Math 163 Exam 2 Lehigh University 2-29-2008

This is closed book, closed notes etc. You have 50 minutes to take this exam.

Points for each problem are indicated as $[\cdot]$.

Be sure to do the problems that you know well first.

There are 6 problems plus 2 bonus problems

When you are asked to pick a part or parts of a problem to answer clearly indicate your selection. If you attempt more than is asked those with lowest scores will be used.

1: [15] Pick *one* of the following and prove it:

(a) Let $a_1, a_2, \ldots, a_k, c$ be positive integers. $a_1x_1 + a_2x_2 + \cdots + a_kx_k = c$ has a an integer solution *only if* c is a multiple of the greatest common divisor $\gcd(a_1, \ldots, a_k)$.

(b) Let $a_1, a_2, \ldots, a_k, c$ be positive integers. *At most one* of the following holds:

(I) $a_1x_1 + \cdots + a_kx_k = c$ has an integer solution $x_1, x_2, \ldots, x_k$;

(II) ya_i integral for $i = 1, 2, \ldots, k$ and yc non-integral has a solution y.

2: [20] Pick *one* of the following and prove it:

(a) Let $a \geq b$ be non-negative integers and let $g = \gcd(a, b)$. There in an integer solution to $ax + by = g$.

(b) Let $a_1, a_2, \ldots, a_k$ be positive integers and let $g = \gcd(a_1, a_2, \ldots, a_k)$. There is an integer solution to $a_1x_1 + a_2x_2 + \cdots + a_kx_k = g$. You may assume the $k = 2$ case (which is part (a)) and also you may use $\gcd(\gcd(a_1, \ldots, a_{k-1}), a_k) = \gcd(a_1, \ldots, a_k)$.

3: [15] Prove the following: If the Euclidean algorithm applied to $a > b \geq 1$ requires k steps then $b \geq F_{k+1}$ and $a \geq F_{k+2}$.

4: [15] Do *one* of the following. Be sure to illustrate the steps in your work using the algorithm.

(a) Use the Euclidean algorithm to find the greatest common divisor of 252 and 102 and find x^*, y^* such that $252x^* + 102y^* = \gcd(252, 102)$.

(b) Use Fourier-Motzkin elimination to find multipliers showing that the system

$$\begin{array}{rcrcr} 3x_1 & + & 6x_2 & \leq & 12 \\ -x_1 & + & x_2 & \leq & 2 \\ -2x_1 & - & 6x_2 & \leq & -14 \end{array}$$

has no solution.

5: [20] Consider the variants for strong duality listed below. Pick *one* implication (e.g., A' implies C', B' implies A' etc.) and prove it. State clearly which implication you are proving, if you state that you are proving A' implies C' and give a proof for C' implies A' you will not get full credit.

Strong Duality for Linear Programming variants:

A': If both problems are feasible then :
$\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}\} = \min\{\boldsymbol{yb} | \boldsymbol{y}A = \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$

B': If both problems are feasible then :
$\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0}\}$

C': If both problems are feasible then :
$\max\{\boldsymbol{cx} | A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{yb} | \boldsymbol{y}A \geq \boldsymbol{c}\}$

6: [15] Consider the linear programming problem:

$$
\begin{array}{rrcrcrcrcl}
\max & x_1 & + & 2x_2 & - & 3x_3 & + & 5x_4 & & \\
\text{s.t.} & 8x_1 & - & 13x_2 & & & + & 21x_4 & \leq & 0 \\
 & 34x_1 & + & x_2 & - & x_3 & + & x_4 & = & 42 \\
 & x_1 & , & x_2 & & & & & \geq & 0
\end{array}
$$

Write down an equivalent problem that is in the form $\max\{\boldsymbol{cx} | A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$. Write out the equations as above and also give the matrix A and vectors $\boldsymbol{b}, \boldsymbol{c}$ for the new version.

7: [10] Consider the linear programming problem in problem 6. Write down an equivalent problem that is in the form $\max\{\boldsymbol{cx} | A\boldsymbol{x} \leq \boldsymbol{b},\}$. Write out the equations as above and also give the matrix A and vectors $\boldsymbol{b}, \boldsymbol{c}$ for the new version.

8: [5] Consider the following form of Farkas' lemma:

Exactly one of the following holds:

(I) $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ has a solution $\boldsymbol{x}$ *or* (II) $\boldsymbol{y}A \geq \boldsymbol{0}, \boldsymbol{yb} < 0$ has a solution $\boldsymbol{y}$

Fill in the blank in the following equivalent statement (you do not need to show any work): (I) $A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}$ has a solution $\boldsymbol{x}$ if and only if $\boldsymbol{yb}$ ______ for each $\boldsymbol{y}$ such that $\boldsymbol{y}A \geq \boldsymbol{0}$. You should fill in one of $\geq, \leq, >, <, =$ and something for the right side of the resulting expression.