Math 242 fall 2008 notes on problem session for week of 11-17-08
This is a short overview of problems that we covered.

1. What is the graph of $9x^2 + 4y^2 = 1$? It is an ellipse with points on the major and minor axes $(1/3, 0), (-1/3, 0), (0, 1/2), (0, -1/2)$. Writing the equation in matrix form we have $(x, y)\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$.

2. Graph $8x^2 - 4xy + 5y^2 = 1$. We write this in matrix form as $(x, y)\begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$.

   We first diagonalize the matrix $A = \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$. A is symmetric so we will be able to write $A = SDS^{-1}$ where S is orthogonal and D is diagonal.

   $det(A - \lambda I) = det\begin{pmatrix} 8-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix} = (8-\lambda)(5-\lambda) - (-2)(-2) = \lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4)$. So eigenvalues are 9 and 4.

   To find an eigenvector associated with $\lambda = 9$ we solve the homogeneous system $(A - 9I)\boldsymbol{x} = \mathbf{0}$. The matrix is $\begin{pmatrix} 8-9 & -2 \\ -2 & 5-9 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}$. After Gaussian elimination this becomes $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ with solutions $(x, y) = (2, -1)y$. So multiples of $(2, -1)$ are eigenvalues associated with eigenvalue 9. Similarly we get eigenvector $(1, 2)$ associated with eigenvalue 4. We normalize these and put them as columns of S and make D the diagonal matrix with the eigenvalues on the diagonal. Then we have $A = \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}^{-1} = SDS^{-1}$. The graph is the same ellipse as before but now with axes given by the eigenvectors. To see this let $\boldsymbol{u}_1$ be the eigenvector with eigenvalue 9 and $\boldsymbol{u}_2$ the eigenvector with eigenvalue 4. These are orthogonal and hence forma a basis for $\mathbb{R}^2$. So for any $\boldsymbol{x}$ we can find r, s so that $\boldsymbol{x} = r\boldsymbol{u}_1 + s\boldsymbol{u}_2$. Using $A\boldsymbol{u}_1 = 9\boldsymbol{u}_1$ and $A\boldsymbol{u}_2 = 4\boldsymbol{u}_2$ as these are eigenvectors and using $\boldsymbol{u}_1^T\boldsymbol{u}_2 = \boldsymbol{u}_2^T\boldsymbol{u}_1 = 0$ since they are orthogonal and $\boldsymbol{u}_1^T\boldsymbol{u}_1 = \boldsymbol{u}_2^T\boldsymbol{u}_2 = 1$ since they have been normalized we get that our graph is
   $1 = \boldsymbol{x}^T A\boldsymbol{x} = (r\boldsymbol{u}_1 + s\boldsymbol{u}_2)^T A(r\boldsymbol{u}_1 + s\boldsymbol{u}_2) = (r\boldsymbol{u}_1^T + s\boldsymbol{u}_2^T)(9r\boldsymbol{u}_1 + 4s\boldsymbol{u}_2) = 9r^2\boldsymbol{u}_1^T\boldsymbol{u}_1 + 4rs\boldsymbol{u}_1^T\boldsymbol{u}_2 + 4rs\boldsymbol{u}_2^T\boldsymbol{u}_1 + 4s^2\boldsymbol{u}_2^T\boldsymbol{u}_2 = 9r^2 + 4s^2$.

   So the graph is an ellipse with points on the major and minor axes $\pm 1/3$ of a unit in the direction $\boldsymbol{u}_1 = (\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}})$ and $\pm 1/2$ of a unit in the direction $\boldsymbol{u}_2 = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$.

3. Let y_0 be the population of California at time $t = 0$ and z_0 the population of the rest of the world. If during any year 1/10 of California's population moves out of

the state and 2/10 of the population of the rest of the world moves into the state write equations for the populations y_1, z_1 after 1 year. We get $y_1 = .9y_0 + .2z_0$ and $z_1 = .1y_0 + .8z_0$. In matrix form these are $\begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} .9 & .2 \\ .1 & .8 \end{pmatrix} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}$. Calling the matrix A and the vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_0$ we can extend this to time t. The populations are given by $\boldsymbol{x}_t = A^t \boldsymbol{x}_0$. The matrix A has eigenvalues 1 and 0.7 with associated eigenvectors $(2/3, 1/3)$ and $(1/3, -1/3)$. So we diagonalize $A = \begin{pmatrix} .9 & .2 \\ .1 & .8 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & .7 \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} = SDS^{-1}$. In this case $S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$. Observe that $A^t = (SDS^{-1})^T = (SDS^{-1})(SDS^{-1}) \cdots (SDS^{-1}) = SD(S^{-1}S)D(S^{-1}S)D \cdots S^{-1} = SD^tS^{-1}$. So we can compute $A^t = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1^t & 0 \\ 0 & (.7)^t \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} = \begin{pmatrix} 2/3 + 1/3(.7)^t & 2/3 - 2/3(.7)^t \\ 1/3 - 1/3(.7)^t & 1/3 + 2/3(.7)^t \end{pmatrix}$. As $t \to \infty$ this becomes $\begin{pmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}$, the matrix with the eigenvector associated with 1 in each column. The long term state is $y_t = 2/3$ and $z_t = 1/3$. This is an example of what is called a Markov chain.

4. Let B have eigenvalue λ and associated eigenvector $\boldsymbol{v}$. If B^{-1} exists show that $\boldsymbol{v}$ is an eigenvector of B^{-1} associated with eigenvalue $1/\lambda$. We have $B\boldsymbol{v} = \lambda\boldsymbol{v}$. Then $\frac{1}{\lambda}\boldsymbol{v} = \frac{1}{\lambda}I\boldsymbol{v} = \frac{1}{\lambda}B^{-1}B\boldsymbol{v} = \frac{1}{\lambda}B^{-1}\lambda\boldsymbol{v} = B^{-1}\boldsymbol{v}$.

   Let A have eigenvalue λ and associated eigenvector $\boldsymbol{v}$. Show that $(I-A)$ and $(I-A)^{-1}$ (assuming it exists) also have $\boldsymbol{v}$ as an eigenvector and determine the eigenvalues.

   $(I - A)\boldsymbol{v} = I\boldsymbol{v} - A\boldsymbol{v} = \boldsymbol{v} - \lambda\boldsymbol{v} = (1 - \lambda)\boldsymbol{v}$. So the eigenvalue is $(1 - \lambda)$. Then, from the previous paragraph $\boldsymbol{v}$ is an eigenvector with eigenvalue $\frac{1}{1-\lambda}$ for $(I - A)^{-1}$.

   Finally we noted the matrix equation (similar to the Taylor expansion of $\frac{1}{1-x} = (1 - x)^{-1}$) $(I-A)^{-1} = I + A + A^2 + A^3 + \cdots$ and noted (comparing to the previous example) that we get convergence when the powers of A go to the zero matrix, which occurs if all eigenvalues have magnitude less than 1. This equation appears in the Leontief model in economics.