Math 163 Exam 1 Lehigh University 2-1-2008

This is closed book, closed notes etc. You have 50 minutes to take this exam.

Points for each problem are indicated as $[\cdot]$.

Be sure to do the problems that you know well first.

There are 6 problems plus a bonus problem

When you are asked to pick a part or parts of a problem to answer clearly indicate your selection. If you attempt more than is asked those with lowest scores will be used.

1: [25] Pick *two* of the following and for each give a proof by *induction*.

(a) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n = 1, 2, \ldots$

(b) The Fibonacci numbers F_n given by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$ satisfy $1 + \sum_{i=0}^{n} F_i = F_{n+2}$

(c) If $D_n = D_{n-1} + 12D_{n-2}$ for $n \geq 2$ with $D_0 = 3$ and $D_1 = -2$ then $D_n = 2 \cdot (-3)^n + 4^n$.

2: [17] Pick *two* of the following and give combinatorial (in terms of sets) proofs. You should just use the interpretation of $\binom{n}{k}$ as the number of size k subsets of an n set. You should not write any formula using $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

(a) (Pascal's identity) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

(b) $\binom{n}{k} = \binom{n}{n-k}$

(c) $k\binom{n}{k} = n\binom{n-1}{k-1}$

3: [11] Prove the Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ in any way you like.

4: [13] Make use of the binomial theorem to prove that the number of subsets of an n element set is 2^n and that the number of even sized subsets of an n element set is equal to the number of odd sized subsets.

5: [15] Make use of $n^3 = \sum_{i=1}^{n}(i^3 - (i-1)^3)$ to show that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. You may use $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

6: [19] Consider a bipartite graph G with parts X, Y such that $|X| = |Y|$.
Hall's condition is $|T| \leq |N(T)|$ for all $T \subseteq X$
The marriage condition is $|C| \geq n$ for every vertex cover. Each is a necessary and sufficient condition for G to have a perfect matching. Consider the following four statements:
(1) Hall's condition is a necessary condition for G to have a perfect matching
(2) Hall's condition is a sufficient condition for G to have a perfect matching
(3) The marriage condition is a necessary condition for G to have a perfect matching
(4) The marriage condition is a sufficient condition for G to have a perfect matching

(a) Which one of (1) or (2) is the same as: If G does not have a perfect matching then there is a $T \subseteq X$ such that $|T| > |N(T)|$?
(b) Which one of (1) or (2) is the same as: if G has a perfect matching then $|T| \leq |N(T)|$ for all $T \subseteq X$?
(c) State the contrapositive of the statement in (a)
(d) State the contrapositive of the statement in (b)
(e) State a condition and its contrapositive for (3) analogous to those described in (a) and (b)
(f) State a condition and its contrapositive for (4) analogous to those described in (a) and (b)

7: [16] Pick *one* of the following and prove it:
(a) (If G does not have a perfect matching then there is a $T \subseteq X$ such that $|T| > |N(T)|$) implies (If G does not have a perfect matching then G has a vertex cover C with $|C| < n$).
(b) (If G does not have a perfect matching then G has a vertex cover C with $|C| < n$) implies (If G does not have a perfect matching then there is a $T \subseteq X$ such that $|T| > |N(T)|$)