Math 163 Introductory Seminar - Lehigh University - Spring 2008 - Assignment 8
Due Monday March 31

29. Consider the variants for strong duality listed below.
B': If both problems are feasible then :
$\max\{\boldsymbol{cx}|A\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0}\} = \min\{\boldsymbol{yb}|\boldsymbol{y}A \ge \boldsymbol{c}, \boldsymbol{y} \ge \boldsymbol{0}\}$
C': If both problems are feasible then :
$\max\{\boldsymbol{cx}|A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0}\} = \min\{\boldsymbol{yb}|\boldsymbol{y}A \ge \boldsymbol{c}\}$

(a) Prove that B' implies C'
(b) Prove that C' implies B'

Indicate clearly which is part (a) and which is part (b) in your solution.

30. Consider the following version of strong duality:
A': If both problems are feasible then : $\max\{\boldsymbol{cx}|A\boldsymbol{x} \le \boldsymbol{b}\} = \min\{\boldsymbol{yb}|\boldsymbol{y}A = \boldsymbol{c}, \boldsymbol{y} \ge \boldsymbol{0}\}$
Use this to prove the following version of Farkas' Lemma:

A: Exactly one of the following holds: (I) $A\boldsymbol{x} \le \boldsymbol{b}$, has a solution $\boldsymbol{x}$
(II) $\boldsymbol{y}A = \boldsymbol{0}, \boldsymbol{y} \ge \boldsymbol{0}, \boldsymbol{yb} < 0$ has a solution $\boldsymbol{y}$

Hints: Consider $A\boldsymbol{x} \le \boldsymbol{b}$ in the statement of Farkas' lemma. Introduce a new variable z and subtract it from each inequality and maximize z subject to these new constraints. Write down what new A', $\boldsymbol{c}'$, $\boldsymbol{x}'$, $\boldsymbol{b}'$ are for this. Be careful to include the coefficients for the original $\boldsymbol{x}$ in $\boldsymbol{c}'$. Explain why this new problem is feasible and why $A\boldsymbol{x} \le \boldsymbol{b}$ has a solution if and only if the maximum is at least 0. So then, if $A\boldsymbol{x} \le \boldsymbol{b}$ does not have a solution then the maximum is negative and by duality so is the minimum in the dual problem. Write down the dual for the new problem with the A', $\boldsymbol{c}'$, $\boldsymbol{x}'$, $\boldsymbol{b}'$ and show that a negative solution for this provides a solution to $\boldsymbol{y}A = \boldsymbol{0}, \boldsymbol{y} \ge \boldsymbol{0}, \boldsymbol{yb} < 0$.

31. Given pairs of data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ consider approximating lines of the form $y = mx + b$. The error e_i for the i^{th} pair is the distance between y_i and the height (y value) of the line at x_i. This is $e_i = y_i - (mx_i + b)$. If we consider the equations $b + x_i m = y_i$ for $i = 1, 2, \ldots, n$ in the variables b and m we can think of this as a system of equations $A\boldsymbol{x} = \boldsymbol{b}$ where $A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} b \\ m \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$. The best least squares approximation for this system (which gives the intercept b and slope m of the best least squares line for the data) is the solution to the normal equations $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$. Determine $A^T A$ (a 2×2 matrix) and $A^T\boldsymbol{b}$ (a 2×1 matrix). The entries will be sums of terms involving the x_i and y_i. Write these, first using σ notation and then simplify the notation using $\overline{x} = \frac{\sum_{i=1}^n x_i}{n}$, $\overline{y} = \frac{\sum_{i=1}^n y_i}{n}$, $\overline{x}^2 = \frac{\sum_{i=1}^n x_i^2}{n}$ and $\overline{xy} = \frac{\sum_{i=1}^n x_i y_i}{n}$. Write down the system of 2 equations in the 2 unknowns m, b with coefficients in terms of the expressions in the previous sentence. Solve this system first for m and then determine b in terms of m (and the coefficients). Determine the least squares line for the points $(0, 2), (1, 1), (3, 4)$ using your results.