi^{(\lambda)}\big((\mathbf{0},h),\mathcal{P}^{(\lambda)}\big)\big]\\ &\qquad\times(1-\lambda^{-\gamma}h)^{d-1}h^\delta\,dh\,d\sigma_{d-1}(u).\end{aligned}\tag{7.12}$$

Notice that $\mathbb{E}[\lambda^{\eta[\xi]}\xi^{(\lambda)}((\mathbf{0},h),\mathcal{P}^{(\lambda)}]h^\delta$ is dominated by an integrable function of h, as the contribution coming from large h is well controllable as in Lemma 3.2 in [30]—in particular we exploit that $\xi(x,\mathcal{X}) = 0$ whenever x is nonextreme in $\mathcal{X}$ and, roughly speaking, only points close enough to the boundary $\mathbb{S}^{d-1}$ have a nonnegligible chance of being extreme in $\mathcal{P}_\lambda$. Thus letting $\lambda\to\infty$ in (7.12), applying the first part of Lemma 7.2, using $\lim_{\lambda\to\infty}(1-\lambda^{-\gamma}h)^{d-1} = 1$ and $\lim_{\lambda\to\infty} g(u(1-\lambda^{-\gamma}h)) = g(u)$ for all $u\in\mathbb{S}^{d-1}$ and applying the dominated convergence theorem as in, for example, Section 3.2 in [30], we finally get from (7.12) the required relation (7.6).

Proof of variance convergence (7.7). We have for $g\in\mathcal{C}(\mathbb{B}^d)$ and $\xi\in\Xi$, that

$$\begin{aligned}&\lambda^{-\tau+2\eta[\xi]}\operatorname{Var}[\langle g,\overline{\mu}_\lambda^\xi\rangle]\\ &\quad= \lambda^{-\tau+2\eta[\xi]+1}\int_{\mathbb{B}^d} g^2(x)\mathbb{E}[\xi(x,\mathcal{P}_\lambda)^2](1-|x|)^\delta\,dx\\ &\qquad+\lambda^{-\tau+2\eta[\xi]+2}\int_{\mathbb{B}^d}\int_{\mathbb{B}^d} g(x)g(y)\big(\mathbb{E}[\xi(x,\mathcal{P}_\lambda\cup y)\xi(y,\mathcal{P}_\lambda\cup x)]\\ &\qquad\qquad-\mathbb{E}[\xi(x,\mathcal{P}_\lambda)]\mathbb{E}[\xi(y,\mathcal{P}_\lambda)]\big)\\ &\qquad\qquad\times(1-|x|)^\delta(1-|y|)^\delta\,dx\,dy\\ &\quad:= I+II.\end{aligned}$$

As in (7.11), we write term I as

$$I = \lambda^{-\tau+2\eta[\xi]+1}\int_{\mathbb{B}^d} g^2(x)\mathbb{E}\big[\xi^{(\lambda)}\big((\mathbf{0},\lambda^\gamma(1-|x|)),\mathcal{P}^{(\lambda)}\big)^2\big](1-|x|)^\delta\,dx.$$

Now put $h := \lambda^\gamma(1-|x|)$, and write $dx = (1-\lambda^{-\gamma}h)^{d-1}\,d\sigma_{d-1}(u)\lambda^{-\gamma}\,dh$. This transforms I as follows:

$$\begin{aligned}I = \lambda^{-\tau+1-\gamma-\delta\gamma}\int_{\mathbb{S}^{d-1}}\int_0^{\lambda^\gamma} & g^2\big(u(1-\lambda^{-\gamma}h)\big)\mathbb{E}\big[\big(\lambda^{\eta[\xi]}\xi^{(\lambda)}\big((\mathbf{0},h),\mathcal{P}^{(\lambda)}\big)\big)^2\big]\\ &\times h^\delta(1-\lambda^{-\gamma}h)^{d-1}\,dh\,d\sigma_{d-1}(u).\end{aligned}$$

Lemma 7.2 and the moment bounds of Lemma 7.1 give

$$\lim_{\lambda\to\infty} \mathbb{E}\big[\big(\lambda^{\eta[\xi]}\xi^{(\lambda)}\big((\mathbf{0},h),\mathcal{P}^{(\lambda)}\big)\big)^2\big] = \mathbb{E}\big[\xi^{(\infty)}((\mathbf{0},h),\mathcal{P}))^2\big] := \varsigma_{\xi^{(\infty)}}((\mathbf{0},h)).$$

Since $\tau := 1-\delta\gamma-\gamma$, by the dominated convergence theorem, we obtain, as $\lambda\to\infty$, that

$$I \to \int_{\mathbb{S}^{d-1}}\int_0^\infty g^2(u)\varsigma_{\xi^{(\infty)}}((\mathbf{0},h))h^\delta\,dh\,d\sigma_{d-1}(u). \tag{7.13}$$

We now consider term II. Recall $x := (u_x,h_x)\in\mathbb{B}^d$ and $y := (u_y,h_y)\in\mathbb{B}^d$. We rotate all points in $\mathcal{P}_\lambda\cup\{x,y\}$ in such a way that $x/|x|$ gets sent to u_0. Denote the rotated point set by $\mathcal{P}'_\lambda\cup\{x',y'\}$, where $x' := (\mathbf{0},h_{x'})$, $y' := (v_{y'},h_{y'})$, with $h_{x'} = 1-|x'|$, $h_{y'} = 1-|y'|$.

We write term II as

$$II = \lambda^{-\tau+2-2\gamma\delta}\int_{\mathbb{B}^d}\int_{\mathbb{B}^d} g(x')g(y')[\cdots]h^\delta_{x'}h^\delta_{y'}\,dx'\,dy',$$

where

$$[\cdots] := \mathbb{E}[\xi(x',\mathcal{P}_\lambda\cup\{y'\})\xi(y',\mathcal{P}_\lambda\cup\{x'\})] - \mathbb{E}\xi(x',\mathcal{P}_\lambda)\mathbb{E}\xi(y',\mathcal{P}_\lambda). \tag{7.14}$$

Write

$$T^{(\lambda)}(x') := (\mathbf{0},\lambda^\gamma h_{x'}) := (\mathbf{0},h); \qquad T^{(\lambda)}(y') := (\lambda^\beta v_{y'},\lambda^\gamma h_{y'}) := (v',h');$$
$$T^{(\lambda)}(\mathcal{P}'_\lambda) := \mathcal{P}'^{(\lambda)}.$$

Under these transformations, the expression $[\cdots]$ in (7.14) transforms to

$$\begin{aligned}[\cdots]' ={}& \mathbb{E}\big[\lambda^{\eta[\xi]}\xi^{(\lambda)}\big((\mathbf{0},h),\mathcal{P}'^{(\lambda)}\cup(v',h')\big)\lambda^{\eta[\xi]}\xi^{(\lambda)}\big((v',h'),\mathcal{P}'^{(\lambda)}\cup(\mathbf{0},h)\big)\big]\\ &- \mathbb{E}\lambda^{\eta[\xi]}\xi^{(\lambda)}\big((\mathbf{0},h),\mathcal{P}'^{(\lambda)}\big)\lambda^{\eta[\xi]}\mathbb{E}\xi^{(\lambda)}\big((v',h'),\mathcal{P}^{(\lambda)}\big).\end{aligned} \tag{7.15}$$

Recalling the definitions of x' and y', we obtain $h^\delta_{x'} = \lambda^{-\gamma\delta}h^\delta$, $h^\delta_{y'} = \lambda^{-\gamma\delta}(h')^\delta$, with

$$dx' = (1-\lambda^{-\gamma}h)^{d-1}\,d\sigma_{d-1}(u)\lambda^{-\gamma}\,dh,$$

and

$$dy' = \lambda^{-\beta(d-1)}\,dv'\lambda^{-\gamma}\,dh'.$$

Thus the polynomial λ multiplier in term II gets replaced by $\lambda^{-\tau+2-2\gamma\delta}\times\lambda^{-2\gamma}\lambda^{-\beta(d-1)}$, and so the differential $\lambda^{-\tau+2-2\gamma\delta}\,dx'\,dy'$ on $\mathbb{B}^d\times\mathbb{B}^d$ in term II transforms to the differential

$$\lambda^{-\tau+2-2\gamma\delta}\lambda^{-\beta(d-1)}\lambda^{-2\gamma}(1-\lambda^{-\gamma}h)^{d-1}\,d\sigma_{d-1}(u)\,dv'\,dh'\,dh \tag{7.16}$$

on $\mathbb{S}^{d-1} \times T^{(\lambda)}(\mathbb{S}^{d-1}) \times [0, \lambda^\gamma] \times [0, \lambda^\gamma]$. The pre-factor in (7.16) involving powers of λ reduces to unity in view of the identity $\tau = 2 - 2\gamma - 2\gamma\delta - \beta(d-1)$. Thus (7.16) transforms to

$$(1 - \lambda^{-\gamma} h)^{d-1} \, d\sigma_{d-1}(u) \lambda^{-\gamma} \, dv' \, dh' \, dh. \tag{7.17}$$

For all triples $(v', h', h) \in T^{(\lambda)}(\mathbb{S}^{d-1}) \times [0, \lambda^\gamma] \times [0, \lambda^\gamma]$, the covariance term $[\cdots]'$ at (7.15) may be expressed as

$$[\cdots]' = c^{(\lambda)}((\mathbf{0}, h), (v', h')). \tag{7.18}$$

By Lemma 7.2 we have for all triples $(v', h', h) \in T^{(\lambda)}(\mathbb{S}^{d-1}) \times [0, \lambda^\gamma] \times [0, \lambda^\gamma]$, that as $\lambda \to \infty$

$$[\cdots]' = c^{(\lambda)}((\mathbf{0}, h), (v', h')) \to \varsigma_{\xi^{(\infty)}}((\mathbf{0}, h), (v', h')). \tag{7.19}$$

Finally, for all $y' \in \mathbb{B}^d$, consider the factor $g(y')$ in the integrand of term II. The factor $g(y')$ transforms to $g((T^{(\lambda)})^{-1}(v', h'))$. For all pairs $(v', h') \in T^{(\lambda)}(\mathbb{S}^{d-1}) \times [0, \lambda^\gamma]$, we have $(T^{(\lambda)})^{-1}(v', h') \to (u_{x'}, 0)$ as $\lambda \to \infty$. By continuity of g we obtain

$$g\big((T^{(\lambda)})^{-1}(v', h')\big) \to g(u_{x'}, 0) \tag{7.20}$$

as $\lambda \to \infty$.

Therefore, combining (7.17), (7.18), we may rewrite term II as

$$\begin{aligned} II = \int_{\mathbb{S}^{d-1}} \int_{T^{(\lambda)}(\mathbb{S}^{d-1})} \int_0^{\lambda^\gamma} \int_0^{\lambda^\gamma} & g\big((u(1 - \lambda^{-\gamma} h)\big) g\big((T^{(\lambda)})^{-1}(v', h')\big) \\ & \times c^{(\lambda)}((\mathbf{0}, h), (v', h')) \\ & \times h^\delta h'^\delta (1 - \lambda^{-\gamma} h)^{d-1} \, dh' \, dh \, dv' \, d\sigma_{d-1}(u). \end{aligned}$$

By Lemma 7.3, the integrand is dominated by the function

$$(u, v', h', h) \mapsto C h^\delta h'^\delta \exp\bigg(\frac{-1}{C} \max(|v'|, h, h')\bigg),$$

which is integrable on $\mathbb{S}^{d-1} \times \mathbb{R}^{d-1} \times (0, \infty)^2$. The dominated convergence theorem, combined with the limits (7.19) and (7.20), together with $T^{(\lambda)}(\mathbb{S}^{d-1}) \uparrow \mathbb{R}^{d-1}$, show that as $\lambda \to \infty$, we have

$$\begin{aligned} II \to \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}^{d-1}} \int_0^\infty \int_0^\infty & g(u)^2 \varsigma_{\xi^{(\infty)}}((\mathbf{0}, h), (v', h')) \\ & \times h^\delta h'^\delta \, dh' \, dh \, dv' \, d\sigma_{d-1}(u). \end{aligned} \tag{7.21}$$

The second part of Lemma 7.3 implies that the integral in (7.21) is finite. Combining (7.13) with (7.21) gives the desired limit (7.7).

Proof of Gaussian convergence (7.8). The proof uses the Stein method for dependency graphs and is inspired by the proof of Theorem 2.1 of [19], which involves a dependency graph structure on nonscaled sample points in a rectangular solid. Since the sample points of this paper belong to the unit ball, we find it more convenient to put a dependency graph on the re-scaled points $\mathcal{P}^{(\lambda)}$ in $\mathcal{R}_\lambda$. Additionally, we do not use all of the re-scaled points $\mathcal{P}^{(\lambda)}$, but only those with a small height coordinate. These differences complicate the approach and, in an effort to make this paper reader friendly, we include the details. As in [19], the argument makes use of the following lemma of Chen and Shao [8]. For any random variable X and any $p > 0$, let $\|X\|_p := (\mathbb{E}|X|^p)^{1/p}$. Let Φ denote the cumulative distribution function of the standard normal.

LEMMA 7.4 (See Theorem 2.7 of [8]). *Let $2 < q \le 3$. Let $W_i, i \in \mathcal{V}$, be random variables indexed by the vertices of a dependency graph. Let $W = \sum_{i\in\mathcal{V}} W_i$. Assume that $\mathbb{E}W^2 = 1, \mathbb{E}W_i = 0$, and $\|W_i\|_q \le \theta$ for all $i \in \mathcal{V}$ and for some $\theta > 0$. Then*

$$\sup_t |P[W \le t] - \Phi(t)| \le 75 D^{5(q-1)} |\mathcal{V}| \theta^q. \tag{7.22}$$

Fix $\xi \in \Xi$ to be one of the basic functionals discussed in Section 6. For all $g \in \mathcal{C}(\mathbb{B}^d)$, we have

$$\langle g, \lambda^{\eta[\xi]} \mu_\lambda^\xi \rangle = \sum_{x \in \mathcal{P}_\lambda} \lambda^{\eta[\xi]} \xi(x, \mathcal{P}_\lambda) g(x) = \sum_{x' \in \mathcal{P}^{(\lambda)}} \lambda^{\eta[\xi]} \xi^{(\lambda)}(x', \mathcal{P}^{(\lambda)}) g([T^\lambda]^{-1} x').$$

Recalling that $x' := (\lambda^\beta v_x, h)$, we define for all $L > 0$ and $g \in \mathcal{C}(\mathbb{B}^d)$,

$$T_\lambda^\xi(L, g) := \sum_{x' \in \mathcal{P}^{(\lambda)}, h \le L \log \lambda} \lambda^{\eta[\xi]} \xi^{(\lambda)}(x', \mathcal{P}^{(\lambda)}) g([T^\lambda]^{-1} x').$$

By the analog of Lemma 3.2 of [30], given $K > 0$ and large, we may choose $L := L(K)$ large so that $\langle g, \lambda^{\eta[\xi]} \mu_\lambda^\xi \rangle$ and $T_\lambda^\xi(L, g)$ coincide everywhere except on a set with probability $O(\lambda^{-K})$. It follows that $\langle g, \lambda^{\eta[\xi]} \mu_\lambda^\xi \rangle$ and $T_\lambda^\xi(L, g)$ have the same asymptotic distribution as $\lambda \to \infty$, and it may be shown that they share the same variance asymptotics. It suffices to find a rate of convergence to $N(0, 1)$ for $(T_\lambda^\xi(L, g) - \mathbb{E}T_\lambda^\xi(L, g))/\sqrt{\mathrm{Var}[T_\lambda^\xi(L, g)]}$.

To prepare for dependency graph arguments, we put $\rho_\lambda := L \log \lambda$, L a constant, and we subdivide $\lambda^\beta \mathbb{B}_{d-1}(\pi)$ into $V(\lambda) := (2\pi\lambda^\beta)^{d-1} (\rho_\lambda)^{-(d-1)}$ sub-cubes $Q_i, i = 1, \dots, V(\lambda)$, of edge length ρ_λ and of volume $(\rho_\lambda)^{d-1}$. Enumerate the points $\mathcal{P}^{(\lambda)} \cap [Q_i \times L \log \lambda]$ by $\{X'_{ij}\}_{j=1}^{N_i}$ so that

$$T_\lambda^\xi(L, g) = \sum_{i=1}^{V(\lambda)} \sum_{j=1}^{N_i} \lambda^{\eta[\xi]} \xi^{(\lambda)}(X'_{ij}, \mathcal{P}^{(\lambda)}) g([T^\lambda]^{-1} X'_{ij}).$$

The random variable N_i is Poisson whose mean ν_i equals the $\mathcal{P}^{(\lambda)}$ intensity measure of the rectangular solid $Q_i \times L \log \lambda$, and thus ν_i is bounded by the product of $\mathrm{Vol}(Q_i \times L \log \lambda)$ and the maximum of the intensity of $\mathcal{P}^{(\lambda)}$ on this solid.

Recalling the intensity of $\mathcal{P}^{(\lambda)}$ at (2.14), we obtain

$$\nu_i := \mathbb{E}N_i \le C\big(\mathrm{Vol}(Q_i \times L\log\lambda)\big)(L\log\lambda)^{\delta} = C(\rho_\lambda)^{d+\delta}.$$

The following result is the analog of Lemma 4.3 of [19] and is proved similarly. For $1 \le i \le V(\lambda)$, and for $j \in \{1, 2, \dots\}$, we define for the fixed functional ξ,

$$\xi_{i,j} := \begin{cases} \lambda^{\eta[\xi]}\xi^{(\lambda)}\big(X'_{i,j}, \mathcal{P}^{(\lambda)}\big), & \text{if } N_i \ge j,\ X'_{i,j} \in \lambda^{\beta}\mathbb{B}_{d-1}(\pi) \times [0, L\log\lambda], \\ 0, & \text{otherwise.} \end{cases}$$

With $\xi \in \Xi$ still fixed, note that ξ satisfies the moment condition (7.9) for all $p > q \ge 1$.

LEMMA 7.5. *With $p > q \ge 1$, there exists $C := C(p, q)$, such that for $1 \le i \le V(\lambda)$, we have*

$$\Bigg\| \sum_{j=1}^{\infty} |\xi_{i,j}| \Bigg\|_q \le C\rho_\lambda^{(d+\delta)(p+1)/p}.$$

Continuing with the dependency graph arguments, we let $p > q$ and $q \in (2, 3]$. Recall from Section 6 that $R^{\xi^{(\lambda)}}(x)$ is localization radius for the functional $\xi^{(\lambda)}$ if and only if a.s.

$$\xi^{(\lambda)}\big(x, \mathcal{P}^{(\lambda)}\big) = \xi^{(\lambda)}_{[r]}\big(x, \mathcal{P}^{(\lambda)}\big)$$

for all $r \ge R^{\xi^{(\lambda)}}(x)$. Put $U(t) := \sup_{\lambda \ge 1, x \in \mathcal{R}_\lambda} P[R^{\xi^{(\lambda)}}(x) > t]$, which is the analog of the τ function defined at (2.2) of [19]. With the choice $\rho_\lambda = L\log\lambda$, Lemma 6.1 implies that for L large, we have that $U(\rho_\lambda)$ has polynomial decay of high order, and so we have

$$\begin{aligned} &V(\lambda)\rho_\lambda^{(d+\delta)(p+1)/p}\big(\lambda^{\beta(d-1)}(\log\lambda)^{1+\delta}U(\rho_\lambda)\big)^{(q-2)/2q} < \lambda^{-3-(\beta d/2)} \quad \text{and} \\ &\qquad U(\rho_\lambda) < \lambda^{-\beta(d-1)-3}, \end{aligned} \tag{7.23}$$

which is the analog of display (4.8) in [19]. We also have $\rho_\lambda^{d+\delta} < C\lambda^{p/(p+2)}$, the analog of display (4.9) in [19].

For all $1 \le i \le V(\lambda)$ and all $j = 1, 2, \dots,$ let $R^{(\lambda)}_{i,j}$ denote the radius of stabilization of $\xi^{(\lambda)}$ at $X'_{i,j}$ if $1 \le j \le N_i$ and $X'_{i,j} \in \lambda^{\beta}\mathbb{B}_{d-1}(\pi) \times [0, \lambda^{\gamma}]$; let $R_{i,j}$ be zero otherwise. Let $E_{i,j} := \{R^{(\lambda)}_{i,j} \le \rho_\lambda\}$. Let $E_\lambda := \bigcap_{i=1}^{V(\lambda)} \bigcap_{j=1}^{\infty} E_{i,j}$. Then

$$\begin{aligned} P[E_\lambda^c] &\le \mathbb{E}\Bigg[\sum_{i=1}^{V(\lambda)} \sum_{j=1}^{N_i} \mathbf{1}(E^c_{i,j})\Bigg] \\ &= \int_{\lambda^{\beta}\mathbb{B}_{d-1}(\pi) \times [0, L\log\lambda]} P\big[R^{\xi^{(\lambda)}}(x) \ge \rho_\lambda\big] h^{\delta}\, dv\, dh \\ &\le C\lambda^{\beta(d-1)}(\log\lambda)^{1+\delta}U(\rho_\lambda). \end{aligned} \tag{7.24}$$

We have thus

$$T_\lambda^\xi(L,g) := \sum_{i=1}^{V(\lambda)} \sum_{j=1}^{N_i} \lambda^{\eta[\xi]} \xi^{(\lambda)}\big(X'_{i,j}, \mathcal{P}^{(\lambda)}\big) g([T^\lambda]^{-1}(X'_{i,j})).$$

To obtain rates of normal approximation, we consider a modified version of $T_\lambda^\xi(L,g)$ having more independence between terms, namely

$$T'_\lambda(L,g) := \sum_{i=1}^{V(\lambda)} \sum_{j=1}^{N_i} \lambda^{\eta[\xi]} \xi^{(\lambda)}\big(X'_{i,j}, \mathcal{P}^{(\lambda)}\big) \mathbf{1}(E_{i,j}) g([T^\lambda]^{-1}(X'_{i,j})).$$

For all $1 \le i \le V(\lambda)$, define

$$S_i := S_{Q_i} := (\operatorname{Var} T'_\lambda(L,g))^{-1/2} \sum_{j=1}^{N_i} \lambda^{\eta[\xi]} \xi^{(\lambda)}\big(X'_{i,j}, \mathcal{P}^{(\lambda)}\big) \mathbf{1}(E_{i,j}) g([T^\lambda]^{-1}(X'_{i,j}))$$

and put

$$S := (\operatorname{Var} T'_\lambda(L,g))^{-1/2}\big(T'_\lambda(L,g) - \mathbb{E}T'_\lambda(L,g)\big) = \sum_{i=1}^{V(\lambda)} (S_i - \mathbb{E}S_i). \tag{7.25}$$

Note that $\operatorname{Var} S = \mathbb{E}S^2 = 1$.

We define a graph $G_\lambda := (\mathcal{V}_\lambda, \mathcal{E}_\lambda)$ as follows. The set $\mathcal{V}_\lambda$ consists of the subcubes $Q_1, \dots, Q_{V(\lambda)}$ and edges (Q_i, Q_j) belong to $\mathcal{E}_\lambda$ if $d(Q_i, Q_j) \le 2\rho_\lambda$, where $d(Q_i, Q_j) := \inf\{|x-y|, x \in Q_i, y \in Q_j\}$. To prepare for dependency graph arguments, we make the following five observations, paralleling those in [19]:

(i) $V(\lambda) := |\mathcal{V}_\lambda|$.

(ii) Since the number of cubes in $Q_1, \dots, Q_V$ distant at most $2\rho_\lambda$ from a given cube is bounded by 5^d, it follows that the maximal degree D of G_λ satisfies $D := D_\lambda \le 5^d$.

(iii) For all $1 \le i \le V(\lambda)$ and all $q \ge 1$, we have, by Lemma 7.5,

$$\begin{aligned} \|S_i\|_q &\le C(\operatorname{Var} T'_\lambda(l,g))^{-1/2} \left\| \sum_{j=1}^\infty |\xi_{i,j}| \right\|_q \\ &\le C(\operatorname{Var} T'_\lambda(L,g))^{-1/2} \rho_\lambda^{(d+\delta)(p+1)/p}. \end{aligned} \tag{7.26}$$

(iv) $T'_\lambda(L,g)$ is the sum of $V(\lambda)$ random variables, which, by the case $q=2$ of Lemma 7.5, each have a variance bounded by a constant multiple of $\rho_\lambda^{2(d+\delta)(p+1)/p}$. The covariance of any pair of the $V(\lambda)$ random variables is zero when the indices of the random variables correspond to nonadjacent cubes. For adjacent cubes, the Cauchy–Schwarz inequality implies that the covariance is also bounded by a constant multiple of $\rho_\lambda^{2(d+\delta)(p+1)/p}$. This gives the analog of (4.13) of [19], namely

$$\operatorname{Var}[T'_\lambda(L,g)] = O\big(\rho_\lambda^{2(d+\delta)(p+1)/p} V(\lambda)\big). \tag{7.27}$$

(v) $\mathrm{Var}[T'_\lambda(L,g)]$ is close to $\mathrm{Var}[T^\xi_\lambda(L,g)]$ for λ large. We require more estimates to show this. Note that $|T'_\lambda(L,g) - T^\xi_\lambda(L,g)| = 0$ except possibly on the set E^c_λ. Lemma 7.5, along with Minkowski's inequality, yields the upper bound

$$
\begin{aligned}
\left\| \sum_{i=1}^{V(\lambda)} \sum_{j=1}^{N_i} \left|\lambda^{\eta[\xi]}\xi^{(\lambda)}\big(X'_{i,j}, \mathcal{P}^{(\lambda)}\big)\right| \right\|_q &\le C V(\lambda)\rho_\lambda^{(d+\delta)(p+1)/p} \\
&= C\lambda^{\beta(d-1)}\rho_\lambda^{-(d-1)}\rho_\lambda^{(d+\delta)(p+1)/p}.
\end{aligned}
\tag{7.28}
$$

Since $T^\xi_\lambda(L,g) = T'_\lambda(L,g)$ on the event E_λ, as in [19], the Hölder and Minkowski inequalities yield

$$
\begin{aligned}
\|T^\xi_\lambda(L,g) - T'_\lambda(L,g)\|_2 &\le \|T^\xi_\lambda(L,g) - T'_\lambda(L,g)\|_q P[E^c_\lambda]^{(1/2)-(1/q)} \\
&\le \big(\|T^\xi_\lambda(L,g)\|_q + \|T'_\lambda(L,g)\|_q\big) P[E^c_\lambda]^{(q-2)/(2q)}.
\end{aligned}
$$

Hence, by (7.24) and the first inequality in (7.29),

$$
\begin{aligned}
&\|T^\xi_\lambda(L,g) - T'_\lambda(L,g)\|_2 \\
&\quad\le C V(\lambda)\rho_\lambda^{(d+\delta)(p+1)/p}\big(\lambda^{\beta(d-1)}(\log\lambda)^{1+\delta}U(\rho_\lambda)\big)^{(q-2)/2q}.
\end{aligned}
$$

By (7.23) this yields

$$
\|T^\xi_\lambda(L,g) - T'_\lambda(L,g)\|_2 \le C\lambda^{-3-\beta d/2}
\tag{7.29}
$$

which clearly implies

$$
\mathbb{E}[|T'_\lambda(L,g) - T_\lambda(L,g)|] \le C\lambda^{-3},
\tag{7.30}
$$

which we use later. As in [19], we obtain the analog of (4.17) of [19], that is,

$$
|\operatorname{Var}[T^\xi_\lambda(L,g)] - \operatorname{Var}[T'_\lambda(L,g)]| \le C\lambda^{-2},
\tag{7.31}
$$

concluding observation (v).

We may now use Lemma 7.4 and dependency graph arguments to establish the error bound (7.8). We apply the bound (7.22) of Lemma 7.4 to $W_i := S_i - \mathbb{E}S_i$, $1 \le i \le V(\lambda)$, with

$$
\theta := C(\operatorname{Var} T'_\lambda(L,g))^{-1/2}\rho_\lambda^{(d+\delta)(p+1)/p}.
$$

Observe that $\mathbb{E}W_i = 0$, $\mathbb{E}(\sum_{i=1}^{V(\lambda)} W_i)^2 = 1$, $\|W_i\|_q \le \theta$ and recall from (7.25) that $S = \sum_{i=1}^{V(\lambda)} W_i$. Lemma 7.4 along with observation (i) above yields the counterpart of (4.18) of [19], namely

$$
\begin{aligned}
\sup_t |P[S \le t] - \Phi(t)| &\le C V(\lambda)(\operatorname{Var} T'_\lambda(L,g))^{-q/2}\rho_\lambda^{q(d+\delta)(p+1)/p} \\
&\le C V(\lambda)(\operatorname{Var} T^\xi_\lambda(L,g))^{-q/2}\rho_\lambda^{q(d+\delta)(p+1)/p}.
\end{aligned}
\tag{7.32}
$$

The last line follows since by (7.7) we have $\mathrm{Var}[T^{\xi}_{\lambda}(L,g)] = \Theta(\lambda^{\tau})$, $\tau \in (0,1)$, and thus by (7.31) we get for λ large that $\mathrm{Var}[T'_{\lambda}(L,g)] \geq \mathrm{Var}[T^{\xi}_{\lambda}(L,g)]/2$. Now put $q = 3$ in (7.32). Since $T^{\xi}_{\lambda}(L,g)$ and $\langle g, \lambda^{\eta[\xi]}\mu^{\xi}_{\lambda}\rangle$ have the same variance asymptotics, it follows from the assumption $\sigma^2(\xi^{(\infty)}) > 0$ that when $q = 3$, we have $(\mathrm{Var}\, T^{\xi}_{\lambda}(L,g))^{-q/2} = \Theta(\lambda^{-3\tau/2})$. Since $V(\lambda) = \lambda^{\tau}\rho_{\lambda}^{-(d-1)}$ and since $q/p < 1$, display (7.32) becomes

$$\sup_t |P[S \leq t] - \Phi(t)| \leq C\lambda^{-\tau/2}(\log\lambda)^{3d+4\delta+1}.$$

This gives a rate of convergence for S, as defined at (7.25). Following verbatim the last part of the proof of Theorem 2.1 of [19] [starting three lines after (4.18) of that paper], we deduce a rate of convergence for $T^{\xi}_{\lambda}(L,g)$, namely

$$\sup_t \left| P\left[\frac{T^{\xi}_{\lambda}(L,g) - \mathbb{E}T^{\xi}_{\lambda}(L,g)}{\sqrt{\mathrm{Var}\, T^{\xi}_{\lambda}(L,g)}} \leq t \right] - P[N(0,1) \leq t] \right| \leq C\lambda^{-\tau/2}(\log\lambda)^{3d+4\delta+1}.$$

This yields (7.8), concluding the proof of Theorem 7.1.

Positivity of asymptotic variances. For $\xi^{(\infty)} \in \Xi^{(\infty)}$, we now consider the question whether $\sigma^2(\xi^{\infty})$ is strictly positive. Fortunately, the variances $\sigma^2(\xi^{(\infty)}_r)$, $\sigma^2(\xi^{(\infty)}_s)$ and $\sigma^2(\xi^{(\infty)}_{\vartheta_k})$, $k \in \{1,\dots,d-1\}$, admit alternative expressions enjoying monotonicity properties in the underlying Poisson input process $\mathcal{P}$, enabling us to use suitable positive correlation inequalities and to conclude the required positivity for variance densities. The underlying Poisson input process $\mathcal{P}$ depends on the parameter δ [recall (3.1)], and the following lemma holds for all $\delta > 0$. More precisely, we have:

LEMMA 7.6. *We have*

$$\sigma_s^2 := \sigma^2(\xi_s^{(\infty)}) = \int_{\mathbb{R}^{d-1}} \mathrm{Cov}(\partial\Psi(0), \partial\Psi(v))\,dv,$$

$$\sigma_r^2 := \sigma^2(\xi_r^{(\infty)}) = \int_{\mathbb{R}^{d-1}} \mathrm{Cov}(\partial\Phi(0), \partial\Phi(v))\,dv$$

and

$$\sigma_k^2 := \sigma^2(\xi^{(\infty)}_{\vartheta_k}) = \int_{\mathbb{R}^{d-1}} \mathrm{Cov}\left(\int_0^{\Phi(0)} \vartheta_k^{\infty}((\mathbf{0},h))\,dh, \int_0^{\Phi(v)} \vartheta_k^{\infty}((v,h))\,dh\right)dv.$$

PROOF. We only consider the functional $\xi_s^{(\infty)}$, as the remaining cases are analogous. Recalling (7.2)–(7.4), the general theory of stabilizing functionals (see, e.g., [4], [15]) shows that if $\xi^{(\infty)}$ is a generic exponentially stabilizing functional on the Poisson input $\mathcal{P}$ on $\mathbb{R}^{d-1} \times \mathbb{R}_+$, then

$$\lim_{T\to\infty} \frac{1}{T^{d-1}} \mathrm{Var}\left(\sum_{x=(v,h)\in\mathcal{P}, v\in[0,T]^{d-1}} \xi^{(\infty)}(x,\mathcal{P})\right) = \sigma^2(\xi^{(\infty)}),$$

that is to say, the scaled variance limit of $\sum_{x=(v,h)\in\mathcal{P}, v\in[0,T]^{d-1}} \xi^{(\infty)}(x,\mathcal{P})$ coincides with

$$\lim_{\lambda\to\infty} \lambda^{-\tau} \operatorname{Var}\biggl(\sum_{x\in\mathcal{P}^{(\lambda)}} \xi^{(\lambda)}\bigl(x,\mathcal{P}^{(\lambda)}\bigr)\biggr).$$

Since $\xi_s^{(\infty)}$ is an exponentially stabilizing functional on the Poisson input $\mathcal{P}$ (recall Lemma 6.1), it follows that $\sigma^2(\xi_s^{(\infty)})$ is the asymptotic variance density for $\xi_s^{(\infty)}$, that is to say,

$$\sigma^2\bigl(\xi_s^{(\infty)}\bigr) = \lim_{T\to\infty} \frac{1}{T^{d-1}} \operatorname{Var}\biggl(\sum_{x=(v,h)\in\mathcal{P}, v\in[0,T]^{d-1}} \xi_s^{(\infty)}(x,\mathcal{P})\biggr).$$

For $x := (v,h) \in \operatorname{ext}(\Psi) = \operatorname{Vertices}(\Phi)$ denote by $V[x] := V[x;\mathcal{P}]$ the set of all $v' \in \mathbb{R}^{d-1}$ for which there exists h' with $(v',h') \in \mathcal{F}^{\infty}(x,\mathcal{P})$—in other words, $V[x]$ is the spatial projection of all faces f of Φ with $x = \operatorname{Top}(f)$. Clearly, $\{V[x], x \in \operatorname{ext}(\Psi)\}$, forms a tessellation of $\mathbb{R}^{d-1}$. Thus, by definition of $\xi_s^{(\infty)}$,

$$\sigma^2\bigl(\xi_s^{(\infty)}\bigr) = \lim_{T\to\infty} \frac{1}{T^{d-1}} \operatorname{Var}\biggl(\sum_{x=(v,h)\in\operatorname{Vertices}(\Psi), v\in[0,T]^{d-1}} \int_{V[x]} \partial\Psi(u)\,du\biggr).$$

Consequently,

$$\begin{aligned}\sigma^2\bigl(\xi_s^{(\infty)}\bigr) &= \lim_{T\to\infty} \frac{1}{T^{d-1}} \operatorname{Var}\biggl(\int_{[0,T]^{d-1}} \partial\Psi(u)\,du\biggr)\\ &= \lim_{T\to\infty} \frac{1}{T^{d-1}} \int_{([0,T]^{d-1})^2} \operatorname{Cov}(\partial\Psi(u), \partial\Psi(u'))\,du'\,du,\end{aligned}$$

where the existence of the integrals follows from the exponential localization of $\xi_s^{(\infty)}$, as stated in Lemma 6.1, implying the exponential decay of correlations. Further, by stationarity of the process $\partial\Psi(\cdot)$, we obtain

$$\begin{aligned}\sigma^2(\xi_s^{(\infty)}) &= \lim_{T\to\infty} \frac{1}{T^{d-1}} \int_{[0,T]^{d-1}} \int_{[0,T]^{d-1}} \operatorname{Cov}\bigl(\partial\Psi(\mathbf{0}), \partial\Psi(u'-u)\bigr)\,du'\,du\\ &= \lim_{T\to\infty} \int_{[-T,T]^{d-1}} \frac{\operatorname{Vol}([0,T]^{d-1} \cap ([0,T]^{d-1} + w))}{T^{d-1}}\\ &\qquad\qquad \times \operatorname{Cov}(\partial\Psi(\mathbf{0}), \partial\Psi(w))\,dw\\ &= \int_{\mathbb{R}^{d-1}} \operatorname{Cov}(\partial\Psi(\mathbf{0}), \partial\Psi(w))\,dw\end{aligned}$$

as required, with the penultimate equality following again by exponential localization of $\xi_s^{(\infty)}$, implying the exponential decay of correlations and thus allowing us to apply dominated convergence theorem to determine the limit of integrals. This completes the proof of Lemma 7.6. □

Observe that for each v, $\partial\Psi(v), \partial\Phi(v)$ as well as $\int_0^{\Phi(v)} \vartheta_k^\infty((v,h))\,dh$ are all nonincreasing functionals of $\mathcal{P}$ and therefore

$$\mathrm{Cov}(\partial\Psi(\mathbf{0}), \partial\Psi(v)) \geq 0, \qquad \mathrm{Cov}(\partial\Phi(\mathbf{0}), \partial\Phi(v)) \geq 0$$

and

$$\mathrm{Cov}\biggl(\int_0^{\Phi(\mathbf{0})} \vartheta_k^\infty((\mathbf{0},h))\,dh, \int_0^{\Phi(v)} \vartheta_k^\infty((v,h))\,dh\biggr) \geq 0$$

for all $v \in \mathbb{R}^{d-1}$ in view of the positive correlations property of Poisson point processes; see Proposition 5.31 in [24]. It is also readily seen that these covariances are not identically zero, because for $v = 0$ they are just variances of nonconstant random variables and, depending continuously on v, they are strictly positive on a nonzero measure set of v's. Thus, the integrals in the variance expressions given in Lemma 7.6 are all strictly positive. Consequently, we have

COROLLARY 7.1. *For all $\delta > 0$, the variance densities $\sigma^2(\xi_r^{(\infty)}), \sigma^2(\xi_s^{(\infty)})$ and $\sigma^2(\xi_{\vartheta_k}^{(\infty)})$, $k \in \{1, \ldots, d-1\}$ are all strictly positive.*

REMARK. When $\delta = 0$ the variance positivity for $\sigma^2(\xi_{\vartheta_k}^{(\infty)})$ has been established in a slightly different, but presumably equivalent, context (binomial input) in [2], Theorem 1.

We also believe that for all $\delta > 0$, the variance density $\sigma^2(\xi_{f_k}^{(\infty)})$ is strictly positive as well—this is because of the asymptotic nondegeneracy of the corresponding so-called add-one cost functional [4, 15, 17, 19]. However, making this intuition precise requires additional technical considerations, as does extending the important work of Reitzner [21] to the case $\delta > 0$, which shows strict variance positivity for $\delta = 0$.

Variance asymptotics and central limit theorems for mean widths, volumes, intrinsic volumes and k-face functionals. We now deduce from Theorem 7.1 and Corollary 7.1 the limit theory for the convex hull functionals described at the outset of this paper. We require some preliminary observations which will also be needed in Section 8. Define for $v \in \mathbb{R}^{d-1}$ the defect width functional

$$H_\lambda^{\xi_s}(v) := \sum_{x \in \mathcal{P}_\lambda, x/|x| \in \exp([\mathbf{0},v])} \xi_s(x; \mathcal{P}_\lambda) \tag{7.33}$$

and the defect volume functional

$$H_\lambda^{\xi_r}(v) := \sum_{x \in \mathcal{P}_\lambda, x/|x| \in \exp([\mathbf{0},v])} \xi_r(x, \mathcal{P}_\lambda). \tag{7.34}$$

The next lemma shows that the centered defect width functional approximates its asymptotic counterpart W_λ and likewise for the centered defect volume functional.

LEMMA 7.7. *We have, uniformly in $v \in \mathbb{R}^{d-1}$,*

$$\lim_{\lambda \to \infty} \lambda^{\zeta/2} \big| \big(H_\lambda^{\xi_s}(v) - \mathbb{E} H_\lambda^{\xi_s}(v) \big) - \big(W_\lambda(v) - \mathbb{E} W_\lambda(v) \big) \big| \stackrel{P}{=} 0 \tag{7.35}$$

and

$$\lim_{\lambda \to \infty} \lambda^{\zeta/2} \big| \big(H_\lambda^{\xi_r}(v) - \mathbb{E} H_\lambda^{\xi_r}(v) \big) - \big(V_\lambda(v) - \mathbb{E} V_\lambda(v) \big) \big| \stackrel{P}{=} 0. \tag{7.36}$$

PROOF. We first prove (7.35). It is enough to show the two following limits:

$$\lim_{\lambda \to \infty} \lambda^{\zeta/2} \big| \big(H_\lambda^{\xi_s}(v) - \mathbb{E} H_\lambda^{\xi_s}(v) \big) - \big(\mathrm{Vol}(\mathcal{C}(v)) - \mathbb{E}\, \mathrm{Vol}(\mathcal{C}(v)) \big) \big| \stackrel{P}{=} 0 \tag{7.37}$$

and

$$\lim_{\lambda \to \infty} \lambda^{\zeta/2} \big| \big(\mathrm{Vol}(\mathcal{C}(v)) - \mathbb{E}\, \mathrm{Vol}(\mathcal{C}(v)) \big) - \big(W_\lambda(v) - \mathbb{E} W_\lambda(v) \big) \big| \stackrel{P}{=} 0, \tag{7.38}$$

where $\mathcal{C}(v) := [\mathbb{B}^d \setminus F(\mathcal{P}_\lambda)] \cap \mathrm{cone}(\exp([\mathbf{0}, v]))$.

We start by proving (7.37). For all $v \in \mathbb{R}^{d-1}$, $|H_\lambda^{\xi_s}(v) - \mathrm{Vol}(\mathcal{C}(v))|$ is bounded by the volume of the set

$$\Delta_\lambda(v) := [\mathbb{B}^d \setminus F(\mathcal{P}_\lambda)] \\ \cap \left(\mathrm{cone}(\exp([\mathbf{0}, v])) \Delta \bigcup_{x \in \mathcal{P}_\lambda \cap \mathrm{cone}(\exp([\mathbf{0}, v]))} \mathrm{cone}(\mathcal{F}(x, \mathcal{P}_\lambda)) \right).$$

Let

$$\mathcal{F}_\lambda(v) := \bigcup \{ f \in \mathcal{F}_{d-1}(K_\lambda) : f \cap \partial\, \mathrm{cone}(\exp([\mathbf{0}, v])) \neq \varnothing \}.$$

By the usual scaling via the transformation T^λ, we get that $T^\lambda(\Delta_\lambda(v))$ is a solid $D_\lambda(v)$, with $D_\lambda(v) \subset T^\lambda(\mathcal{F}_\lambda(v))$. Consider the $(d-2)$-dimensional surface given by

$$S_\lambda(v) := \partial \big(T^\lambda[\mathrm{cone}(\exp([\mathbf{0}, v])) \cap \mathbb{S}^{d-1}] \big).$$

Then the maximal height coordinate of $D_\lambda(v)$, with respect to the surface $S_\lambda(v)$, satisfies the exponential decay (4.5). Also, the maximal spatial distance between $T^\lambda(\mathcal{F}_\lambda(v))$ and $S_\lambda(v)$ has exponentially decaying tails, as in Lemma 6.1. By mimicking the proof of Theorem 7.1, but with now τ taken to be $\beta(d-2)$ instead of $\beta(d-1)$, it follows that $\lambda^{-\beta(d-2)/2+\eta}(\Delta_\lambda(v) - \mathbb{E}\Delta_\lambda(v))$ converges to a normal random variable. Since $\lambda^{\zeta/2} := \lambda^{\beta(d-1)/2+\gamma} = o(\lambda^{-\beta(d-2)/2+\eta})$, this gives the convergence (7.37).

We prove now (7.38). We deduce from (2.25) that

$$\begin{aligned} W_\lambda(v) - \mathrm{Vol}(\mathcal{C}(v)) &= \int_{\exp([\mathbf{0}, v])} \left(s_\lambda(u) - \frac{1 - (1 - s_\lambda(u))^d}{d} \right) d\sigma_{d-1}(u) \\ &= O\left(\int_{\exp([\mathbf{0}, v])} s_\lambda^2(u)\, d\sigma_{d-1}(u) \right). \end{aligned}$$

For every $x \in \mathcal{P}_\lambda$, let

$$\widetilde{\xi}_s(x,\mathcal{P}_\lambda) := \int_{\mathrm{cone}(\mathcal{F}(x,\mathcal{P}_\lambda)\cap\exp([\mathbf{0},v]))} s_\lambda^2(u,\mathcal{P}_\lambda)\,d\sigma_{d-1}(u).$$

In particular, we have

$$\int_{\exp([\mathbf{0},v])} s_\lambda^2(u)\,d\sigma_{d-1}(u) = \sum_{x\in\mathcal{P}_\lambda} \widetilde{\xi}_s(x,\mathcal{P}_\lambda).$$

Exactly as for $\xi \in \Xi$, where Ξ is the class of functionals defined in Section 6, $\widetilde{\xi}_s$ has an associated scaling prefactor $\lambda^{\eta[\widetilde{\xi}_s]}$ with $\eta[\widetilde{\xi}_s] = \beta(d-1)+2\gamma$ (recall that s_λ is of order λ^γ). Moreover $\widetilde{\xi}_s$ is seen to satisfy the exponential decay (6.6). Following verbatim the proof of Theorem 7.1, we obtain that $\lambda^{-\tau/2+\eta[\widetilde{\xi}_s]}\int_{\exp([\mathbf{0},v])}(s^2(u,\mathcal{P}_\lambda) - \mathbb{E}s^2(u,\mathcal{P}_\lambda))\,d\sigma_{d-1}(u)$ converges to a normal random variable. Since $\tau/2 - \eta[\widetilde{\xi}_s] < -\zeta/2$, the asserted limits (7.38) and (7.35) follow.

To prove (7.36), it suffices to recall from (2.23) that

$$V_\lambda(v) := \int_{\exp([\mathbf{0},v])} r_\lambda(u)\,d\sigma_{d-1}(u)$$

and to follow arguments similar to those given above for $H_\lambda^{\xi_s}$. This completes the proof of Lemma 7.7. $\square$

Letting $d\kappa_d$ be the total surface measure of $\mathbb{S}^{d-1}$ and recalling $\zeta := (d+3)/(d+1+2\delta)$ from (2.26), the following theorem gives scalar variance asymptotics and scalar central limit theorems for the basic functionals discussed in the Introduction.

THEOREM 7.2. (i) *The volume functional $V(K_\lambda)$ satisfies*

$$\lim_{\lambda\to\infty} \lambda^\zeta \operatorname{Var}[V(K_\lambda)] = \sigma_V^2 := \sigma^2\bigl(\xi_r^{(\infty)}\bigr)\,d\kappa_d \tag{7.39}$$

and

$$\lambda^{\zeta/2}\bigl(V(K_\lambda) - \mathbb{E}V(K_\lambda)\bigr) \xrightarrow{\mathcal{D}} N(0,\sigma_V^2), \tag{7.40}$$

where σ_V^2 is strictly positive.

(ii) *The volume functional $V_\lambda(\infty)$ satisfies the identical asymptotics whereas the mean width functional $W_\lambda(\infty)$ and the intrinsic volume functionals $V_k(K_\lambda)$, $k \in \{1,\dots,d-1\}$ satisfy (7.39) and (7.40) with strictly positive variances $\sigma_W^2 := \sigma^2(\xi_s^{(\infty)})\,d\kappa_d$ and $\sigma_{V_k}^2 := \sigma^2(\xi_{\vartheta_k}^{(\infty)})\,d\kappa_d$, respectively.*

REMARK. Recalling (2.27) and setting $\delta = 0$, Theorem 7.2 yields the asserted variance limits (1.1), (1.2) and (1.4). In Section 8 we shall show convergence of the $\mathbb{R}^{d-1}$-indexed processes $W_\lambda(\cdot)$ and $V_\lambda(\cdot)$.

PROOF OF THEOREM 7.2. To prove the assertion for $V(K_\lambda)$, it suffices to put $g \equiv 1$ and $\xi \equiv \xi_r$ in Theorem 7.1, to recall that $\mathrm{Vol}(\mathbb{B}^d \setminus K_\lambda) = \sum_{x \in \mathcal{P}_\lambda} \xi_r(x, \mathcal{P}_\lambda)$, and to use $\lambda^{-\tau}\lambda^{2\eta[\xi_r]} = \lambda^{(d+3)/(d+1+2\delta)}$. Corollary 7.1 yields positivity of the limiting variance σ_V^2. The limit theory for $V_\lambda(\infty)$ holds since we may follow verbatim the proof of Lemma 7.7 to show that $\sum_{x \in \mathcal{P}_\lambda} \xi_r(x, \mathcal{P}_\lambda)$ approximates $V_\lambda(\infty)$.

Similarly, to prove the asserted limit theory for $W_\lambda(\infty)$, we put $g \equiv 1$ and $\xi \equiv \xi_s$ in Theorem 7.1, we use $\lambda^{-\tau}\lambda^{2\eta[\xi_s]} = \lambda^{(d+3)/(d+1+2\delta)}$, and we follow verbatim the proof of Lemma 7.7 to show that $\sum_{x \in \mathcal{P}_\lambda} \xi_s(x, \mathcal{P}_\lambda)$ approximates $W_\lambda(\infty)$. Corollary 7.1 yields positivity of the limiting variance σ_W^2. Finally, the asserted limit theory for $V_k(K_\lambda)$ follows by putting $g \equiv 1$ and $\xi \equiv \xi_{\vartheta_k}$ in Theorem 7.1 and using Corollary 7.1 to deduce the positivity of the limiting variance. $\square$

Next, using (6.3) and Theorem 7.1 we obtain the limit theory for the k-face empirical measures $\mu_\lambda^{f_k}$ defined at (2.5).

THEOREM 7.3. *For each $k \in \{0, \dots, d-1\}$, the k-face empirical measures $\mu_\lambda^{f_k}$ satisfy the measure-level variance asymptotics and central limit theorem with scaling exponent $\tau/2$ and with variance density $\sigma^2(\xi_{f_k}^{(\infty)})$ where $\tau := (d-1)/(d+1+2\delta)$. In particular, the total number $f_k(K_\lambda)$ of k-faces for K_λ satisfies the scalar variance asymptotics and central limit theorem with scaling exponent $\tau/2$ and variance $\sigma_{f_k}^2 := \sigma^2(\xi_{f_k}^{(\infty)})\, d\kappa_d$.*

REMARKS. (i) Setting $\delta = 0$ in Theorem 7.3 gives the asserted variance limit (1.3).

(ii) We expect that the variance asymptotics of Theorems 7.2 and 7.3 can be de-Poissonized, that is to say, that there are analogous variance limits when the polytope K_λ is replaced by the polytope K_n generated by n i.i.d. uniformly distributed points in $\mathbb{B}^d$. We leave these issues for further study.

8. Global regime and Brownian limits. In this section we establish a functional central limit theorem for the integrated convex hull processes $\hat{W}_\lambda$ and $\hat{V}_\lambda$, defined at (2.27). The methods extend to yield functional central limit theorems for stabilizing functionals in general, thus extending [31].

For any $\sigma^2 > 0$ let B^{σ^2} be the Brownian sheet of variance coefficient σ^2 on the injectivity region $\mathbb{B}_{d-1}(\pi)$ of $\exp := \exp_{\mathbb{S}^{d-1}}$; that is to say, B^{σ^2} is the mean zero continuous path Gaussian process indexed by $\mathbb{R}^{d-1}$ with

$$\mathrm{Cov}(B^{\sigma^2}(v), B^{\sigma^2}(w)) = \sigma^2 \cdot \sigma_{d-1}\big(\exp([\mathbf{0}, v] \cap [\mathbf{0}, w])\big),$$

where, recall, σ_{d-1} is the $(d-1)$-dimensional surface measure on $\mathbb{S}^{d-1}$. Recalling from Lemma 7.6 the shorthand notation $\sigma_s^2 := \sigma^2(\xi_s^{(\infty)})$ and $\sigma_r^2 := \sigma^2(\xi_r^{(\infty)})$, we have the following limit result, the main result of this section. Via Lemma 7.7

and Theorem 7.2, this theorem also yields Brownian sheet limits for the defect width and volume functionals given at (7.33) and (7.34), respectively. We remark that the same holds for the two processes $\mathrm{Vol}([\mathbb{B}^d \setminus K_\lambda] \cap \mathrm{cone}(\exp([\mathbf{0}, v])))$ and $\mathrm{Vol}([\mathbb{B}^d \setminus F(\mathcal{P}_\lambda)] \cap \mathrm{cone}(\exp([\mathbf{0}, v])))$, $v \in \mathbb{R}^{d-1}$.

THEOREM 8.1. *As $\lambda \to \infty$, the random functions $\hat{W}_\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$ converge in law to $B^{\sigma_s^2}$ in the space $\mathcal{C}(\mathbb{R}^{d-1})$. Likewise, the random functions $\hat{V}_\lambda : \mathbb{R}^{d-1} \to \mathbb{R}$ converge in law to $B^{\sigma_r^2}$ in $\mathcal{C}(\mathbb{R}^{d-1})$.*

PROOF OF THEOREM 8.1. Our argument relies heavily on the theory developed in [30] and is further extended in Section 6. For $v \in \mathbb{R}^{d-1}$ and $x \in \mathbb{B}^d$, define

$$\mathbf{1}_{\mathbb{B}^d}^{[\mathbf{0},v]}(x) := \begin{cases} 1, & \text{if } x/|x| \in \exp([\mathbf{0}, v]), \\ 0, & \text{otherwise}. \end{cases} \tag{8.1}$$

We thus have the identities

$$\lambda^{\zeta/2}\big(H_\lambda^{\xi_s}(v) - \mathbb{E}H_\lambda^{\xi_s}(v)\big) = \lambda^{\zeta/2}\langle \mathbf{1}_{\mathbb{B}^d}^{[\mathbf{0},v]}, \bar{\mu}_\lambda^{\xi_s}\rangle$$

and

$$\lambda^{\zeta/2}\big(H_\lambda^{\xi_r}(v) - \mathbb{E}H_\lambda^{\xi_r}(v)\big) = \lambda^{\zeta/2}\langle \mathbf{1}_{\mathbb{B}^d}^{[\mathbf{0},v]}, \bar{\mu}_\lambda^{\xi_r}\rangle.$$

Recalling from (2.27) that $\hat{W}_\lambda(v) := \lambda^{\zeta/2}(W_\lambda(v) - \mathbb{E}W_\lambda(v))$ and $\hat{V}_\lambda(v) := \lambda^{\zeta/2}(V_\lambda(v) - \mathbb{E}V_\lambda(v))$, and using (7.35) and (7.36) from Lemma 7.7, we obtain, uniformly in v,

$$\begin{aligned} &\lim_{\lambda\to\infty} \big|\hat{W}_\lambda(v) - \lambda^{\zeta/2}\langle \mathbf{1}_{\mathbb{B}^d}^{[\mathbf{0},v]}, \bar{\mu}_\lambda^{\xi_s}\rangle\big| \stackrel{P}{=} 0, \\ &\lim_{\lambda\to\infty} \big|\hat{V}_\lambda(v) - \lambda^{\zeta/2}\langle \mathbf{1}_{\mathbb{B}^d}^{[\mathbf{0},v]}, \bar{\mu}_\lambda^{\xi_r}\rangle\big| \stackrel{P}{=} 0. \end{aligned} \tag{8.2}$$

Even though $\mathbf{1}_{\mathbb{B}^d}^{[\mathbf{0},v]}$ is not a continuous function, it is easily seen that the proofs in [30] hold for functions which are almost everywhere continuous with respect to the uniform measure on $\mathbb{B}^d$, and, in fact, the central limit theorems and variance asymptotics of [30] hold for all bounded functions on $\mathbb{B}^d$. Thus Theorem 7.1 for ξ_s and ξ_r remain valid upon setting the test function g to $\mathbf{1}_{\mathbb{B}^d}^{[\mathbf{0},v]}$. This application of Theorem 7.1, combined with (8.2), yields that the fidis of $(\hat{W}_\lambda(v))_{v\in\mathbb{R}^{d-1}}$ converge to those of $(B^{\sigma^2(\xi_s^{(\infty)})}(v))_{v\in\mathbb{R}^{d-1}}$ and, likewise, the fidis of $(\hat{V}_\lambda(v))_{v\in\mathbb{R}^{d-1}}$ converge to those of $(B^{\sigma^2(\xi_r^{(\infty)})}(v))_{v\in\mathbb{R}^{d-1}}$. Additionally, for all $v \in \mathbb{R}^{d-1}$, we have

$$\lim_{\lambda\to\infty} \mathrm{Var}[\hat{W}_\lambda(v)] = \sigma^2\big(\xi_s^{(\infty)}\big)(v),$$

with similar variance asymptotics for $\hat{V}_\lambda(v)$; see also Theorems 1.2 and 1.3 in [30].

We claim that the fidis convergence of $\hat{W}_\lambda$ and $\hat{V}_\lambda$ can be strengthened to convergence in law in $\mathcal{C}(\mathbb{R}^{d-1})$. It suffices to establish the tightness of the processes

$(\hat{W}_\lambda(v))_{v\in\mathbb{R}^{d-1}}$ and $(\hat{V}_\lambda(v))_{v\in\mathbb{R}^{d-1}}$. We shall focus on $\hat{W}_\lambda$, the argument for $\hat{V}_\lambda$ being analogous, and we shall proceed to some extent along the lines of the proof of Theorem 8.2 in [9], which is based on [5]. We extend the definition of W_λ to subsets of $\mathbb{R}^{d-1}$ putting for measurable $B \subseteq \mathbb{R}^{d-1}$

$$W_\lambda(B) := \int_{\exp_{d-1}(B)} s_\lambda(u)\, d\sigma_{d-1}(u)$$

and letting

$$\hat{W}_\lambda(B) := \lambda^{\zeta/2}\big(W_\lambda(B) - \mathbb{E}W_\lambda(B)\big). \tag{8.3}$$

It is enough to show

$$\mathbb{E}(\hat{W}_\lambda([v,v']))^4 = O(\mathrm{Vol}([v,v'])^2), \qquad v, v' \in \mathbb{R}^{d-1}, \tag{8.4}$$

for then $\hat{W}_\lambda$ satisfies condition (2) on page 1658 of [5], thus belongs to the class $\mathcal{C}(2,4)$ of [5] and is tight in view of Theorem 3 on page 1665 of [5].

To this end, we put

$$W^\#_\lambda(B) := \lambda^{\eta[\xi_s]} W_\lambda(B) = \lambda^{\beta(d-1)+\gamma} W_\lambda(B), \tag{8.5}$$

where we recall from the definition of $\Xi^{(\lambda)}$ in Section 6 that $\eta[\xi_s] = \beta(d-1)+\gamma$ is the proper scaling exponent for ξ_s. The crucial point now is that in analogy to the proof of Lemma 5.3 in [4], and similar to (3.24) in the proof of Theorem 1.3 in [30], by a stabilization-based argument all cumulants of $W^\#_\lambda([v,w])$ over rectangles $[v,w]$ are at most linear in $\lambda^\tau\,\mathrm{Vol}([v,w])$ with $\tau := \beta(d-1)$ as in (7.5). In other words, for all $k \geq 1$, we have

$$|c^k(W^\#_\lambda([v,w]))| \leq C_k \lambda^\tau\,\mathrm{Vol}([v,w]), \qquad v, w \in \mathbb{R}^{d-1}, \tag{8.6}$$

where $c^k(Y)$ stands for the kth order cumulant of the random variable Y and where C_k is a constant. Thus, putting (8.3) and (8.5) together, we get from (8.6)

$$\begin{aligned}|c^k(\hat{W}_\lambda([v,w]))| &\leq C_k \lambda^{k[\zeta/2-\eta[\xi_s]]}\lambda^\tau\,\mathrm{Vol}([v,w]) \\ &= C_k \lambda^{k[\zeta/2-\beta(d-1)-\gamma]}\lambda^{\beta(d-1)}\,\mathrm{Vol}([v,w]).\end{aligned} \tag{8.7}$$

To proceed, we use the identity $\mathbb{E}(Y-\mathbb{E}Y)^4 = c^4(Y) + 3(c^2(Y))^2$ valid for any random variable Y. Recalling that $\gamma = 2\beta$ and $\zeta = \beta(d-1)+2\gamma$, as in (2.11) and (2.26), respectively, we obtain from (8.4)–(8.7) that for $v, w \in \mathbb{R}^{d-1}$,

$$\begin{aligned}\mathbb{E}(\hat{W}_\lambda([v,w]))^4 &= O\big(\lambda^{4[\zeta/2-\beta(d-1)-\gamma]}\lambda^{\beta(d-1)}\,\mathrm{Vol}([v,w])\big) \\ &\quad + O\big(\big[\lambda^{2[\zeta/2-\beta(d-1)-\gamma]}\lambda^{\beta(d-1)}\,\mathrm{Vol}([v,w])\big]^2\big) \\ &= O\big(\lambda^{-\beta(d-1)}\,\mathrm{Vol}([v,w])\big) + O(\mathrm{Vol}([v,w])^2),\end{aligned} \tag{8.8}$$

which is of the required order $O(\mathrm{Vol}([v,w])^2)$ as soon as $\mathrm{Vol}([v,w]) = \Omega(\lambda^{-\beta(d-1)})$. Thus we have shown (8.4) for $\mathrm{Vol}([v,w]) = \Omega(\lambda^{-\beta(d-1)})$, and we have to show it holds for $\mathrm{Vol}([v,w]) = O(\lambda^{-\beta(d-1)})$ as well. To this end, we use that $W_\lambda([v,w]) = \lambda^{-\gamma} O_P(\mathrm{Vol}([v,w]))$ with γ being the height coordinate re-scaling exponent, and $\mathbb{E}[W_\lambda([v,w]) - \mathbb{E}W_\lambda([v,w])]^4 = \lambda^{-4\gamma} O(\mathrm{Vol}([v,w])^4)$. Thus by (8.3)

$$\mathbb{E}(\hat{W}_\lambda([v,w]))^4 = \lambda^{2\zeta}\lambda^{-4\gamma} O(\mathrm{Vol}([v,w])^4).$$

Recalling $\zeta = \beta(d-1) + 2\gamma$ and using that $\mathrm{Vol}([v,w]) = O(\lambda^{-\beta(d-1)})$, we conclude that

$$\mathbb{E}(\hat{W}_\lambda([v,w]))^4 = O\big(\lambda^{2\beta(d-1)}\mathrm{Vol}([v,w])^4\big) = O(\mathrm{Vol}([v,w]^2))$$

as required, which completes the proof of the required relation (8.4). Having obtained the required tightness, we get the convergence in law of $(\hat{W}_\lambda(v))_{v\in\mathbb{R}^{d-1}}$ to $(B^{\sigma_s^2}(v))_{v\in\mathbb{R}^{d-1}}$ and, likewise, of $(\hat{V}_\lambda(v))_{v\in\mathbb{R}^{d-1}}$ to $(B^{\sigma_r^2}(v))_{v\in\mathbb{R}^{d-1}}$in $\mathcal{C}(\mathbb{R}^{d-1})$. This completes the proof of Theorem 8.1. $\square$

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