18. Apply Fourier-Motzkin elimination to the following systems. Using this method, either
determine one solution or give a certificate showing the original system is inconsistent. Use
Fourier-Motzkin elimination to get your answers, showing how it could work on much larger
systems.

(a) \[ \begin{align*}
    x_1 + x_2 & \leq 3 \\
    2x_1 - 8x_2 & \leq 8 \\
    -3x_1 + 9x_2 & \leq -15
\end{align*} \]

(b) \[ \begin{align*}
    x_1 + x_2 & \leq 3 \\
    2x_1 - 8x_2 & \leq 4 \\
    -3x_1 + 9x_2 & \leq -15
\end{align*} \]

We will eliminate the variable \( x_1 \). Rewriting we get the following ‘equivalent’ systems (where
equivalent means either both have solutions or both do not):

For (a)

\[ \begin{align*}
    x_1 + x_2 & \leq 3 \\
    2x_1 - 8x_2 & \leq 8 \quad \Rightarrow \quad x_1 \leq 4 + 4x_2 \quad \Rightarrow \quad 5 + 3x_2 \leq 3 - x_2 \quad \Rightarrow \quad 4x_2 \leq -2 \\
    -3x_1 + 9x_2 & \leq -15 \quad \Rightarrow \quad 5 + 3x_2 \leq x_1
\end{align*} \]

The last system is inconsistent as seen by the multipliers \((1/4, 1)\). (Multiply the first in-
equality by \(1/4\) and the second by 1 and combine to get \(0 \leq -3/2\).) This correspond to
multipliers \((1/4, 1)\) in the third system, multipliers \((1/4, 1, 1 + 1/4) = (1/4, 1, 5/4)\) in the
second system and \((1/4, 1/2, 5/12)\) in the original, yielding the same inconsistency \(0 \leq -3/2\).

For (b)

\[ \begin{align*}
    x_1 + x_2 & \leq 3 \\
    2x_1 - 8x_2 & \leq 4 \quad \Rightarrow \quad x_1 \leq 2 + 4x_2 \quad \Rightarrow \quad 5 + 3x_2 \leq 3 - x_2 \quad \Rightarrow \quad 4x_2 \leq -2 \\
    -3x_1 + 9x_2 & \leq -15 \quad \Rightarrow \quad 5 + 3x_2 \leq x_1
\end{align*} \]

The last system is inconsistent as seen by the multipliers \((1/4, 1)\). (Multiply the first in-
equality by \(1/4\) and the second by 1 and combine to get \(0 \leq -7/2\).) This correspond to
multipliers \((1/4, 1)\) in the third system, multipliers \((1/4, 1, 1 + 1/4) = (1/4, 1, 5/4)\) in the
second system and \((1/4, 1/2, 5/12)\) in the original, yielding the same inconsistency \(0 \leq -7/2\).

19. Prove the following version of weak duality:
If both problems are feasible then \[ \max\{cx | Ax \leq b\} \leq \min\{yb | yA = c, y \geq 0\} \]

Give two proofs. One using matrix notation and one using \( \sum \) notation.

For any feasible \( x^* \) and \( y^* \) we have

\[ cx^* = (y^*A)x^* = y^*(Ax^*) \leq y^*b \]

where the first inequality follows since \( y^*A = c \), the second inequality follows from associa-
tivity and the inequality follows since \( y^* \geq 0 \) and \( Ax^* \leq b \).
We give an alternate proof using summation notation. For any feasible \( x^*_i \) and \( y^*_j \) we have
\[
\sum_{j=1}^n c_j x^*_j = \sum_{j=1}^n \left( \sum_{i=1}^m a_{i,j} y^*_i \right) x^*_j = \sum_{i=1}^m \left( \sum_{j=1}^n a_{i,j} x^*_j \right) y^*_i \leq \sum_{i=1}^m b_i y^*_i.
\]

20. Consider the following statement: If the dual is unbounded then the primal is infeasible. Prove this for the versions of linear programming problems in problem 19 by first stating the contrapositive and then proving that. You may use the result of problem 19.

The contrapositive is: If the primal is feasible then the dual is bounded. To show this note that since the primal is feasible there is a feasible solution \( x^* \). That is, \( x^* \) is such that
\[
A x^* \leq b.
\]
From weak duality, for any feasible \( \mathbf{cx} \leq \mathbf{y}^* \mathbf{b} \). Thus the dual is bounded below by \( \mathbf{cx}^* \).

21. Consider the linear programming problem:
\[
\begin{align*}
\text{max} & \quad x_1 + 2x_2 + 3x_3 \\
\text{s.t.} & \quad 4x_1 - 5x_2 + 6x_3 = 7 \\
& \quad 8x_1 + 9x_3 = 10 \\
& \quad x_1 + x_2 + x_3 \leq 0 \\
& \quad x_1 - x_2 + 13x_3 \leq 1 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

(a) Write down an equivalent problem that is in the form \( \text{max}\{\mathbf{cx} | A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\} \). Write out the equations as above. Also give the matrix \( \mathbf{A} \) and vectors \( \mathbf{b}, \mathbf{c} \) for this.

(b) Write down the duals to both the original problem and the problem in part (a).

(a)
\[
\begin{align*}
\text{max} & \quad x_1 + 2x_2 + 3x_3^+ - 3x_3^- \\
\text{s.t.} & \quad 4x_1 - 5x_2 + 6x_3^+ - 6x_3^- = 7 \\
& \quad 8x_1^+ + 9x_3^- = 10 \\
& \quad x_1 + x_2 + x_3^+ - x_3^- + s_1 = 0 \\
& \quad x_1 - x_2 + 13x_3^+ - 13x_3^- + s_2 = 1 \\
& \quad x_1, x_2, x_3^+, x_3^-, s_1, s_2 \geq 0
\end{align*}
\]

\[
\mathbf{A} = \begin{bmatrix} 4 & -5 & 6 & -6 & 0 & 0 \\ 8 & 0 & 9 & -9 & 0 & 0 \\ 1 & 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 13 & -13 & 0 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3^+ \\ x_3^- \\ s_1 \\ s_2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 10 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 & 2 & 3 & -3 & 0 & 0 \end{bmatrix}
\]

(b)
\[
\begin{align*}
\text{min} & \quad 7y_1 + 10y_2 + y_4 \\
\text{s.t.} & \quad 4y_1 + 8y_2 + y_3 + y_4 \geq 1 \\
& \quad -5y_1 + y_3 - y_4 \geq 2 \\
& \quad 6y_1 + 9y_2 + y_3 + 13y_4 = 3 \\
& \quad y_3, y_4 \geq 0
\end{align*}
\]
\[
\begin{align*}
\text{min} & \quad 7u_1 + 10u_2 + u_4 \\
\text{s.t.} & \quad 4u_1 + 8u_2 + u_3 + u_4 \geq 1 \\
& \quad -5u_1 + u_3 - u_4 \geq 2 \\
& \quad 6u_1 + 9u_2 + u_3 + 13u_4 \geq 3 \\
& \quad -6u_1 - 9u_2 - u_3 - 13u_4 \geq -3 \\
& \quad u_3 \geq 0 \\
& \quad u_4 \geq 0
\end{align*}
\]

22. Prove the equivalence of B and C:

B: Exactly one of the following holds:
(1) \(Ax \leq b, x \geq 0\) has a solution \(x\) or (II) \(y\). \(A \geq 0, y \geq 0, yb < 0\) has a solution \(y\)

C: Exactly one of the following holds:
(1) \(Ax = b, x \geq 0\) has a solution \(x\) or (II) \(y\). \(A \geq 0, yb < 0\) has a solution \(y\)

Note - there are at least two ways to take care of the ‘at most one of the systems has a solution’ part of the statements. While it is a bit redundant we will show both ways below. First we show it directly and we also show it by the equivalent systems. If the ‘at most one system holds’ is shown first then only the \(\Leftarrow\)'s are needed for the equivalent systems.

First we note that for each it is easy to show that at most one of the systems holds.

If both IB and IIB hold then
\[0 = 00 \leq (yA)x = y(Ax) \leq yb < 0\]
a contradiction. We have used \(y \geq 0\) in the second \(\leq\).

If both IC and IIC hold then
\[0 = 00 \leq (yA)x = y(Ax) = yb < 0\]
a contradiction.

So for the remainder we will seek to show at least one of the following holds.

\((B \Rightarrow C)\): Note the following equivalences.

\[(IC)\quad Ax = b, x \geq 0 \quad \Leftrightarrow \quad Ax \leq b, x \geq 0 \quad \Leftrightarrow \quad \begin{bmatrix} A \\ -A \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \end{bmatrix} \quad (IC')\]

and

\[(IIC)\quad yA \geq 0, yb < 0 \quad \Leftrightarrow \quad (u - v)A \geq 0, (u - v)b < 0 \quad \Leftrightarrow \quad \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} A \\ -A \end{bmatrix} \geq 0 \quad \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} b \\ -b \end{bmatrix} < 0 \quad (IIC')\]
In the second line, given \( y \) one can easily pick non-negative \( u, v \) such that \( y = u - v \) so the first \( \iff \) in the second line does hold.

Applying B, we get that exactly one of (IC') and (IIC') has a solution. The equivalences then show that exactly one of (IC) and (IIC) has a solution.

\( (C \Rightarrow B) \): Note following equivalences.

\[
\begin{align*}
(IB) \quad & Ax \leq b \quad \iff \quad Ax + Is = b \quad \iff \quad \begin{bmatrix} A & I \\ x & s \end{bmatrix} \geq 0 \\
\text{and} \quad & yA \geq 0 \quad y \geq 0 \quad \iff \quad yI \geq 0 \quad yb < 0 (IIB')
\end{align*}
\]

Applying C, we get that exactly one of (IB') and (IIB') has a solution. The equivalences then show that exactly one of (IB) and (IIB) has a solution.

23. Prove the equivalence of A' and B':

A': If both problems are feasible then:

\[
\max \{ cx \mid Ax \leq b, x \geq 0 \} = \min \{ yb \mid yA = c, y \geq 0 \}
\]

B': If both problems are feasible then:

\[
\max \{ cx \mid Ax \leq b, x \geq 0 \} = \min \{ yb \mid yA \geq c, y \geq 0 \}
\]

To show \( (A') \) implies \( (B') \): Assuming the first and last LPs below are feasible we have

\[
\begin{align*}
\max \{ cx \mid Ax \leq b, x \geq 0 \} &= \max \left\{ cx \left| \begin{array}{c} Ax \\ -I \end{array} \right| x \leq \begin{bmatrix} b \\ 0 \end{bmatrix} \right\} \\
&= \min \left\{ \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} b \\ 0 \end{bmatrix} \mid \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} A \\ -I \end{bmatrix} = c, \begin{bmatrix} u & v \end{bmatrix} \geq 0 \right\} \\
&= \min \{ yb \mid yA \geq c, y \geq 0 \}
\end{align*}
\]

The first and third equalities follow from basic manipulations. The second follows from \( (A') \).

To show \( (B') \) implies \( (A') \): Assuming the first and last LPs below are feasible we have

\[
\begin{align*}
\max \{ cx \mid Ax \leq b \} &= \max \left\{ c \mid \begin{bmatrix} A & -A \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \leq \begin{bmatrix} b \\ 0 \end{bmatrix} \right\} \\
&= \min \{ yb \mid yA \geq c, y \geq 0 \}
\end{align*}
\]

The first and third equalities follow from basic manipulations. The second follows from \( (B') \).