Examples of Combinatorial Duality

Garth Isaak*

Draft from around 2005, updated 2015

1 Introduction

Our goal in this paper is to illustrate the idea of combinatorial duality: the combinatorial manifestation of theorems of the alternative for linear systems that arise when a combinatorial problem is formulated as such a system. This is a powerful paradigm for viewing such problems even if shorter direct proofs can be found without this perspective.

We will consider two easily stated combinatorial problems, determining if a given sequence of integers could arise as the numbers of wins in a round robin tournament and determining if a diagram of relations can be realized as 'comes before' for a set of intervals in time. Both problems will be given equivalent formulations of determining if a system of inequalities has a solution. These particular linear systems relate to circulation and distance problems in digraphs. Theorems for the digraph problems will yield proofs of the combinatorial problems. Necessary and sufficient conditions for the digraph problems will be derived directly from Farkas' Lemma. Farkas' lemma is a theorem of the alternative for systems of linear inequalities, stating that either such a system has a solution or it has a simple certificate of inconsistency.

Thus, our goal is to show that in some sense the combinatorial theorems we examine are in effect instances of the statement that either a system of linear inequalities has a solution or it is inconsistent.

We begin with a story to motivate the problems.

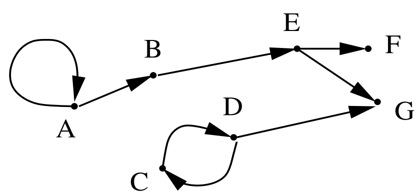
The night before a hypothetical mathematics conference Abel (A), Bolzano (B), Cayley (C), Dantzig (D), Erdos (E), Fourier (F) and Gauss (G) get together for some

*Department of Mathematics, Lehigh University, Bethlehem, PA 18015 gisaak@lehigh.edu

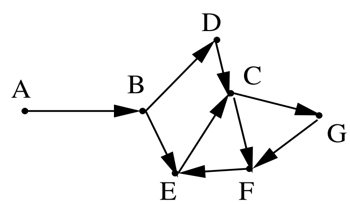
recreation. They play a round-robin ping-pong tournament, with each pair of players playing exactly one game and a winner declared for each game. The score for a player is the number of wins and the *score sequence* for the tournament is a list of the scores for (A, B, C, D, E, F, G). Some other sequences of numbers get written down over the course of the evening and the next morning the task is to determine which of the following is the score sequence for the ping-pong tournament:

(a) $(7, 5, 4\frac{1}{3}, 4, 2\frac{3}{7}, 0, -2)$
(b) $(5, 4, 3, 3, 3, 1, 0)$
(c) $(3, 3, 3, 3, 3, 3, 3)$
(d) $(6, 6, 4, 2, 1, 1, 1)$.

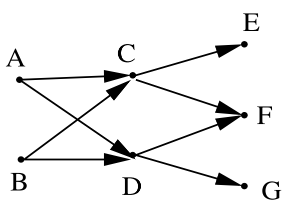
After the ping-pong tournament the mathematicians get together and diagram the sequence of talks for the next day as a digraph. The digraph is drawn with a directed arc from speaker X to speaker Y if the talk X gives ends before the talk Y gives begins. Arcs that are implied by transitivity are omitted so the digraph does not get too cluttered. We call such a digraph an interval digraph. Some other diagrams get drawn over the course of the evening and the next morning the task is to determine which of the following is the interval digraph for the talks:



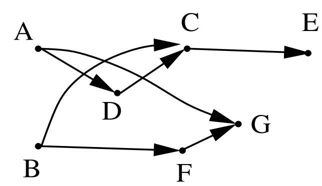
(a)



(b)



(c)



(d)

2 Necessary conditions

Using some straightforward necessary conditions it is not difficult to answer the questions.

For score sequences, (a) is easily eliminated. Scores must be non-negative integers. It is impossible for Cayley to win $4\frac{1}{3}$ games or for Gauss to win -2 games. For (b), note that with 7 players, $\binom{7}{2} = (7)(6)/2 = 21$ games are played. The sum of the scores must be 21 and thus (b) is not a score sequence. Finally observe that the four players with lowest scores, Dantzig, Erdos, Fourier and Gauss play $\binom{4}{2} = (4)(3)/2 = 6$ games amongst themselves. The sum of the scores for these four players must be at least 6 and thus (d) is not a score sequence. So (c) remains as the only possible score sequence. It is not difficult to construct outcomes of a round-robin tournament with these scores.

For interval digraphs (a) is easily eliminated. Abel's talk cannot end before it begins and Cayley and Dantzig cannot each end their talk before the other's talk begins. For (b) note that the cycle C, G, F, E, C along with the assumed transitivity would imply that each of these person's talks ends before it begins. Thus (b) is not an interval digraph. Finally, observe that if X's talk ends before Y's begins and each of Z and W have talks that overlap the talks of X and Y then the talks of Z and W must also overlap (during the gap between the end of X's talk and the beginning of Y's). So it is not possible for the only arcs (including those implied by transitivity) between four speakers X,Y,Z,W to be $X \to Y$ and $Z \to W$. Looking at Dantzig, Erdos, Fourier and Gauss we see that (d) is not an interval digraph for this reason. So (c) remains as the only possible interval digraph. It is not difficult to describe time intervals for talks that result in this interval digraph.

3 Necessary conditions in general

We have described necessary conditions in the specific instances above. Some of these go naturally into the formal definitions of score sequences and interval digraphs and the others are easy to extend to the general versions below.

Score sequence*: A sequence $(s_1, s_2, \ldots, s_n)$ of non-negative integers is a score sequence if s_i records the number of wins for player i in a round-robin tournament (where for each pair of players there is exactly one game and one winner).*

Score sequences are usually defined in the context of outdegrees in complete directed graphs, or tournaments. They are also often defined to be non-increasing.

A *digraph* or directed graph consists of a set V of vertices along with a set A of arcs, which are ordered pairs of distinct vertices. Sometimes digraphs are defined to allow loops, arcs from a vertex to itself or multiple arcs. A walk from x_1 to x_t in a digraph is a sequence of (not necessarily distinct vertices) $x_1, x_2, \ldots, x_t$ such that $x_i x_{i+1}$ is an arc for each $i = 1, 2, \ldots, t-1$. A *closed walk* is a walk from a vertex to itself. That is, such that $x_1 = x_t$. An *acyclic digraph* is a digraph with no non-trivial closed walks. The term comes from no cycles, a cycle is a closed walk with all vertices distinct except $x_1 = x_t$. It is not difficult to show that a digraph with a non-trivial closed walk contains a cycle. The length of a walk or cycle is the number of arcs in the walk. If the digraph has weights associated with the arcs, then the length is the sum of the weights of the arcs. A shortest walk is a walk with minimum length.

Interval digraph: *An acyclic digraph is an interval digraph if there is a set of time intervals for the vertices so that there is a walk from x_i to x_j if and only if the interval for x_i ends before the interval for x_j begins.*

What we have described as interval digraphs are usually called interval orders and discussed in the context of ordered sets. The binary relation described by walks in an acyclic digraph corresponds to a strict partially ordered set. The transitive closure of an acyclic digraph is is the digraph obtained by adding an arc from x to y if there is a walk from x to y. A transitive acyclic digraph is a digraph that is its own transitive closure and the binary relation determined by the arcs is a strict partial order.

It is straightforward to check that the necessary conditions in the specific instances above extend to the following necessary conditions.

Necessary conditions for score sequences: If non-negative integers $s_1, s_2, \ldots, s_n$ are a score sequence of a tournament then

$$\sum_{i \in I} s_i \geq \binom{|I|}{2} \text{ for any } I \subseteq \{1, 2, \ldots, n\} \text{ with equality when } I = \{1, 2, \ldots, n\}.$$

This simply states that the number of wins among any subset of players must be at least as large as the number games they play and the total number of wins must equal the total number of games. There is an equivalent condition that any subset of players cannot have too many wins. However, if a subset has too many wins then its compliment has too few wins, violating the condition above. So we do not write down the too many wins condition. Observe also that the non-negativity condition is implied by the case $|I| = 1$ but it is convenient to state it explicitly.

Necessary conditions for interval digraphs: If a transitive acyclic digraph is an interval digraph then it does not contain a $\mathbf{2} + \mathbf{2}$, a set of four vertices x, y, z, w whose only arcs are $x \to y$ and $z \to w$.

Necessity of this condition is exactly as described in the specific case above. A $\mathbf{2}+\mathbf{2}$ is called this as when the digraph is viewed as a partial order it corresponds to the disjoint union of two chains, each with 2 elements.

4 Sufficiency

The necessary conditions for score sequences and interval digraphs described above turn out to also be sufficient.

Landau's Theorem: Non-negative integers $s_1, s_2, \ldots, s_n$ are the scores of some tournament if and only if

$$\sum_{i \in I} s_i \geq \binom{|I|}{2} \text{ for any } I \subseteq \{1, 2, \ldots, n\} \text{ with equality when } I = \{1, 2, \ldots, n\}. \quad (1)$$

and

Fishburn's Theorem: A transitive acyclic digraph is an interval digraph if and only if it does not contain a $\mathbf{2}+\mathbf{2}$.

Landau's Theorem was first proved by Landau [14] for applications to dominance relations in biology such as pecking order in a group of chickens. Since then there have been multiple proofs. For example the survey [15] describes at least 8 proofs by various authors and [11] gives two more.

Fishburn's Theorem was first proved by Fishurn [4], [5]. It is closely related the characterization of interval graphs by Gilmore and Hoffman [10]. Various proofs can be found, for example in [2]. See [6] for more information on interval orders. Fishburn and Monjerdet [7] have also written a fascinating paper on the 'prehistory' of interval digraphs. Many results were anticipated in several papers published by Norbert Weiner in the period 1914-1921 in his early 20's while working with Bertrand Russell. Weiner's papers use the formal notation of Russell and Whitehead and the paper of Fishburn and Monjerdet 'translates' into terminology that we can understand.

Our aim in the rest of this paper is to describe how both Landau's and Fishburn's Theorems are implied by theorems of the alternative for systems of linear inequalities. That is, by theorems that state that either a system of linear inequalities has a solution or it is inconsistent. Most undergraduate students are familiar with the analogous alternative theorems for systems of equations. It is interesting to see how the paradigm suggested by these theorems of the alternative provide a powerful tool and context for results that on the surface would not seem to be related. The proofs invoking this tool are insightful even though in the end one might prefer the direct proofs that do not require the extra machinery.

5 Linear systems

Returning to our hypothetical mathematicians, Gauss posed the challenge of solving the following systems of linear equations.

$$\begin{array}{rcrcrcr} x & + & 4y & - & z & = & 2 \\ -2x & - & 3y & + & z & = & -1 \\ -3x & - & 2y & + & z & = & 0 \\ 4x & + & y & - & z & = & -1 \end{array} \qquad \begin{array}{rcrcrcr} x & + & 4y & - & z & = & 2 \\ -2x & - & 3y & + & z & = & 1 \\ -3x & - & 2y & + & z & = & 0 \\ 4x & + & y & - & z & = & -1 \end{array}$$

The other mathematicians could easily solve the system on the left. A solution is $x = 0, y = 1, z = 2$. The system on the right has no solution, convincing Gauss that this is the case involves a little more work.

A typical undergraduate might verify that the system on the right has no solution by noting something about the row echelon form of the reduced augmented matrix having a row of zeros with a non-zero right side or some other variant on this. This can be directly stated by producing a certificate of inconsistency such as $(-1, -3, 3, 1)$. Multiplying the first row by -1, the second by -3, the third by 3 and the fourth by 1 and adding we get the inconsistency $0 = -6$. So the system must not have a solution. The fact that a system of linear equations either has a solution or a certificate of inconsistency is presented in in various guises in typical undergraduate linear algebra texts and often proved as a result of the correctness of Gaussian elimination.

Once the mathematicians solved Gauss's challenge, Dantzig posed the challenge of solving the following systems of linear inequalities.

$$\begin{array}{rcrcrcr} x & + & 4y & - & z & \leq & 2 \\ -2x & - & 3y & + & z & \leq & 1 \\ -3x & - & 2y & + & z & \leq & 0 \\ 4x & + & y & - & z & \leq & -1 \end{array} \tag{2}$$

$$\begin{array}{rcrcrcr} x & + & 4y & - & z & \leq & 1 \\ -2x & - & 3y & + & z & \leq & -2 \\ -3x & - & 2y & + & z & \leq & 1 \\ 4x & + & y & - & z & \leq & 1 \end{array} \tag{3}$$

The other mathematicians could easily solve the system (2). A solution is $x = 0, y = 1, z = 2$. The system (3) has no solution, convincing Dantzig that this is the case involves a little more work.

In order to get a certificate of inconsistency consisting of multipliers for the rows as we did for systems of equations we need to be a bit more careful with the multipliers. Try using the same multipliers $(-1, -3, 3, 1)$ from the equations for the inequalities (3). Multiplying the first row by -1, the second by -3, the third by 3 and the fourth by 1 and combining we get $0 \leq 11$. This is not an inconsistency. As before we need a left side of 0 but because of the $\leq$ we need the right side to be negative in order to get an inconsistency. So we try the multipliers $(1, 3, -3, -1)$ and would seem to get the inconsistency $0 \leq -11$. However, this is not a certificate of inconsistency. Recall that multiplying an inequality by a negative number also changes the direction of the inequality. In order for our computations to be valid for a system of inequalities the multipliers must be non-negative.

It is not difficult to check that $(3, 4, 1, 2)$ is a certificate of inconsistency for (3). Multiplying the first row by 3, the second by 4, the third by 1 and the fourth by 2 and combining we get the inconsistency $0 \leq -2$. So the system of inequalities (3) has no solution. In general, for a system of inequalities, a certificate of inconsistency consists of non-negative multipliers and results in $0 \leq b$ with b negative. For a mixed system with equalities and inequalities we can drop the non-negativity constraint on multipliers for the equations.

The fact that a system of linear inequalities either has a solution or a certificate of inconsistency is often called Farkas's lemma. It can be proved by an easy induction using Fourier-Motzkin elimination. Fourier-Motzkin elimination in some respects parallels Gaussian elimination, using (non-negative) linear combinations of inequalities to create a new system in which a variable is eliminated. From a solution to the new system a solution to the original can be determined and a certificate of inconsistency to the new system can be used to determine a certificate of inconsistency to the original.

6 Fourier-Motzkin Elimination

We will give here just a small part of an example of Fourier-Motzkin elimination for illustration. The reader can easily extend these ideas to produce a formal inductive proof. See, for example [16] for more details.

Consider the system of inequalities (3). Rewrite each inequality so that it is of the form $x \geq$ or $x \leq$ depending of the sign of the coefficient of x.

$$\begin{array}{rcrcrcrcrcrcr} & & & & & & x & \leq & 1 & - & 4y & + & z \\ 1 & - & 3y/2 & + & z/2 & \leq & x & & & & & & \\ -1/3 & - & 2y/3 & + & z/3 & \leq & x & & & & & & \\ & & & & & & x & \leq & 1/4 & - & y/4 & + & z/4 \end{array}$$

Then pair each upper bound on x with each lower bound on x.

$$\begin{array}{rcrcrcrcrcr} 1 & - & 3y/2 & + & z/2 & \leq & 1 & - & 4y & + & z \\ 1 & - & 3y/2 & + & z/2 & \leq & 1/4 & - & y/4 & + & z/4 \\ -1/3 & - & 2y/3 & + & z/3 & \leq & 1 & - & 4y & + & z \\ -1/3 & - & 2y/3 & + & z/3 & \leq & 1/4 & - & y/4 & + & z/4 \end{array} \tag{4}$$

Simplify to obtain a new system in which x is eliminated.

$$\begin{array}{rcrcr} 5y/2 & - & z/2 & \leq & 0 \\ -5y/4 & + & z/4 & \leq & -3/4 \\ 10y/3 & - & 2z/3 & \leq & 4/3 \\ -5y/12 & + & z/12 & \leq & 7/12 \end{array} \tag{5}$$

The new system (5) has a solution if and only if the original (3) does. The new system is inconsistent. A certificate of inconsistency is $(2, 6, 1, 2)$. Observe that the first row of (5) is obtained from 1/2 the second row and the first row of (3). Similarly, the second row of (5) comes from 1/2 the second row and 1/4 the fourth row of (3). The other inequalities in (5) do not involve the second row of (3). Using the multipliers 2,6 for the first two rows of (5) we translate to a multiplier of $2 \cdot 1/2 + 6 \cdot 1/2 = 3$ for the second row of (3). Looking at the other rows in a similar manner we translate the certificate of inconsistency $(2, 6, 1, 2)$ for (5) to the certificate of inconsistency $(3, 4, 1, 2)$ for (3).

In a similar manner any certificate of inconsistency for the new system determines a certificate of inconsistency for the original.

Now, consider the system of inequalities (2). Eliminating x as in the previous example we get the system

$$\begin{array}{rcrcr} 5y/2 & - & z/2 & \le & 5/2 \\ -5y/4 & + & z/4 & \le & 1/4 \\ 10y/3 & - & 2z/3 & \le & 2 \\ -5y/12 & + & z/12 & \le & -1/4 \end{array} \tag{6}$$

A solution to (6) is $y = 1$, $z = 2$. Substituting these values into (2) we get

$$\begin{array}{rcl} x & \le & 0 \\ -2x & \le & 2 \\ -3x & \le & 0 \\ 4x & \le & 0 \end{array}$$

So $y = 1$, $z = 2$ along with $x = 0$ gives a solution to (2). In general, each solution to the new system substituted into the original yields an interval of possible values for x. Since we paired upper and lower bounds to the get the new system, this interval will be well defined for each solution to the new system.

An inductive proof of Farkas' lemma can be given following the patterns of the examples above. Eliminate one variable to obtain a new system. A solution to the new system can be used to determine a solution to the original and a certificate of inconsistency to the new system can be used to determine a certificate of inconsistency to the original.

Note that in these example we get the same number of new inequalities. In general, with n inequalities we might get as many as $n^2/4$ new inequalities if upper and lower bounds are evenly split. Iterating to eliminate all variables might then yield an exponential number of inequalities in the end. This is not a practical method for solving systems of inequalities, either by hand or with a computer. It is interesting as it does yield a simple inductive proof of Farkas' Lemma. There are variations of Farkas' lemma for systems of linear equations with non-negative variables etc.

There are efficient methods to solve systems of inequalities similar to those used to solve linear programming problems. Farkas' lemma is closely related to the duality theorem for linear programming. Linear programming refers to the problem of maximizing or minimizing a linear objective function subject to constraints given by a linear system of equations and/or inequalities including possibly bounds on the variables. The word programming in linear programming refers to the terminology used in the original application and not to computer programming. See Dantzig [3]. These results can also be proved using the simplex method developed by George Dantzig to solve linear programming problems and by separating hyperplane theorems, stating for example that a convex polytope can be separated from a point outside by a hyperplane.

7 Digraphs

We will not apply Farkas' lemma directly for our problems. Instead we will indicate how it can be used to prove some 'obvious' results about digraphs which are easily understood. These results will be used to prove Landau's and Fishburn's Theorems.

We now encounter digraphs in a context different from interval digraphs. The digraphs we consider now will in addition have various numbers associated with the arcs.

Circulation in a digraph: *Given a digraph along with upper and lower bounds $u(xy)$ and $l(xy)$ associated with each arc xy (such that $l(xy) \leq u(xy)$) a circulation is an assignment of flows $f(xy)$ to the arcs such that*

$$\sum_{xy \in A} f(xy) - \sum_{yz \in A} f(yz) = 0 \text{ for all vertices } y \in V \tag{7}$$

$$f(xy) \leq u(xy) \text{ for all arcs } xy \in A \tag{8}$$

$$-f(xy) \leq -l(xy) \text{ for all arcs } xy \in A. \tag{9}$$

Flow conservation, the flow into each vertex equals the flow out is given by (7). Upper and lower bounds on the flow in each arc are given by (8) and (9).

For convenience we have not restricted flows to be non-negative. A negative flow on an arc can be thought of as a positive flow of the same absolute value on an arc in the opposite direction.

Summing the flow conservation constraints over a set of vertices S we get a more general flow conservation stating that flow into S equals flow out of S:
$\displaystyle\sum_{x \notin S, y \in S} f(xy) = \sum_{y \in S, z \notin S} f(yz)$.

Apply the upper and lower bounds to get a necessary condition for a circulation

Necessary condition for circulations: If a digraph along with upper and lower bounds $u(xy)$ and $l(xy)$ has a circulation then

$$\sum_{x \notin S, y \in S} u(xy) \geq \sum_{y \in S, z \notin S} l(yz) \text{ for all } S \subset V$$

This simply states that the maximum amount of flow that can enter S must be at least the minimum amount of flow that must leave S. There is similar condition that the maximum amount of flow that can leave S must be at least the minimum amount of flow that must enter S. By looking at complements we see that these conditions are equivalent.

What we will see is that this necessary condition for a circulation is also sufficient. A circulation can be found for any digraph which satisfies this condition. Indeed we will see that this is an immediate consequence of Farkas' lemma applied to the system given by (7), (8), (9). No solution will imply the existence of a set S violating the necessary condition.

Potentials and Shortest walks in digraphs: *Given a digraph along with weights $w(xy)$ associated with each arc xy, a potential function p defined on the vertices is any function satisfying*

$$p(y) - p(x) \leq w(xy) \textit{ for all arcs } xy \in A. \tag{10}$$

Observe that if $d(x)$ is the length (distance) of a shortest walk ending at x (and starting anywhere, including x) then d is a potential: If xy is an arc then the distance to y is at most the distance to x plus the length $w(xy)$ of arc xy and we have $d(y) \leq d(x) + w(xy)$. It is not hard to see that a negative closed walk contains a negative cycle. Summing (10) over arcs on a negative cycle yields the inconsistency $0 \leq w(C)$ where $w(C)$ is the (negative) weight of the cycle. So having a multiplier 1 for the inequalities corresponding to arcs on a negative cycle and 0 for other inequalities shows that the system (10) has no solution if there is a negative cycle. Thus we get

Necessary condition for potentials: If a digraph along with weights $w(xy)$ associated with each arc xy admits a potential function then it does not contain a negative cycle.

So absence of negative cycles is a necessary condition for a potential. What we will see is that this necessary condition is also sufficient. A digraph with no negative cycles has a potential. Indeed we will see that this is an immediate consequence of Farkas' lemma applied (10). No solution will imply the existence of a negative cycle. It can be shown that shortest walks are well defined if and only if there are no negative cycles.

8 Sufficiency for circulations and shortest walks

We will now show how sufficiency of the necessary conditions for circulations and shortest walks follows from Farkas' lemma. The circulation theorem was proved by Hoffman in [12]. It is often proved algorithmically, see for example [17]. It is necessary only to understand the statements of these results in order to prove Landau's and Fishbun's Theorems so the proofs could be skipped. The key idea in each of the proofs in this section is to write down an appropriate linear system and show that if the system is inconsistent then there is a 'nice' inconsistency, corresponding to a violation of the necessary conditions.

The results are stated as follows.

Circulation Theorem: A digraph with upper and lower bounds on each arc has a circulation if and only if

$$\sum_{x\notin S, y\in S} u(xy) \geq \sum_{y\in S, z\notin S} l(yz) \text{ for each vertex subset } S \tag{11}$$

Potential Theorem: A digraph with weights on each arc admits a potential function if and only if the digraph does not contain a negative cycle.

Proof of the circulation theorem:
With a slight abuse of notation we will think of flows $f(xy)$ as variables. Directly from the definition we see that a digraph has a circulation if and only if the system of inequalities given by (7), (8), (9) has a solution.

We have already observed that if there is a circulation then the condition (11) must hold. So it remains to show that if there is no solution to the system (7), (8), (9) then there is some set S violating the necessary condition (11).

By Farkas' Lemma, if there is no solution to this system, then there exists a certificate of inconsistency consisting of multipliers (non-negative for each inequality, arbitrary for each equality). Multiplying each inequality/equality by the corresponding multiplier and combining yields an inconsistency of the form $0 \leq b$ where $b < 0$. Each variable $f(xy)$ appears four times, with coefficient 1 in equation (7) for x and inequality (8) for xy and with coefficient -1 in equation (7) for y and inequality (9) for xy.

Consider a certificate of inconsistency that maximizes the number of equations of type (7) with multiplier 0. We can assume that at most one of the multipliers for a given arc from (8), (9) is non-zero. If both were non-zero reducing each by the smaller value of the two would still result in an inconsistency since $u - l \geq 0$. If all multipliers for (7) are 0 then with at most one of the multipliers (8), (9) for a given arc non-zero, they both must be zero. Then all multipliers would be zero, which cannot occur. Thus we may assume that some multiplier for (7) is non-zero. We will assume there is some positive multiplier. A similar proof follows if we assume a negative multiplier.

Let T be the set of vertices whose corresponding equation in (7) has positive multiplier and let δ be the smallest positive value of such a multiplier. Let $T_{out} = \{xy \in A | x \in T, y \notin T\}$ be the set of arcs 'leaving' T. Similarly let $T_{in} = \{xy \in A | x \notin T, y \in T\}$ be the set of arcs 'entering' T. Consider $xy \in T_{out}$. Since the multiplier for x is at least δ and the multiplier for y is at most 0, the multiplier for xy in (9) is at least δ. Similarly, for $xy \in T_{in}$ the multiplier in (8) is at least δ. Reduce by δ the following multipliers: (7) for vertices in T, (8) for arcs in T_{in} and (9) for arcs in T_{out}. Do not change other multipliers. Multiplying by the revised multipliers and adding still yields a left side of 0. So we get $0 \leq b'$ for some b'. The revised multipliers have a greater number of 0 multipliers for (7). Thus by our original choice, the revised multipliers do not describe an inconsistency and we must have $b' > 0$. It is straightforward to check that

$$b' - b = \delta \left(\sum_{xy \in T_{out}} l(xy) - \sum_{xy \in T_{in}} u(xy) \right).$$

However, since $b < 0$ we have $b' - b > 0$ and the set T violates the necessary condition (11). □

Proof of potential theorem: We have already observed that if there is a negative cycle then (10) has no solution. So it remains to show that if there is no solution to then there is a negative cycle.

By Farkas' Lemma, if there is no solution to (10), then there exists a certificate of inconsistency consisting of non-negative multipliers. Multiplying each inequality by the corresponding multiplier and combining yields an inconsistency of the form $0 \leq b$ where $b < 0$. There is a multiplier corresponding to each arc xy. Call these multipliers $f(xy)$. The condition that the left side of the inconsistency $0 < b$ is 0 states that for each variable the sum of the coefficients in (10) times the corresponding multiplier is 0. Since each variable corresponds to a vertex y and inequalities for arcs xy have coefficient $+1$ and inequalities for arcs yz have coefficient -1 we get

$$\sum_{xy \in A} f(xy) - \sum_{yz \in A} f(yz) = 0 \text{ for all vertices } y \in V.$$

This is just flow conservation (7). So we can think of the multipliers in a certificate of inconsistency as a flow. Since the multipliers are non-negative the flow values are also. The *weight* of the flow is $\sum f(xy)w(xy)$. The condition $b < 0$ is that the flow has negative weight.

Consider a certificate of inconsistency that maximizes the number of 0 multipliers. Take a path $z_1z_2\dots z_t$ maximizing t with each multiplier $f(z_iz_{i+1}) > 0$. Since $f(z_{t-1}z_t) > 0$ and flow conservation holds there must be an arc z_tw with $f(z_tw) > 0$. By maximality of t, w must be z_i for some i. Let $\delta > 0$ be the smallest multiplier for the arcs on the cycle $z_iz_{i+1}\dots z_t$. Reduce the multiplier for each arc on the cycle and do not change the other multipliers. This results in a flow with with more 0 multipliers. That is, multiplying by the revised multipliers and adding still yields a left side of 0. So we get $0 \le b'$ for some b'. The revised multipliers have a greater number of 0 multipliers. Thus by our original choice, the revised multipliers do not describe an inconsistency and we must have $b' > 0$. It is straightforward to check that

$$b - b' = \delta\left(f(z_tz_1) + \sum_{i=1}^{t-1} f(z_iz_{i+1})\right).$$

However, since $b < 0$ we have $b - b' < 0$ and the sum on the right above, which is the weight of the cycle $z_1z_2\dots z_tz_1$ is negative. □

9 Integrality

For our application of the circulation theorem to prove Landau's Theorem we will need more than just a solution to a linear system. We will in fact need an integral solution. An integral circulation has all flow values integral.

Integrality of circulations: If a digraph has integral upper and lower bounds on each arc and has a circulation, then it has an integral circulation.

There is an analogous result for potentials but we do not need it here.

At the end of this section we will outline a direct proof of the integrality theorem for circulations. First we will remark on how this theorem fits into a broader set of results. See [16] for more details on the brief comments in this section.

Consider the questions regarding linear systems posed earlier by the hypothetical Gauss and Dantzig. If integral solutions were required the problems become much more difficult. For integral solutions to systems of equations, there is a theorem of the alternative like that for general systems and like Farkas' lemma. There is also a computationally efficient method for finding solutions or certificates of inconsistency which can be viewed as an extension of the Euclidean algorithm. For integral solutions to systems of linear inequalities there is *not* a theorem analogous to Farkas' lemma and computationally the problem is difficult. It is in the class of NP-complete problems (NP stand for non-deterministic polynomial) for which no efficient (i.e., polynomial time) algorithms are known (and it is expected that none exist).

Fortunately for our purposes, the linear systems that arise in the circulation theorem have a very nice form and we are able to get integrality results.

The matrix of coefficients for the left side of the inequalities and equations (7), (8), (9) has a special property. Every square submatrix has determinant 0, +1 or -1. Such a matrix is called totally unimodular. This is not difficult to prove using induction. Using this, along with the fact that vertices of the polyhedron defined by the system can be found by solving certain subsystems of equations and recalling Cramer's rule for solutions of systems of equations it can be shown that integral right sides in such a system along with total unimodularity yields the integrality result.

We can also use ideas similar to those in the proof of the potential theorem to directly prove the integrality theorem for circulations as follows. Consider a circulation. By flow conservation, if a vertex has at least one incident arc with non-integral flow then it has at least two. Thus, it can be shown that, ignoring the directions on the arcs, if some arc has non-integral flow then there is a cycle of arcs with non-integral flow. Follow such a cycle and increase flow for arcs traversed in a forward direction and decrease flow for arcs traversed in a backward direction by an equal amount 'until' some flow becomes integral. This maintains flow conservation and does not violate the bounds (if they are integral). So we obtain a circulation with fewer arcs having non-integral flow. Repeating this we eventually obtain an integral circulation.

10 Proof of Landau's Theorem

In order to prove Landau's Theorem we will write down a linear system corresponding to a possible score sequence. We can view this system as the system arising for a circulation of a particular digraph. The digraph will have an integral circulation if and only if the sequence is a score sequence of some tournament. It will be straightforward to see that a violation of Landau's necessary condition yields a violation of the necessary condition for circulations. To complete the proof we need to show that if there is a violation of the condition for circulations then there is a 'nice' violation which corresponds to a violation of Landau's condition.

Our proof of Landau's Theorem is closely related to ideas found in both in [1] and [9]. The form we give is probably known to many but it is a bit of a variant to those described in the survey [15].

Consider a potential score sequence $(s_1, s_2, \ldots, s_n)$ of non-negative integers. For each integral pair $1 \leq x < y \leq n$ define a variable $f(xy)$. We will keep in mind the following association between arcs $xy \in A$ for a tournament with vertices $\{1, 2, \ldots, n\}$ and the variables. Recall that for each pair exactly one of xy or yx is an arc.

$$f(xy) \quad = \quad \begin{cases} 1 \text{ iff } xy \in A \\ 0 \text{ iff } yx \in A \end{cases} \tag{12}$$

So when the variable is 1 the player with smaller label wins and when it is 0 the player with larger label wins. Thus, for $x < y$, x wins when $f(xy) = 1$ and y wins when $f(xy) = 0$. Equivalently, y wins when $(1 - f(xy)) = 1$. We thus force the appropriate scores by

$$\sum_{x<y}(1 - f(xy)) + \sum_{y<z} f(yz) = s_y \text{ for } y = 1, 2, \ldots, n \tag{13}$$

Rewriting this we get

$$-(s_y - y + 1) + \sum_{x<y} -f(xy) + \sum_{y<z} f(yz) = 0 \text{ for } y = 1, 2, \ldots, n \tag{14}$$

Given a tournament with scores $s_1, s_2, \ldots, s_n$, set the $f(xy)$ by the rules (12). Then it is easy to see that this yields a $0-1$ solution to system (14). Conversely any $0-1$ solution to the system will determine a tournament with scores $s_1, s_2, \ldots, s_n$.

The system (14) looks very similar to flow conservation constraints except for the terms $s_y - y + 1$. Imagine an extra vertex 0 with arcs $0y$ for $y \in \{1, 2, \ldots, n\}$ having $s_y - y + 1$ as both upper and lower bounds. That is, consider a digraph on vertex set $\{0, 1, \ldots, n\}$ with arcs xy for $x < y$, lower bound 0 and upper bound 1 for all arcs xy with $x \neq 0$ and $l(0y) = u(0y) = s_y - y + 1$. The circulation constraints for this digraph then become exactly (14) after substituting $f(0y) = s_y - y + 1$ (forced since this is both the upper and lower bound) along with a lower bound of 0 and an upper bound of 1 on the variables $f(xy)$. By the integrality theorem for circulations (14) has a 0-1 solution if and only if the digraph has a circulation. By the circulation theorem, if there is no solution then there is a set of vertices violating the necessary condition (11). Consider such a set I with $0 \notin I$. We will show that this is a set with 'too few' wins, violating Landau's necessary condition (1). In a manner similar to what we do below, it is straightforward to check that a set of vertices that contains 0 and violates the necessary condition (11) has a complement that violates Landau's necessary condition.

If I with $0 \notin I$ violates the circulation condition we have $\sum_{x \notin I, y \in I} u(xy) < \sum_{y \in I, z \notin I} l(yz)$. Since $0 \notin I$ all of the lower bounds $l(yz) = 0$. Using the bound $s_y - y + 1$ for $u(0y)$

and the upper bound 1 for $u(xy)$ with $x \neq 0$ we get

$$\sum_{y \in I}(s_y - y + 1) + \sum_{x \neq 0, x \notin I, y \in I} 1 < 0.$$

Equivalently

$$\sum_{y \in I} s_y < \left(\sum_{y \in I}(y-1) - \sum_{x \neq 0, x \notin I, y \in I} 1 \right).$$

The first term on right side above counts the number of arcs in the digraph into vertices $y \in I$ and the second term subtracts those arcs into vertices $y \in I$ from vertices outside of I. So the right side counts the number of arcs with both ends in I. This is $\binom{|I|}{2}$ and we obtain a violation of Landau's necessary condition. $\square$

11 Proof of Fishburn's Theorem

In order to prove Fishburn's Theorem we will write down a system of linear inequalities corresponding to the endpoints in an possible interval representation of a digraph D. We can view this system as the system arising for potentials of a different digraph D'. The new digraph D' admits a potential function if and only if the original digraph D is an interval digraph. It will be straightforward to see that a $\mathbf{2+2}$ in D corresponds to a negative cycle in D', so there can be no potential function in this case. To complete the proof we need to show that if there is no potential function for D', that is, if there is a negative cycle in D' , then there is a negative cycle corresponding to a $\mathbf{2+2}$ in D.

Our proof of Fishburn's Theorem is based on the ideas of proofs of more general results found in [8] and [13] but we are not aware of it being written down before.

Consider a transitive acyclic digraph D with vertex set $V(D)$ and arc set $A(D)$ for which we wish to determine if it has an interval representation. For each vertex $v \in V(D)$ define two variables $p(r_v)$ and $p(l_v)$. We will keep in mind the association that the variables $p(r_v)$ correspond to the placement of right endpoints and the variables $p(l_v)$ correspond to the placement of the left endpoints in an interval representation. For this we need $p(r_v) < p(l_w)$ if and only if vw is an arc of D. We will fix some positive constant ϵ which we think of as the gap between endpoints and hence rewrite $p(r_v) < p(l_w)$ as $p(r_v) - p(l_w) \leq -\epsilon$. For not $p(r_v) < p(l_w)$ we have $p(r_v) \geq p(l_w)$ or $-p(r_v) + p(l_w) \leq 0$. For intervals we also need $p(l_v) \leq p(r_v)$ to put the left endpoint smaller than the right endpoint. Thus we get the following system of inequalities

$$\begin{array}{rcl} p(r_v) - p(l_w) & \leq & -\epsilon \text{ for } vw \in A(D) \\ -p(r_v) + p(l_w) & \leq & 0 \text{ for } vw \notin A(D) \\ -p(r_v) + p(l_v) & \leq & 0 \text{ for } v \in V(D) \end{array} \tag{15}$$

The endpoints of an interval representation of the digraph satisfy (15) and any solution to (15) yields an interval representation using intervals $[p(l_v), p(r_v)]$.

If we think of the variables $p(r_v)$ and $p(l_w)$ as potentials, the system (15) corresponds to the definition of potentials for a new digraph D' with vertex set

$$V(D') = \{r_v | v \in V(D)\} \cup \{l_v | v \in V(D)\}$$

and arc set

$$A(D') = \begin{cases} l_w r_v \text{ with weight } -\epsilon \text{ for } vw \in A(D) \\ r_v l_w \text{ with weight } 0 \text{ for } vw, wv \notin A(D) \\ r_v l_v \text{ with weight } 0 \text{ for } v \in V(D) \end{cases}$$

Observe that if $vw, wv \notin A(D)$ then we have both arcs $r_v l_w$ and $r_w l_v$ with weight 0. A $\mathbf{2}+\mathbf{2}$ in D with arcs uv and xy (and all other pairs non-arcs) then corresponds to a negative cycle $l_y r_x l_v r_u l_y$ in D' as is easily checked.

If D' does not admit a potential function then by the potential theorem, D' contains a negative cycle. We need to show that if D' has a negative cycle then it has one of the form $l_y r_x l_v r_u l_y$ for some $\{x, y, u, v\}$. Observe that all arcs from an 'r-vertex' r_v to an 'l-vertex' l_w have weight $-\epsilon$ and all arcs from an 'l-vertex' l_v to an 'r-vertex' r_w have weight 0 (there are two types of such arcs depending on whether or not $v = w$). Thus cycles alternate between 'r-vertices' and 'l-vertices'. Hence any cycle in D' must contain at least one arc with weight $-\epsilon$ and have negative weight.

Consider a negative cycle $l_y r_x l_v r_u l_w \ldots$ with the fewest number of arcs in D'. If we can show that $y = w$ and x, y, u, v are distinct then there are no other vertices in the cycle and it corresponds to a $\mathbf{2}+\mathbf{2}$ in D with arcs xy and uv.

We have $y \neq x$ and $u \neq v$ since arcs of the form $l_y r_x$ are defined only if xy is an arc of D. We have $y \neq v$ since otherwise the arcs $l_y r_x$ and $r_x l_y$ would imply $xy \in A(D)$ and $xy \notin A(D)$. Similarly $x \neq u$.

If $y = u$ then we have arcs xy and yv in D and by transitivity in D we have the arc xv. But then $r_x l_v$ would not be an arc of D'. So $y \neq u$.

If $x = v$ then we have arcs xy and ux in D and by transitivity in D we have the arc uy. Hence $l_y r_u$ is an arc in D'. Replacing $l_y r_x l_v r_u$ with the arc $l_y r_u$ would yield a cycle in D' with fewer arcs, contradicting our choice of a cycle. So $x \neq v$.

Finally, we need to show that $y = w$. Consider y and u. In D exactly one of the the following three possibilities holds: (i) $uy \in A(D)$, (ii) $yu \in A(D)$, (iii) $uy \notin A(D)$ and $yu \notin A(D)$. If (i) then $l_y r_u \in A(D')$ and we could replace $l_y r_x l_v r_u$ with $l_y r_u$ to a obtain a shorter cycle, a contradiction. If (ii) then $xy, yu, uv \in A(D)$ and by transitivity we have $xv \in A(D)$ contradicting $xv \notin A(D)$ since we have arc $r_x l_v$. Thus (iii) must hold and in particular we have $r_u l_y \in A(D')$ and we could take $y = w$. By minimality this is the case. $\square$

12 Conclusion

The idea of combinatorial duality is that certain combinatorial results can be viewed as specific instances of the easily stated fact that either a system of linear inequalities has a solution or it is inconsistent. This perspective provides a powerful tool for combinatorial problems. We have illustrated these ideas with two easily stated combinatorial problems along with two digraph problems that can be used as intermediate tools.

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