Trigonometric functions

1. Trigonometric functions often arise in physical applications with periodic motion. They do not arise often in business examples. We will still cover them but with less detail than other functions.

2. Use radian measure rather than degrees for calculus since this makes computation of derivatives easier. See the unit circle diagram text page 35. For radians we specify an angle in terms terms of the arc length along the unit circle, a circle of radius 1 centered at the origin.

   For example, the angle formed by the positive x-axis and the positive y-axis (horizontal line going right and vertical line going up) is 90 degrees. The corresponding arc on the circle is $1/4^{th}$ of the circumference, 2π. So the radian measure of a 90 degree angle is $\frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$.

   In a similar manner the ratio of d degrees to a 'complete turn' of 360 degrees is the same as the ratio of the radian measure of the angle to going around the unit circle once, 2π.

   So we have the conversion d in degrees is $s = \frac{2\pi}{360} = \frac{\pi}{180}\, d$ in radians.

3. Values of $\cos\theta$ and $\sin\theta$ are the x and y coordinates respectively of point on the unit circle at distance θ along the circle from $(1,0)$. See the figure text page 36 for basic values you should know.

   Note that you really only need to memorize 4 values:
   0 degrees: $(\cos 0, \sin 0) = (1, 0)$
   30 degree angles: $(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$
   45 degrees: $(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$
   60 degrees: $(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

   The rest can be obtained by symmetry

4. The notation $\sin x$ etc represents a function. That is, $\sin\frac{\pi}{6}$ is the value of the sine function evaluated at the angle $\frac{\pi}{6}$. It is *not* 'sin' times $\frac{\pi}{6}$. In particular we cannot distribute, $\sin(x+y)$ is *not* $\sin x + \sin y$. Also sin by itself has no meaning. It is a function and needs and argument. If you find yourself writing just 'sin' you are doing something wrong.

We use particular shorthand notation for powers, writing, for example, $\sin^2 x$ for $(\sin x)^2$.
We can do this for any power except -1, $\sin^{-1} x$ represents the inverse function to sine (described below). There is a special term for the reciprocal of the sine function $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$, the cosecant function. Similarly, secant is the reciprocal of cosine, $\frac{1}{\cos x} = \sec x$ and cotangent is the reciprocal of tangent, $\frac{1}{\tan x} = \cot x$.

Recall also that tangent is the ratio of sine to cosine: $\tan x = \frac{\sin x}{\cos x}$ so cotangent is the ratio of cosine to sine, $\cot x = \frac{\cos x}{\sin x}$.

5. Using the Pythagorean theorem on a triangle with hypotenuse 1 yields the basic identity $\sin^2\theta + \cos^2\theta = 1$. As other trigonometric functions are defined in terms of $\sin\theta$ and $\cos\theta$ we can obtain other Pythagorean identities. Text page 37.

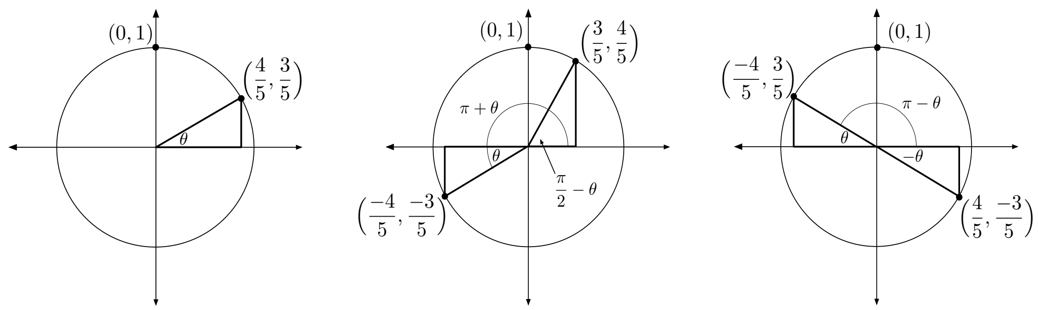
   For example, to show the identity $1 + \cot^2\theta = \csc^2\theta$ multiply $\sin^2\theta + \cos^2\theta = 1$ by $\frac{1}{\sin^2\theta}$ and use $\cot\theta = \frac{\cos\theta}{\sin\theta}$ and $\csc\theta = \frac{1}{\sin\theta}$.

   $\frac{1}{\sin^2\theta}\left(\sin^2\theta + \cos^2\theta = 1\right) \Rightarrow 1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Rightarrow 1 + \cot^2\theta = \csc^2\theta$

   Be aware that double and half angle identities exist but do not memorize them. We will not use them in Math 81. If you ever need them you can look them up.

6. If $\cos(\theta) = 4/5$ for $0 < \theta < \pi/2$, use basic trigonometric identities and the unit circle to determine $\sin(\theta)$, $\tan(\theta)$, $\cos(\pi/2 - \theta)$, $\cos(\pi - \theta)$, $\cos(\pi + \theta)$ and $\sin(-\theta)$.

   Since $1 = \sin^2(\theta) + \cos^2(\theta) = \sin^2(\theta) + (4/5)^2$ we get $\sin^2(\theta) = 1 - (16/25) = 9/25$. Hence $\sin(\theta) = \pm 3/5$ and since $0 < \theta < \pi/2$, $\sin(\theta) > 0$ so we have $\sin(\theta) = 3/5$. $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{3/5}{4/5} = 3/4$. Using the similar triangles shown below for the various angles we get $\cos(\pi/2 - \theta) = 3/5$, $\cos(\pi - \theta) = -4/5$, $\cos(\pi + \theta) = -4/5$ and $\sin(-\theta) = -3/5$.



7. Identities that follow from moving a basic triangle on the unit circle in a manner similar to the previous example include:

   (a) $\sin(\pi - \theta) = \sin(\theta)$ and $\cos(\pi - \theta) = -\cos(\theta)$

   (b) $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$

   (c) $\cos(\theta) = \sin(\theta + \frac{\pi}{2})$

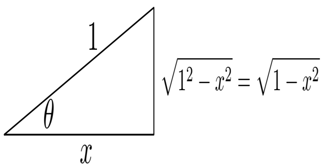
8. Inverse trigonometric functions are described in the text pages 39-40. What angles have $\sin\theta = \frac{\sqrt{2}}{2}$? One answer is $\pi/4$. But also $\ldots, -4\pi + \pi/4, -2\pi + \pi/4, \pi/4, 2\pi + \pi/4, 4\pi+\pi/4, \ldots$ as well as $\ldots, -4\pi+3\pi/4, -2\pi+3\pi/4, 3\pi/4, 2\pi+3\pi/4, 4\pi+3\pi/4, \ldots$. To define the inverse function we pick a particular value. For inverse sine functions we use angles between $-\pi/2$ and $\pi/2$ and for inverse cosine functions we use angles between 0 and π.

   Recall the notation for inverse functions from lecture 01. Inverse trigonometric functions are denoted, for example, $\sin^{-1} x$ (the term arcsine is also used). Note that this is *not* $\frac{1}{\sin x}$. It is the angle θ between $-\pi/2$ and $\pi/2$ such that $\sin\theta = x$.

9. Derivatives of inverse trigonometric functions arise in engineering applications. To determine these we will use computations like the following example and example 4b in the text page 41.

   Simplify $\tan(\cos^{-1} x)$

   Let $\theta = \cos^{-1} x$, then $\cos\theta = x$. This is adjacent/hypotenuse in the triangle below. Using the Pythagorean Theorem gives the remaining side as $\sqrt{1^2 - x^2} = \sqrt{1 - x^2}$. Then $\tan(\cos^{-1} x) = \tan\theta = \frac{\sqrt{1-x^2}}{x}$ as it is opposite/adjacent in the triangle.



Supplementary problems

Degrees and radians

P5.1 Convert the following radian measures to degrees: $\frac{2\pi}{3}, \frac{7\pi}{6}, \frac{\pi}{4}, \frac{\pi}{12}$

P5.2 Convert the following degrees to radians: 10°, 75°, 150°, 180°

Basic trigonometric values

P5.3 Find
$\sin\frac{4\pi}{3}$, $\cos\frac{4\pi}{3}$, $\tan\frac{4\pi}{3}$ and
$\sin\frac{7\pi}{6}$, $\cos\frac{7\pi}{6}$, $\tan\frac{7\pi}{6}$

Trigonometric identities

P5.4 Derive the trigonometric identity $\tan^2\theta + 1 = \sec^2\theta$ from $\sin^2\theta + \cos^2\theta = 1$ and the definitions of $\tan\theta$ and $\sec\theta$ in terms of $\sin\theta$ and $\cos\theta$

P5.5 If $\sin\theta = \frac{5}{13}$ and $0 \le \theta \le \frac{\pi}{2}$ determine $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$

P5.6 If $\sin\theta = \frac{1}{4}$ and $0 \le \theta \le \frac{\pi}{2}$ determine $\cos(\theta + \pi)$ and $\cos(\theta - \frac{\pi}{2})$. Explain your computations using a unit circle.

P5.7 Use a right triangle to write the expression in terms of x: $\sin\left(\cos^{-1}\frac{3}{x}\right)$

Solutions to supplementary problems

Degrees and radians

S5.1 We get the following degrees $\frac{2\pi}{3} \cdot \frac{180}{\pi} = 120$, $\frac{7\pi}{6} \cdot \frac{180}{\pi} = 210$, $\frac{\pi}{4} \cdot \frac{180}{\pi} = 45$, $\frac{\pi}{12} \cdot \frac{180}{\pi} = 15$

S5.2 We get the following radians $10 \cdot \frac{\pi}{180} = \frac{\pi}{18}$, $75 \cdot \frac{\pi}{180} = \frac{5\pi}{12}$, $150 \cdot \frac{\pi}{180} = \frac{5\pi}{6}$, $180 \cdot \frac{\pi}{180} = \pi$

Basic trigonometric values

S5.3 See the unit circle diagram in the text:
$\sin\frac{4\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{4\pi}{3} = \frac{-1}{2}$, $\tan\frac{4\pi}{3} = -\sqrt{3}$ and
$\sin\frac{7\pi}{6} = \frac{-1}{2}$, $\cos\frac{7\pi}{6} = \frac{-\sqrt{3}}{2}$, $\tan\frac{7\pi}{6} = \frac{1}{\sqrt{3}}$

Trigonometric identities

S5.4 To show the identity $\tan^2\theta + 1 = \sec^2\theta$ multiply $\sin^2\theta + \cos^2\theta = 1$ by $\frac{1}{\cos^2\theta}$ and use $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sec\theta = \frac{1}{\cos\theta}$.

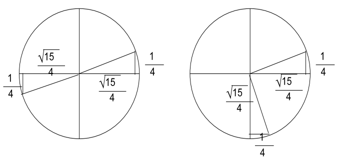
$\frac{1}{\cos^2\theta}\left(\sin^2\theta + \cos^2\theta = 1\right) \Rightarrow \frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta} \Rightarrow \tan^2\theta + 1 = \sec^2\theta$

S5.5 Use the Pythagorean theorem $\sin^2\theta + \cos^2\theta = 1$ with $\sin\theta = \frac{5}{13}$:
$\left(\frac{5}{13}\right)^2 + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \frac{25}{169} = \frac{144}{169} \Rightarrow \cos\theta = \sqrt{144/169} = \frac{12}{13}$.
We take the positive root since cosine is positive for $0 < \theta < \frac{\pi}{2}$.

Then, from the definitions:
$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{5/13}{12/13} = \frac{5}{12}$, $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{12/13}{5/13} = \frac{12}{5}$,
$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{12/13} = \frac{13}{12}$, $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{5/13} = \frac{13}{5}$.

S5.6 Use the Pythagorean theorem $\sin^2\theta + \cos^2\theta = 1$ with $\sin\theta = \frac{1}{4}$: $\left(\frac{1}{4}\right)^2 + \cos^2\theta = 1 \Rightarrow$ $\cos^2\theta = 1 - \frac{1}{16} = \frac{15}{16} \Rightarrow \cos\theta = \sqrt{15/16} = \frac{\sqrt{15}}{4}$.
We take the positive root since cosine is positive for $0 < \theta < \frac{\pi}{2}$.

Similar triangles in the figure show $\cos(\theta + \pi) = \frac{-\sqrt{15}}{4}$ and $\cos(\theta - \frac{\pi}{2}) = \frac{1}{4}$.



S5.7 Let $\theta = \cos^{-1}\frac{3}{x}$ then $\cos\theta = \frac{3}{x}$. This is adjacent/hypotenuse in the triangle below. Using the Pythagorean Theorem gives the remaining side as $\sqrt{x^2 - 3^2} = \sqrt{x^2 - 9}$. Then $\sin\left(\cos^{-1}\frac{3}{x}\right) = \sin\theta = \frac{\sqrt{x^2-9}}{x}$ as it is opposite/hypotenuse in the triangle.