Research Summary and Research Plan for Garth Isaak

Summary of Research Papers

This is a summary of research papers for the tenure packet of Garth Isaak. For this context I will write slightly differently than a regular research summary. I will occasionally use the first person. I will intentionally be somewhat informal, *not* defining terms or giving complete references, as they are contained in the papers. Instead, I will give a brief description and some of the context of the work, but for the sake of brevity will not include all details. For the full story on related work again see the papers.

I include all of my research papers so that others can form their own opinion about the different papers. The summary will give some view of my perspective on the relative contributions. On the multi-author (3 or more) papers [8], [9] and [12] I did the actual writing, synthesizing various parts, as well as making significant content contributions of my own. All of the two author papers are real collaborations.

Roughly speaking, I can divide my papers into four groups. One group consists of individual papers on problems encountered in various contexts that do not fit into any systematic research 'program' of mine. The other three groups are sets of papers from areas that I have maintained continuing investigations over several years; investigations into structures related to interval graphs and orders, investigations into sets of inconsistencies under tournament rankings, and higher dimensional analogues of de Bruijn cycles. I will include a section discussing work in each group. There is no particular order in terms of presenting the groups. On page 5 I will be begin a discussion of research plans.

Interval Graphs and Orders

In [4] I answered a question of Bogart regarding representing an interval order using intervals that have discrete (integral) endpoints and meet specified length bounds. A network/ integer linear programming model is used to give a polynomial algorithm for determining if there is a representation as well as characterizations when there is a constant upper and constant lower bound on interval lengths. A similar network was introduced independently by Golumbic and Shamir [33] in the study of 'sandwich' problems. The network model can also be used to simplify earlier related work of Fishburn (see [30]) which used 'picycles' and linear programming models. The methods can also be used to give alternate proofs of fundamental representation theorems for interval orders and semiorders.

In the early 1980's Golumbic, Monma and Trotter [31], [32] introduced a generalization of interval graphs in which intersections must exceed certain tolerances in order for an edge to occur (this corresponds to allowing some tolerance in applications where intervals represent possible interference). Structural results on tolerance graphs and orders have turned out to be surprisingly difficult. Regular interval graphs/orders can be represented using intervals with no interval properly contained in another if and only if they can be represented with unit length intervals. Golumbic, Monma and Trotter [32] asked if this was the case for tolerance graphs/orders. In [9] we show that this is not the case, giving infinite families that are proper but not unit. In [15] we show that for bitolerance orders the unit and proper classes do coincide.

In [16] I examine powers of Hamiltonian paths in the class of interval graphs. My original interest arose from my work in [12] in which we needed to look at the separation number of a highly structured graph (comparability graph of a complete binary tree) for an application of wavelets to medical imaging. Separation is a natural 'dual' to the well known bandwidth problem and corresponds to looking for powers of Hamiltonian paths in the graph complement. Recently there has been significant work regarding degree conditions and powers of Hamiltonian paths and cycles. My work is in another direction. Over the last ten years there has been some interest in efficient algorithms for detecting Hamiltonian paths/cycles for classes of graphs such as interval, circular arc, cographs, and cocomparability. Chvatal's toughness condition, which is necessary but not sufficient for the existence of a Hamiltonian cycle in general graphs is sufficient (or nearly so) in these and other highly structured graph classes. I show in [16] that a natural toughness condition is also sufficient in the case of powers of Hamiltonian paths in interval graphs and begin investigation of the power analogues of various Hamiltonian completion problems. Not surprisingly the sufficiency of the toughness conditions and various minimax results that hold, for example in cocomparability graphs for Hamiltonian problems, fail for Hamiltonian power problems. In [17], we obtain various such structural results for threshold graphs, arborescent comparability graphs and examine where such results fail in cographs and interval graphs. Some of the work in [17] arose from an undergraduate honors thesis by Sam Donnelly [25] that I directed.

Perfect Maps

Results regarding De Bruijn cycles have a long (100 years) history of discovery and rediscovery and arise in various applications. The study of two and higher dimensional analogues had been suggested over thirty years ago and again various applications have been suggested. In the 1980's initial ground breaking work on these objects appeared in the work of Fan,Fan, Ma and Sui [29], Cock [20], Etzion [27] and others. In [5] we answered a question posed by Chung, Diaconis and Graham [19] by constructing 'square' two dimensional de Bruijn tori in nearly all (except $n = 3, 5, 7, 9$) cases in which such are possible and conjectured more general existence results.

DeBruijn tori have also been called perfect maps and attempts to construct them produce the need for 'factored' and 'multi' versions, called perfect factors and perfect multifactors. These variations are of interest in their own right. Perfect maps (de Bruijn tori) are a special cases of each of these. There are obvious necessary conditions on the various parameters for existence of perfect (multi) factors and it is conjectured that these conditions are (in most cases) sufficient. Paterson [41] showed sufficiency for perfect maps in two dimensions when the base is a prime power. Paterson and Mitchell have written a number of papers developing results for one and two dimensional cases. In [6], [10] and [14] Hurlbert and I focus on perfect factors in higher dimensions, obtaining new results on cases where sufficiency of the conditions can be shown.

While several techniques for perfect factors have been examined, much of the work has built on two techniques, what we call 'integrating' (introduced under the name Lempel's homomorphism by Fan, Fan, Ma and Sui [29]) and 'concatenation' (introduced independently by Ma [40], Cock [20], by Etzion [27] and by others). In [18], I codify the various extensions of these techniques in a unified framework, in the process giving the first general results on 'integrating' in dimensions three and higher. This allows construction of new classes of higher dimensional perfect multifactors. I expect that this general framework will set the stage for even more results regarding sufficiency in higher dimensions.

Perfect factors fit into the general scheme of universal cycles, introduced by Chung, Diaconis and Graham [19] for 'cyclic' representations of combinatorial objects. In [13], we construct a compact type of universal cycle for permutations, answering a question raised by Chung, Diaconis and Graham for such compact representations, although using a different tack than they suggested.

Inconsistencies Under Tournament Ranking

J.P. Barthelemy suggested a different perspective on the old problem of tournament rankings at a conference in 1989. He suggested that instead of examining what a ranking procedure produces on various tournaments, that one look at patterns of inconsistencies under the ranking and ask when or if they can arise. This has intrinsic appeal in terms of understanding ranking as well as producing some mathematically interesting questions and providing insight into older problems. In [8] we began a systematic investigation of this problem for the ranking procedure that minimizes the number of inconsistencies. There are a number of general structural results as well as specific results for some very special cases of inconsistency digraphs. In [1] we look at this problem for ranking by score sequences.

An interesting question which arises and provides important bounds on the 'reversing number' is when 'the' acyclic tournament on n vertices can be a feedback arc set for some larger tournament. In [11] I extend work on a linear programming model that I introduced in [8] and also introduce new techniques which are in some sense similar to latin square construction in order to give lower and upper bounds for the size of the larger tournament. In some cases this this yields an exact answer. The results also touch on questions regarding the very difficult question of when the minimum size of a feedback arc set is equal to the maximum number of arc disjoint cycles.

Other Papers

Other papers which I have worked on include [3], where we examined some special cases of p-intersection representations and [2], which contains elementary mathematics, where we use 'acceptability' graphs for representing possible pairings in the stable matching problem. (We were encouraged to write [2] because of the introduction of this perspective.)

My work in [12], began with my coauthors asking me a combinatorial question that arose from their application of wavelets to medical imaging. While not needed for the specific application, an interesting question arose and I was able to provide a general theoretical result with a new type of argument. The problem turned out to be solving certain cases of the difficult problem of graph separation, which is 'dual' to the bandwidth problem and 'equivalent' to the problem of Hamiltonian path powers in the graph complement (as discussed in the section on interval graphs).

My work in [7] was in answer to a question regarding scheduling with communication delays that Ivan Rival introduced me to. The question regarded the computational complexity of scheduling unit execution time jobs on two processors which forest precedence constraints. (As is usual in scheduling problems only very restricted cases have good algorithms, and the question here was if there was a good algorithm for even one of the simplest cases.) It turns out that the problem was solved about the time I heard it, independently, by Lawler, by Picouleau and by Lenstra, Veldhorst and Veltman. I also (independently) solved it. While the other proofs look more at algorithmic aspects, my work also provides a minimax result, allowing for a short proof that there is a good algorithm as well as providing structural insight.

Research Plan

This is a brief description of research plans for the tenure packet of Garth Isaak. For this context I will write somewhat differently than a more formal research/grant proposal. I will occasionally use first person. I will intentionally be somewhat informal, *not* defining terms or giving exact references, as most are contained in my papers.

My research 'programs' fall roughly along the lines of the groups described in my research summary. There are short descriptions of the relevant terms there and formal ones in my papers, so I will refer the these terms without further description.

Much of what I describe here will be specific questions arising directly from my previous work. In the last section I will mention some other work.

Problems Related to Interval Orders and Graphs and other Highly Structured Graph Families

Classes of graphs related to the tolerance graphs introduced in [31] and [32] have turned out to be surprisingly difficult to characterize. One partial step toward understanding some of these classes is to use forbidden ordered subgraphs. In [15] we do this for unit bitolerance graphs. In [9] (lemmas 2-5) we showed that a proper tolerance graph has an ordering of its vertices for which particular orderings of certain induced subgraphs on 3 and 4 vertices are forbidden. I believe that these necessary conditions, which can be viewed in terms of forbidden ordered subgraphs, are also sufficient (in fact I have several times thought I had proved this only to catch my error when working out the details). This sort of approach may also provide some insight into a problem posed in [32] which has generated some interest. The question is to determine if every cocomparability graph that is also a tolerance graph has a bounded representation.

Along the lines of [16] and [17] there is more that can be investigated in terms of determining where there are boundaries between graph classes with efficient algorithms for Hamiltonian powers and those where the problem is NP-complete. The class of cographs in particular seems worth examining, since this class admits efficient solutions for many optimization problems. Our examples in [17] suggest some potential difficulties even for cographs. There is a natural 'greedy' algorithm based on endpoints in an interval representation for Hamiltonian paths and cycles ([37] or [42] for example) which I extend to Hamiltonian path powers in [16]. However, I have examples to show that the most obvious extensions of the 'greedy' endpoint algorithm for cycles fails for Hamiltonian cycle powers. I plan to examine algorithms based on maximal cliques [22]. In terms of my original motivation for looking at these problems [12], it would be interesting to examine Hamiltonian powers in cocomparability graphs of orders with a rooted forest diagram.

There are several other natural graph classes for which the toughness conditions are sufficient and for which there are efficient algorithms for detecting Hamilton paths/cycles. Among these are cocomparability graphs [23], [24], circular arc graphs [42]. It would be worth examining the complexity of Hamiltonian power problems on these classes in addition to cographs. I should mention another intriguing problem that I have worked on is the complexity of testing for a Hamiltonian path (not path power) on the class of asteroidal triple free graphs [21].

As mentioned in the article summary, the network method from [4] can be used to give proofs of standard representation theorems for interval orders and semiorders. I plan to write an expository article (for something like the Monthly) hitting both the fascinating history of interval orders as well as using this to illustrate the beauty of linear programming/network duality.

Perfect Maps

For perfect maps, two main directions remain in showing sufficiency of the 'obvious' necessary conditions. In one and two dimensions certain cases over non prime power alphabets remain. For three and higher dimensions (and two dimensions for multifactors), constructions have been somewhat limited, even for prime powers. Completion of showing sufficiency will require combining solutions to both of these tasks.

In [18], I codify two main construction techniques that have been used so far. This sets the stage for constructing large classes of two and higher dimensional perfect multifactors (including higher dimensional perfect maps). I plan to begin to work on the many details needed to complete application of the results of [18] to show sufficiency of the conditions for all dimensions in the prime power case.

For non prime powers various methods have been used, although certain classes of parameter for the factors remain. In [35], a technique of puncturing is used in the case that the 'window' size is 2×2. I have preliminary results extending this technique to more general window sizes.

Finally, all of the work on complexity of sequences, which is used to show when the sum of entries in a fundamental block is zero, has been in the context of finite fields. In order to apply the concatenation construction we need to examine sequences with entries from Z_a where a is a prime power or a composite. Here we will need to look not at complexity, but instead at when we can repeatedly 'integrate' such sequences.

Inconsistencies Under Tournament Ranking

In the process of looking at various ranking procedures and their behavior along the line studied in [8] and [1], I have become interested in another ranking procedure (for tournaments) which seems not to be well studied, (I have yet to find a reference), but which, from my perspective is interesting. Ranking by score sequences is equivalent to ranking so as to minimize a weighted sum of inconsistencies. This is probably well known and is written down in [36] and [1]. We could instead seek to minimize the maximum 'weight' inconsistency; that is, find an ordering $1, 2, 3, \ldots$ of the vertices in a tournament for which the largest $j - i$ with $j > i$ and (j, i) an arc is minimized. This, as a directed graph version, fits the general scheme of labeling vertices under a min/max or max/min condition on the edges. The general scheme includes the well known bandwidth problem as well as graph separation ('equivalent' to Hamiltonian path powers in the graph complement). See [39] for another directed version. I have begun preliminary investigation of this ranking procedure to see if there are any interesting structural results.

Several problems from [11] relating when to a particular acyclic digraph is a feedback arc set remain of interest to me. I have a conjecture on the size of a smallest tournament which has 'the' acyclic tournament on n vertices as a feedback arc set. There are two possible values $3n - 2 - \lfloor \log_2(n) \rfloor$ or $3n - 1 - \lfloor \log_2(n) \rfloor$ depending (in a specific way) on the binary representation of n. While this at first appears artificial and a byproduct of my lower bound proof, I think it may be correct. For small values (up to 22) the pattern is correct and the smaller bound $3n - 2 - \lfloor \log_2(n) \rfloor$ is correct for $n = 2^k - 2^t$, hence any other possible solution would seem 'equally' unnatural.

The bounds discussed above arise from cutting planes in a particular integer linear programming problem. I can show that an optimal solution to this ILP is also a solution to the tournament problem and hope to show an analogous result for feedback arc sets that contain a Hamiltonian path. This in turn relates to the more general problem of feedback arc sets and arc disjoint cycles in a tournament. There is no general minimax formula, but I propose a different case where equality may hold: if a tournament T has a minimum feedback arc set which induces a digraph with a Hamiltonian path then is it true that the maximum number of arc disjoint cycles in T is equal to the minimum size of a feedback arc set?

Other Areas

This section might be better described as research 'non-plans'. One of the things I enjoy about the discrete mathematics community is the opportunities to work on somewhat varied problems and problems from unexpected sources. Several of my previous papers arose from unexpected questions of others that caught my interest. A paper on a scheduling problem [7] arose from a question posed by Ivan Rival during a conference dinner. I have some continued interest in scheduling problems that I encountered while preparing this paper. A paper on separation number of a graph [12] arose when a colleague (Jim Driscoll) walked into my office an said 'we are working on using wavelets in medical imaging and encountered this combinatorics problem in our work'. Looking at the separation number in a graph is 'equivalent' to looking for powers of Hamiltonian paths in the graph's complement. This has led to my work in [16] and [17]. Working intensively with a Lehigh graduate student (Darren Narayan) this summer on finishing a project he started during a REU (research experiences for undergraduates) is producing a paper (also with Anthony Evans) on representations of graphs modulo n. This problem arose from work of Erdös and Evans on orthogonal latin square graphs [26] [28]. I expect that in the future some of my research will continue to arise from unexpected sources such as these.

One other area where I have some plans is that of stable matching (my interest began with my work on [2]). Stable matching underlies the assignment of interns to hospitals (see [34] for a comprehensive discussion). Interns and hospitals ranks their choices for pairing and a stable matching is found where there is no hospital h and intern i each of which prefer the other to their match. In the application of this algorithm a problem has arisen as married couples (who are both prospective interns) would like their preferences linked. Determining if a stable matching exists in this situation is NP-complete (see [34]). Examining 'bad' examples for this problem one sees that a variation makes sense in terms of the applied model. If 'instability' for married couples also involves a 'time horizon', looking at whether an intern displaced by one member of the couple might in turn displace the other member of the couple, then perhaps a tractable solution might be found. The process of 'iterating' toward a solution in regular stable matching is solved in [43] and a time horizon version might parallel this.

Of course I should also mention work on 'big' open problems in combinatorics (e.g., Erdös-Faber-Lovasz conjecture, Hadamard matrices, orthogonal latin squares, etc). This is not a formal research 'program' but such problems are often very inviting and I (as probably most discrete mathematicians) spend some time working on them. Ideas from linear programming and more generally polyhedral combinatorics have played an important role in discrete mathematics. My work has often relied on these tools. Many well known theorems (e.g., Erdös-Ko-Rado, Turan's etc.) can be viewed as applications of linear programming duality. My attempts at working on 'big' problems involve understanding the role of polyhedral combinatorics in the problem. It wouldn't surprise me that when solutions to some of these 'big' problems do arise, methods from polyhedral combinatorics will be involved.

References

These references are in two parts, the first includes only my papers, so that the numbering is consistent with my cv. Other papers follow.

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