Algebra review

1. *Inverse functions* are defined in the text pages 24-26. We will use this basic concept when defining logarithms. Otherwise we will stick to the basic algebra for inverses and not focus too much on the formal notation as done in the text.

   (a) Informally, when the function is described by setting the dependent variable equal to an algebraic expression involving the independent variable (i.e., an algebraic function), we solve the resulting equation for the dependent variable in terms of the independent variable. In other words, a function produces an output for each input. Its inverse tells us which input will produce a given output. Note then that for this to be defined we cannot have a function where two different inputs produce the same output. These conditions will be described more carefully in 3 below.

   (b) For transcendental functions the inverse function is simply defined as the function that 'reverses' the operation. There is no direct algebraic manipulation that we do.

   (c) The text notation 'switches' the variable symbols so the inverse has independent variable x and dependent variable y. We will rarely do this.

2. Recall the demand function $q = -5p + 200$ of lecture 01 example 1. Economists often prefer demand functions written as price in terms of demand. This is the inverse demand function (sometimes just called the demand curve). Solving for p in terms of q uses basic algebra: $q = -5p + 200 \Rightarrow q - 200 = -5p \Rightarrow \frac{q-200}{-5} = p \Rightarrow p = \frac{-1}{5}q + 40$. This is the same equation as before, just rearranged. Now we think of q as the independent variable (input) and p as the dependent variable (output).

3. (a) Try to find an inverse for $y = x^2$. For example, what input gives output 9? Here both $+3$ and -3 output 9. As there is no unique input for each output we cannot invert this function. If we restrict to $x \geq 0$ then inverting gives $x = \sqrt{y}$ and if we restrict to $x \leq 0$ then inverting gives $x = -\sqrt{y}$. To have an inverse, the function must be one-to-one on the given interval. See the text for details.

   (b) Text writes $f^{-1}(x)$ and relabels variables. We will rarely use this notation.
   Note that this is different from the exponent notation $x^{-1} = \frac{1}{x}$.

4. On a graph, inverting a function switches the axes. That is, we flip the figure about antidiagonal line $y = x$.

   Note then that the slope at any point of the inverse is the reciprocal of the slope of the original. We will use this to find formulas for derivatives of logarithmic functions.

5. If $P = K\sqrt{Q+7}$ with K a constant, is a function of Q, find the inverse function. That is, solve for Q in terms of P.

   (a) We should specify the domain but will usually skip this in Math 81.

   (b) In this example we should say that $K \neq 0$ in order to avoid division by 0.

   (c) In this class we will assume constants are such that we avoid division by 0, square roots of negative numbers etc. Technically we should note this in each example but we will consider this a blanket assumption.

   $P = K\sqrt{Q+7} \Rightarrow \frac{P}{K} = \sqrt{Q+7} \Rightarrow \left(\frac{P}{K}\right)^2 = Q+7 \Rightarrow Q = \frac{P^2}{K^2} - 7$

6. Recall basic rules for working with exponents:
   $x^{-a} = \frac{1}{x^a}$, $x^a x^b = x^{a+b}$, $(x^a)^b = x^{ab}$.
   We will also rewrite expressions using root notation and exponent notation:
   $\sqrt{x} = x^{1/2}$, $\sqrt[3]{x} = x^{1/3}$ etc.
   More exponent rules are in the margin of text page 23. Basic examples are in the supplementary problems.

   (a) Simplify $\frac{st^{-1/3}+s^{2/3}t^{2/3}}{t^{2/3}s}$.
   $\frac{st^{-1/3}+s^{2/3}t^{2/3}}{t^{2/3}s} = (st^{-1/3} + s^{2/3}t^{2/3})t^{-2/3}s^{-1} = t^{-1} + s^{-1/3} = \frac{1}{t} + \frac{1}{s^{1/3}}$.
   Alternatively we can write this
   $\frac{st^{-1/3}+s^{2/3}t^{2/3}}{t^{2/3}s} = \frac{st^{-1/3}}{t^{2/3}s} + \frac{s^{2/3}t^{2/3}}{t^{2/3}s} = s^{1-1}t^{-1/3-2/3} + s^{2/3-1}t^{2/3-2/3} = s^0t^{-1} + s^{-1/3}t^0 = \frac{1}{t} + \frac{1}{s^{1/3}}$.

   (b) A Cobb-Douglas utility function is $Q = CL^{1/3}K^{2/3}$ where $L, K \geq 0$. Solve for L in terms of the other symbols.
   $Q = CL^{1/3}K^{2/3} \Rightarrow L^{1/3} = \frac{Q}{CK^{2/3}} \Rightarrow L = \left(\frac{Q}{CK^{2/3}}\right)^3 = \left(\frac{Q}{C}\right)^3 \frac{1}{K^2} = \left(\frac{Q}{C}\right)^3 K^{-2}$

7. For optimization problems we will need to solve equations for a particular variable. We start with a numerical example and then move to a generic version of the same form. Note that the generic version is in a sense simpler than that with fractional exponents (which arise in applications).

   More examples are in the supplementary problems.

   We should assume $x \neq 0$ and $x \neq 1$ to avoid division by zero.

   (a) Solve $8x^7(1-x)^5 - 5x^8(1-x)^4 = 0$ for x.
   $8x^7(1-x)^5 - 5x^8(1-x)^4 = 0 \Rightarrow 8x^7(1-x)^5 = 5x^8(1-x)^4 \Rightarrow 8(1-x) = 5x \Rightarrow 8 - 8x = 5x \Rightarrow 8 = 13x \Rightarrow x = \frac{8}{13}$

(b) Solve $\frac{1}{4}x^{-3/4}(1-x)^{3/4} - \frac{3}{4}x^{1/4}(1-x)^{-1/4} = 0$ for x.
$\frac{1}{4}x^{-3/4}(1-x)^{3/4} - \frac{3}{4}x^{1/4}(1-x)^{-1/4} = 0 \Rightarrow \frac{1}{4}x^{-3/4}(1-x)^{3/4} = \frac{3}{4}x^{1/4}(1-x)^{-1/4} \Rightarrow \frac{1}{4}(1-x) = \frac{3}{4}x \Rightarrow \frac{1}{4} - \frac{1}{4}x = \frac{3}{4}x \Rightarrow \frac{1}{4} = \frac{1}{4}x + \frac{3}{4}x = x \Rightarrow x = \frac{1}{4}$

(c) Solve $ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1} = 0$ for x (in terms of a and b).
$ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1} = 0 \Rightarrow ax^{a-1}(1-x)^b = bx^a(1-x)^{b-1} \Rightarrow a(1-x) = bx \Rightarrow a - ax = bx \Rightarrow a = ax + bx = (a+b)x \Rightarrow x = \frac{a}{a+b}$.

(d) Note that if $a+b=1$ as in the second example (and as will happen in certain applications that we will see later) we get a very simple solution $x = a$.

8. Later we will work with slopes of secant lines. The algebra involves simplifying $\frac{f(x+h)-f(x)}{h}$, which we will call a *difference quotient*. It is the slope of the line between two points on the graph of the function $f(x)$.

The first example uses the square of an expression which we will look at in Lecture 03 and the second involves finding a common denominator.

(a) Simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for $f(x) = 5x^2$.
$\frac{f(x+h)-f(x)}{h} = \frac{5(x+h)^2-5x^2}{h} = \frac{5(x^2+2xh+h^2)-5x^2}{h} = \frac{5x^2+10xh+5h^2-5x^2}{h} = \frac{10xh+5h^2}{h} = 10x+5h$

(b) Simplify the difference quotient $\frac{p(x+h)-p(x)}{h} = \frac{1}{h}\left(p(x+h)-p(x)\right)$ for $p(x) = \frac{5}{x}$.
$\frac{1}{h}\left(p(x+h)-p(x)\right) = \frac{1}{h}\left(\frac{5}{(x+h)} - \frac{5}{x}\right) = \frac{1}{h}\left(\frac{5x-5(x+h)}{(x+h)x}\right) = \frac{1}{h}\left(\frac{5x-5x-5h}{(x+h)x}\right) = \frac{1}{h}\left(\frac{-5h}{(x+h)x}\right) = \frac{-5}{(x+h)x}$

9. A useful 'trick' is to make use of the 'difference of two squares'. That is $(x+y)(x-y) = x^2 + xy - xy - y^2 = x^2 - y^2$.
We use this to simplify or change to a useful form, expressions involving square roots.

We use the term 'conjugate' for the other term in the pair $(x+y)$ and $(x-y)$.

(a) Simplify $(\sqrt{3(x+h)} + \sqrt{3x})(\sqrt{3(x+h)} - \sqrt{3x})$
$(\sqrt{3(x+h)} + \sqrt{3x})(\sqrt{3(x+h)} - \sqrt{3x}) =$
$(\sqrt{3x+3h})^2 - (\sqrt{3x})^2 = (3x+3h) - (3x) = 3h$.

(b) Rationalize the denominator for $\frac{1}{\sqrt{2}+\sqrt{7}}$ and $\frac{1}{\sqrt{x+h}-\sqrt{x}}$.
In each case we multiply by the conjugate. Since our new expression must be equal to the original, (we want to write '='), we multiply by 1 (in the form the conjugate divided by itself).
$\frac{1}{\sqrt{2}+\sqrt{7}} = \frac{1}{\sqrt{2}+\sqrt{7}} \cdot \frac{\sqrt{2}-\sqrt{7}}{\sqrt{2}-\sqrt{7}} = \frac{\sqrt{2}-\sqrt{7}}{(\sqrt{2})^2-(\sqrt{7})^2} = \frac{\sqrt{2}-\sqrt{7}}{2-7} = \frac{\sqrt{2}-\sqrt{7}}{-5} = \frac{-\sqrt{2}+\sqrt{7}}{5}$
$\frac{1}{\sqrt{x+h}-\sqrt{x}} = \frac{1}{\sqrt{x+h}-\sqrt{x}} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{\sqrt{x+h}+\sqrt{x}}{(\sqrt{x+h})^2-(\sqrt{x})^2} = \frac{\sqrt{x+h}+\sqrt{x}}{(x+h)-x} = \frac{\sqrt{x+h}+\sqrt{x}}{h}$

Supplementary problems

We assume that constants are such that we do not divide by 0 or take the square root of a negative number. Note the use of $\Rightarrow$ to indicate order of implications in solving equations. You will be expected to clearly indicate such in Math 81. Problems that clearly break into several 'parts' are best solved that way.

Exponentials

P2.1 Simplify $27^{2/3}$ and $4^{-5/2}$

P2.2 Simplify $\dfrac{K^{1/4}L^{-1/3} + \omega KL}{K^{-3/4}L^{2/3}}$

Inverse functions and solving equations

P2.3 Solve $r = \alpha - \beta s$ for s

P2.4 Solve $2\alpha - 10(7 - 5\alpha) = 0$ for α

P2.5 An inventory control problem requires solving $0 = \frac{-QF}{q^2} + \frac{S}{2}$ for q, the order size where Q, F, S are constants (annual demand, fixed costs per order and storage costs). Determine q assuming it is nonnegative.

P2.6 Solve $\frac{3}{4}K^{-1/4}L^{1/4} + \frac{1}{4}K^{3/4}L^{-3/4} = 0$ for L

P2.7 Solve $Q = CL^aK^b$ for L.

Difference quotients

P2.8 Simplify the difference quotient $\frac{g(x+h)-g(x)}{h}$ for $g(x) = ax + b$

P2.9 Simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for $f(x) = \frac{1}{x^2}$

Conjugates

P2.10 Simplify $\sqrt{5(x+h)+7} - \sqrt{5x+7}$

P2.11 Simplify the difference quotient $\frac{f(x+h)-f(x)}{h}$ for $f(x) = \sqrt{5x+7}$

Solutions to supplementary problems

Exponentials

S2.1 $27^{2/3} = (\sqrt[3]{27})^2 = 3^2 = 9$
$4^{-5/2} = \left(\sqrt{4}\right)^{-5} = 2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

S2.2 $\dfrac{K^{1/4}L^{-1/3} + \omega KL}{K^{-3/4}L^{2/3}} = \dfrac{K^{1/4}L^{-1/3}}{K^{-3/4}L^{2/3}} + \dfrac{\omega KL}{K^{-3/4}L^{2/3}} = K^{1/4-(-3/4)}L^{-1/3-2/3} + \omega K^{1-(-3/4)}L^{1-2/3}$

$= K^1L^{-1} + \omega K^{7/4}L^{1/3} = \dfrac{K}{L} + \omega K^{7/4}L^{1/3}$

Inverse functions and solving equations

S2.3 $r = \alpha - \beta s \Rightarrow \beta s = \alpha - r \Rightarrow s = \frac{\alpha - r}{\beta}$

S2.4 $2\alpha - 10(7-5\alpha) = 0 \Rightarrow 2\alpha - 70 + 50\alpha = 0 \Rightarrow 52\alpha = 70 \Rightarrow \alpha = \frac{70}{52} = \frac{35}{26}$

S2.5 $0 = \frac{-QF}{q^2} + \frac{S}{2} \Rightarrow \frac{S}{2} = \frac{QF}{q^2} \Rightarrow q^2 = \frac{2QF}{S} \Rightarrow q = \sqrt{\frac{2QF}{S}}$

S2.6 $\frac{3}{4}K^{-1/4}L^{1/4} + \frac{1}{4}K^{3/4}L^{-3/4} = 0 \Rightarrow \frac{3}{4}K^{-1/4}L^{1/4} = -\frac{1}{4}K^{3/4}L^{-3/4}$
$\Rightarrow \frac{3}{4}L^{1/4}L^{3/4} = -\frac{1}{4}K^{3/4}K^{1/4} \Rightarrow L = \frac{-1}{3}K$

S2.7 $Q = CL^aK^b \Rightarrow L^a = \frac{Q}{CK^b} \Rightarrow L = \left(\frac{Q}{CK^b}\right)^{1/a} = \left(\frac{Q}{C}\right)^{1/a} K^{-b/a}$

Difference quotients

S2.8 $\frac{g(x+h)-g(x)}{h} = \frac{[a(x+h)+b]-(ax+b)}{h} = \frac{ax+ah+b-ax-b}{h} = \frac{ah}{h} = a$

S2.9 Note first that $f(x+h) - f(x) = \frac{1}{(x+h)^2} - \frac{1}{x^2} = \frac{x^2}{(x+h)^2x^2} - \frac{(x+h)^2}{(x+h)^2x^2} = \frac{x^2-(x+h)^2}{(x+h)^2x^2} = \frac{x^2-(x^2+2xh+h^2)}{(x+h)^2x^2} = \frac{-h(2x+h)}{(x+h)^2x^2}$

Then $\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left(f(x+h)-f(x)\right) = \frac{1}{h}\left(\frac{-h(2x+h)}{(x+h)^2x^2}\right) = \frac{-(2x+h)}{(x+h)^2x^2}$

Conjugates

S2.10 $\sqrt{5(x+h)+7} - \sqrt{5x+7} = \sqrt{5(x+h)+7} - \sqrt{5x+7}\,\frac{\sqrt{5(x+h)+7}+\sqrt{5x+7}}{\sqrt{5(x+h)+7}+\sqrt{5x+7}} = \frac{(5(x+h)+7)-(5x+7)}{\sqrt{5(x+h)+7}+\sqrt{5x+7}} = \frac{5h}{\sqrt{5(x+h)+7}+\sqrt{5x+7}}$

S2.11 Note that $f(x+h) - f(x) = \sqrt{5(x+h)+7} - \sqrt{5x+7} = \frac{5h}{\sqrt{5(x+h)+7}+\sqrt{5x+7}}$
from the previous problem. Then
$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left(f(x+h)-f(x)\right) = \frac{1}{h}\left(\frac{5h}{\sqrt{5(x+h)+7}+\sqrt{5x+7}}\right) = \frac{5}{\sqrt{5(x+h)+7}+\sqrt{5x+7}}$