Exponentials and logarithms

1. Exponential functions are described in the text pages 23-24. For all bases the graph contains the point $(0,1)$. For different bases, the slope of the curve has different values at this point. The slope of 2^x at $x=0$ is less than 1 and the slope of 3^x is greater than 1. The number $e = 2.718\ldots$ has slope exactly 1 when $x=0$. Because of this, e is a more convenient base for exponential functions in calculus. Also e is the number that is convenient for use in computations with continuously compounded interest.

2. Note the differences in the horizontal and vertical scales in figure 1.42, page 23. Using a scale of 1 inch per unit, at 6 inches horizontally the vertical coordinate on 2^x would be over 5 feet and at 12 inches horizontally the vertical coordinate would be over 341 feet. At 24 inches (2 feet) horizontally the vertical coordinate is at 264 miles, at 36 inches (3 feet) horizontally the vertical coordinate is over 1 million miles (four times the distance to the moon). Hence the term exponential growth.

3. Many scales are based on logarithms. The Richter scale for earthquakes, magnitudes of stars, Ph scales for acidity and the decibel scale for sound are all based on logarithms.

4. Logarithms are inverses of exponentials.

   (a) Basic exponent rules (text page 23) translate into basic logarithm rules (text page 29). We use these rules for many of our exercises.

   For example, $16 \cdot 32 = 2^4 \cdot 2^5 = 2^{4+5} = 2^9 = 512$. From the definition of logarithms, $2^4 = 16$ means $\log_2 16 = 4$ and $2^5 = 32$ means $\log_2 32 = 5$ and $2^9 = 512$ means $\log_2 512 = 9$. Now the adding of the exponents $4+5=9$ is $\log_2 16 + \log_2 32 = \log_2 512 = \log_2(16 \cdot 32)$. We will show the general rule more formally below.

   (b) In words the rules are 'the log of a product is the sum of the logs' and 'the log of a quotient is the difference of the logs' and 'the log of a power is the exponent times the log of the base'.

   (c) To show $\log_b(xy) = \log_b x + \log_b y$ we use $b^{\log_b z} = z$ with z set to x, y, xy in turn. Then $b^{\log_b(xy)} = xy = x \cdot y = b^{\log_b x} \cdot b^{\log_b y} = b^{\log_b x + \log_b y}$. Equating exponents gives the result.

   (d) Note the change of base rules for logarithms text page 31. We will rarely use these in Math 81. Do not memorize them.

(e) Historically (before computers) logarithms were used to simplify multiplication and division of large numbers using either logarithm tables or a slide rule. Log transformations are still useful in many applications.

5. We begin with a few examples to get used to the logarithm definition.

   Find $\log_3 81$, $\log_3 \frac{1}{81}$, $\log_5(\sqrt{5})^3$ and $\ln e^{3x}$.

   Recall that $\log_b x = y$ means $b^y = x$. Then

   $\log_3 81 = 4$ since $3^4 = 81$
   $\log_3 \frac{1}{81} = -4$ since $3^{-4} = \frac{1}{3^4} = \frac{1}{81}$
   $\log_5(\sqrt{5})^3 = \frac{3}{2}$ since $5^{3/2} = (\sqrt{5})^3$
   $\ln e^{3x} = 3x$

6. We will use logarithm rules to either condense or expand various expressions. In different settings each process can be useful.

   (a) Condense $\log_6 9 + \log_6 4$ and $7\ln y + 3\ln z$.
   $\log_6 9 + \log_6 4 = \log_6(9 \cdot 4) = \log_6(36) = 2$
   $7\ln y + 3\ln z = \ln y^7 + \ln z^3 = \ln(y^7 z^3)$

   (b) Expand $\ln\left(K^{1/3}L^{2/3}\right)$ and $\ln\left(\frac{x^2\sqrt{x^2+9}}{x-3}\right)$
   $\ln\left(K^{1/3}L^{2/3}\right) = \ln K^{1/3} + \ln L^{2/3} = \frac{1}{3}\ln K + \frac{2}{3}\ln L.$
   $\ln\left(\frac{x^2\sqrt{x^2+9}}{x-3}\right) = \ln(x^2(x^2+9)^{1/2}(x-3)^{-1}) = \ln x^2 + \ln(x^2+9)^{1/2} + \ln(x-3)^{-1} =$
   $2\ln x + \frac{1}{2}\ln(x^2+9) - \ln(x-3)$

7. The function $Q = K^{1/3}L^{2/3}$ in the previous example is an instance of a Cobb-Douglas function modelling economic output in terms of K capital and L labor.

   In this example of a log transform we see that the Cobb-Douglas function is log-linear. That is, if we use the logarithms of capital and labor as the independent variables we get a linear function.

   Indeed Cobb and Douglas originally used data from the U.S. economy in the early 1900's and plotted the logarithms. The slopes gave the coefficients 1/4 and 3/4 for this particular economic data. Data in other economies have different slopes but empirically the sum of the these values is often 1.

8. Solve the following equations for x using rules of exponents.
   $2^{(x-9)/6} = \sqrt{2}$ and $e^{x+10} = \frac{1}{e^6}$

   $2^{(x-9)/6} = \sqrt{2} \Rightarrow 2^{(x-9)/6} = 2^{1/2} \Rightarrow \frac{x-9}{6} = \frac{1}{2} \Rightarrow x - 9 = 3 \Rightarrow x = 12$
   $e^{x+10} = \frac{1}{e^6} \Rightarrow e^{x+10} = e^{-6} \Rightarrow x + 10 = -6 \Rightarrow x = -16$

9. Solve the following equations for x using rules of logarithms
$\log_5 x^3 - \log_5 5x = 5$ and $\ln(x-7) - \ln(x+10) = \ln(x-1) - \ln(x+1)$
$\log_5 x^3 - \log_5 5x = 5 \Rightarrow 3\log_5 x - \log_5 5 - \log_5 x = 5 \Rightarrow 2\log_5 x - 1 = 5$
$\Rightarrow 2\log_5 x = 6 \Rightarrow \log_5 x = 3 \Rightarrow x = 5^3 = 125$
$\ln(x-7) - \ln(x+10) = \ln(x-1) - \ln(x+1) \Rightarrow \ln\frac{x-7}{x+10} = \ln\frac{x-1}{x+1} \Rightarrow \frac{x-7}{x+10} = \frac{x-1}{x+1}$
$\Rightarrow (x-7)(x+1) = (x-1)(x+10) \Rightarrow x^2-6x-7 = x^2+9x-10 \Rightarrow 3 = 15x \Rightarrow x = \frac{3}{15} = \frac{1}{5}$

10. Equations that arise in models will often have solutions expressed as a logarithm. We usually start by taking the logarithm of each side of the equation.

While it is interesting to see the numerical answer, we will leave answer in terms of logarithms unless it is possible to simplify easily without a calculator.

(a) Solve $5^{7x} = 12$ for x
$5^{7x} = 12 \Rightarrow \log_5 5^{7x} = \log_5 12 \Rightarrow 7x = \log_5 12 \Rightarrow x = \frac{1}{7}\log_5 12$

(b) For continuous compounding the balance after t years on an initial investment P (principal) with interest rate r (expressed as a decimal) is $B(t) = Pe^{rt}$. How long does it take for an investment of \$1000 to grow to \$1300 at an annual rate of 5%?
We need to solve $1300 = 1000e^{.05t}$ for t. Taking logarithms
$1300 = 1000e^{.05t} \Rightarrow \frac{1300}{1000} = e^{.05t} \Rightarrow \ln\frac{1300}{1000} = \ln e^{.05t} = .05t = \frac{5}{100}t = \frac{1}{20}t$
$\Rightarrow t = 20\ln\frac{13}{10}$

(c) Solve $Ne^{Rn} = Se^{rn}$ for R. Here N, S, n, r, R represent net proceeds, sales proceeds, time and old and new tax rates.
$Ne^{Rn} = Se^{rn} \Rightarrow e^{Rn} = \frac{S}{N}e^{rn} \Rightarrow \ln e^{Rn} = \ln\frac{S}{N}e^{rn} = \ln\frac{S}{N} + \ln e^{rn}$
$\Rightarrow Rn = \ln\frac{S}{N} + rn \Rightarrow R = \frac{1}{n}\ln\frac{S}{N} + r$

(d) The half-life of Carbon-14 is 5730 years. This is the time it takes for half of the carbon-14 to decay. An object is found to have 25% of the carbon present in living objects. How old is it?

i. There is a quick answer without writing any equations. After 5730 years 50% of the original is left. After another 5730 years half of this 50%, i.e., 25% of the original is left. So the object is $2 \cdot 5730 = 11560$ years old.

ii. If the amount left was 30% we cannot determine the age quite as easily. However using logarithms it is not too hard. We assume decay is exponential so we model the percent remaining as $A(t) = e^{kt}$ where k is a particular constant determined by the properties of Carbon-14. Since the half-life is 5730 years, 50% remains at time $t = 5730$ we have $.5 = e^{k \cdot 5730}$. We solve this for k then solve $.3 = e^{kt}$ for t using this value of k. We do this in the supplementary exercises.

Supplementary problems

Simplify exponentials and logarithms numerically

P4.1 Find the exact value (express as an integer or a simple fraction):
(a) $\log_2 32$ (b) $\ln(e^7)$ (c) $\log_{10}(.001)$ (d) $\log_{25}\frac{1}{5}$ (e) $\log_4 8$

P4.2 Simplify $3^{2\log_3 9}$

P4.3 Solve for x: $\log_x 27 = \frac{3}{4}$

Condense and expand logarithmic expressions

P4.4 Condense and simplify:
(a) $2\log_2 6 - \log_2 9$ (b) $\log(x^2 - 36) - \log(x+6)$ (c) $\ln 6x^3y^2 + \ln\left(\frac{x}{y}\right) - \ln 6y$

P4.5 Expand $\ln\left(\frac{K^3\sqrt{K^2+L^2}}{L^3}\right)$

Solve equations involving logarithms and exponentials

P4.6 Solve each of the following for x:
(a) $2^{x-5} = 3$ (b) $\log_3 x - \log_3(x-1) = 2$ (c) $e^{4x+3} - 4 = 0$
(d) $\log_2 x + \log_2 6 = 3$ (e) $\log_2 x + \log_2 K = 3$ for x.
Your answer to (e) will be in terms of K.

P4.7 Solve $.5 = e^{k\cdot 5730}$ for k and then use this value to solve $.3 = e^{kt}$ for t.
This determines the age of the object in the example from example 10d.

P4.8 An investment of \$1000 grows as $B(t) = e^{rt}$ where r is the interest rate and t is time in years.
(a) If the rate is 4% how long does it take double?
(b) If the account doubles after 10 years, what is the rate?

Solutions to supplementary problems

Simplify exponentials and logarithms numerically

S4.1 (a) $\log_2 32 = 5$ since $2^5 = 32$
(b) $\ln(e^7) = 7$ since $e^7 = e^7$ and e is the base of natural logarithms
(c) $\log_{10}(.001) = \log_{10}(\frac{1}{1000}) = -3$ since $10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$
(d) $\log_{25} \frac{1}{5} = \frac{-1}{2}$ since $25^{-1/2} = \frac{1}{25^{1/2}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$
(e) $\log_4 8 = \frac{3}{2}$ since $4^{3/2} = (4^{1/2})^3 = (\sqrt{4})^3 = 2^3 = 8$.

S4.2 $3^{2\log_3 9} = 3^{\log_3(9^2)} = 9^2 = 81$

S4.3 $\log_x 27 = \frac{3}{4} \Rightarrow x^{3/4} = 27 \Rightarrow x = 27^{4/3} \Rightarrow x = (\sqrt[3]{27})^4 = 3^4 = 81$

Condense and expand logarithmic expressions

S4.4 (a) $2\log_2 6 - \log_2 9 = \log_2(6^2) - \log_2(9) = \log_2(6^2/9) = \log_2(4) = 2$
(b) $\log(x^2 - 36) - \log(x+6) = \log\left(\frac{x^2-36}{x+6}\right) = \log\left(\frac{(x+6)(x-6)}{x+6}\right) = \log(x-6)$
(c) $\ln 6x^3y^2 + \ln\left(\frac{x}{y}\right) - \ln 6y = (\ln 6 + \ln x^3 + \ln y^2) + (\ln x - \ln y) - (\ln 6 + \ln y) =$
$3\ln x + 2\ln y + \ln x - \ln y - \ln y = 4\ln x = \ln x^4$

S4.5 $\ln\left(\frac{K^3\sqrt{K^2+L^2}}{L^3}\right) = \ln K^3 + \ln(K^2+L^2)^{1/2} - \ln L^3 = 3\ln K + \frac{1}{2}\ln(K^2+L^2) - 3\ln L$

Solve equations involving logarithms and exponentials

S4.6 (a) $2^{x-5} = 3 \Rightarrow x - 5 = \log_2 3 \Rightarrow x = 5 + \log_2 3$
(b) $\log_3 x - \log_3(x-1) = 2 \Rightarrow \log_3(x/(x-1)) = 2 \Rightarrow x/(x-1) = 3^2 = 9$
$\Rightarrow x = 9x - 9 \Rightarrow 9/8 = x$
(c) $e^{4x+3} - 4 = 0 \Rightarrow e^{4x+3} = 4 \Rightarrow 4x + 3 = \ln 4 \Rightarrow x = (\ln(4) - 3)/4$
(d) $\log_2 x + \log_2 6 = 3 \Rightarrow \log_2(6x) = 3 \Rightarrow 6x = 2^3 = 8 \Rightarrow x = \frac{4}{3}$.
(e) $\log_2 x + \log_2 K = 3 \Rightarrow \log_2(Kx) = 3 \Rightarrow Kx = 2^3 = 8 \Rightarrow x = \frac{8}{K}$

S4.7 $.5 = e^{k\cdot 5730} \Rightarrow \ln .5 = \ln(e^{k\cdot 5730}) = k \cdot 5730 \Rightarrow k = \frac{\ln .5}{5730}$.
It simplifies notation to wait to substitute the value of k until the end.
$.3 = e^{kt} \Rightarrow \ln .3 = \ln(e^{kt}) = kt \Rightarrow t = \frac{\ln .3}{k} = \frac{\ln .3}{(\ln .5)/5730} = \frac{5730 \ln .3}{\ln .5}$.
Using a calculator this is approximately 9953 years.

S4.8 (a) Solve $2000 = 1000e^{.04t}$ for t.
$2000 = 1000e^{.04t} \Rightarrow 2 = e^{.04t} \Rightarrow \ln 2 = \ln(e^{.04t}) = .04t \Rightarrow \frac{\ln 2}{.04} = t$ (this is $t \approx 17.33$)
(b) Solve $2000 = 1000e^{r\cdot 10}$ for r.
$2000 = 1000e^{r\cdot 10} \Rightarrow 2 = e^{r\cdot 10} \Rightarrow \ln 2 = \ln(e^{r\cdot 10}) = r \cdot 10 \Rightarrow \frac{\ln 2}{10} = r$ (this is $r \approx .069$),
i.e., a rate of 6.9%