Catalan Multijections

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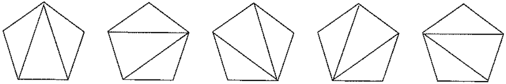
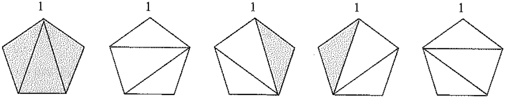
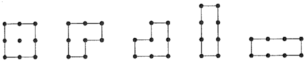
We now come to the core of this monograph, the 214 combinatorial interpretations of Catalan numbers. We illustrate each item with the case $n = 3$, hoping that these illustrations will make any undefined terminology clear. Some of this terminology also appears in the Glossary. Solutions, hints, and references are given in the next section. In some instances two different items will agree as sets, but the descriptions of the sets will be different. Readers seeking to become experts on Catalan numbers are invited to take each pair (i_1, i_2) of distinct items and find a bijection (valid for all n) from the sets counted by i_1 to the sets counted by i_2, so $214 \cdot 213 = 45582$ bijections in all!

1. Triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors.

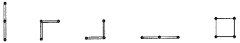


2. Total number of triangles with vertices $1, i, i+1$, $2 \leq i \leq n+1$, among all triangulations of a convex $(n+2)$-gon with vertices $1, 2, \ldots, n+2$ in clockwise order.

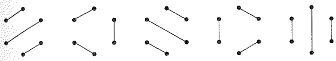




58. (Unordered) pairs of lattice paths with $n-1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, such that one path never rises above the other path.

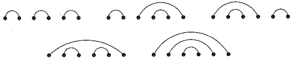


59. n nonintersecting chords joining $2n$ points on the circumference of a circle.



60. Joining some of the vertices of a convex $(n-1)$-gon by disjoint line segments, and circling a subset of the remaining vertices.

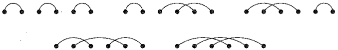
61. *Noncrossing (complete) matchings* on $2n$ vertices, i.e., ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points.



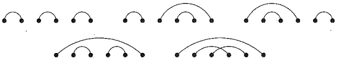
62. Ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (a) the arcs do not pass below L, (b) the graph thus formed is a tree, (c) no two arcs intersect in their interiors (i.e., the arcs are noncrossing), and (d) at every vertex, all the arcs exit in the same direction (left or right).

63. Ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (a) the arcs do not pass below L, (b) the graph thus formed is a tree, (c) no arc (including its endpoints) lies strictly below another arc, and (d) at every vertex, all the arcs exit in the same direction (left or right).

64. *Nonnesting matchings* on $[2n]$, i.e., ways of connecting $2n$ points in the plane lying on a horizontal line by n arcs, each arc connecting two of the points and lying above the points, such that no arc is contained entirely below another.



65. Ways of connecting $2n$ points in the plane lying on a horizontal line by n vertex-disjoint arcs, each arc connecting two of the points and lying above the points, such that the following condition holds: for every edge e let $n(e)$ be the number of edges e' that nest e (i.e., e lies below e'), and let $c(e)$ be the number of edges e' that begin to the left of e and that cross e. Then $n(e)-c(e)=0$ or 1.



66. Ways of connecting any number of points in the plane lying on a horizontal line by nonintersecting arcs lying above the points, such that the total number of arcs and isolated points is $n-1$ and no isolated point lies below an arc.

202. Maximal chains $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = [n]$ of subsets of $[n]$ such that $S_i - S_{i-1} = \{m\}$ if and only if m belongs to the rightmost maximal set of consecutive integers contained in S_i.

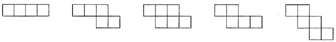
$$\emptyset \subset 1 \subset 12 \subset 123,\quad \emptyset \subset 2 \subset 12 \subset 123,\quad \emptyset \subset 1 \subset 13 \subset 123$$

$$\emptyset \subset 2 \subset 23 \subset 123,\quad \emptyset \subset 3 \subset 23 \subset 123$$

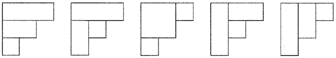
203. Ways to write $(1,1,\ldots,1,-n) \in \mathbb{Z}^{n+1}$ as a sum of vectors $e_i - e_{i+1}$ and $e_j - e_{n+1}$, without regard to order, where e_k is the kth unit coordinate vector in $\mathbb{Z}^{n+1}$.

$$(1,-1,0,0)+2(0,1,-1,0)+3(0,0,1,-1)$$
$$(1,0,0,-1)+(0,1,-1,0)+2(0,0,1,-1)$$
$$(1,-1,0,0)+(0,1,-1,0)+(0,1,0,-1)+2(0,0,1,-1)$$
$$(1,-1,0,0)+2(0,1,0,-1)+(0,0,1,-1)$$
$$(1,0,0,-1)+(0,1,0,-1)+(0,0,1,-1)$$

204. Horizontally convex polyominoes (as defined in [64, §4.7.5]) of width (number of columns) $n+1$ such that each row begins strictly to the right of the beginning of the previous row and ends strictly to the right of the end of the previous row.



205. Tilings of the staircase shape $(n, n-1, \ldots, 1)$ with n rectangles.



206. Complete matchings of the triangular benzenoid graph T_{n-1} of order $n-1$. The graph T_n is a planar graph whose bounded regions are hexagons, with i hexagons in row i (from the top) and n rows in all, as illustrated for $n = 4$ in Figure 2.4.

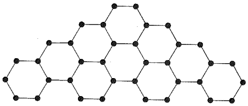
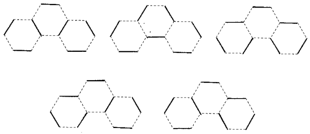


Figure 2.4. The triangular benzenoid graph T_4.



207. n-tuples $(a_1, \ldots a_n)$ of positive integers such that the tridiagonal matrix

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & a_3 & 1 & \cdots & 0 & 0 \\ & & & & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_n \end{bmatrix}$$

is positive definite with determinant one.

131 122 221 213 312

208. $n \times n$ $\mathbb{N}$-matrices $M = (m_{ij})$ where $m_{ij} = 0$ unless $i = n$ or $i = j$ or $i = j-1$, with row and column sum vector $(1, 2, \ldots, n)$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Ballot Lists: n 0 's; n 1's, # 0's $\geq$ # 1's in initial segments

$$111000, 110100, 110010, 101100, 101010$$

Parentheses: n (, n) , well formed pairs

$$((())), (()()), (())(), ()(()), ()()()$$

(n)-Multisets: $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n$ with $a_i \leq i-1$

$$000, 001, 002, 011, 012$$

Easy bijections between these as well as the larger collections:
balanced binary strings, balanced parentheses,
n element multisets from an $n+1$ set

Catalan Numbers

$$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\left(\!\!\binom{n+1}{n}\!\!\right)$$

Catalan counts we will refer to:

Ballot Lists: n 0 's; n 1's, # 1's $\geq$ # 0's in initial segments

Parentheses: n (, n) , well formed pairs

(n)-Multisets: $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n$ with $a_i \leq i-1$

Proof versions:

- Reflection: count bad lists and subtract
- Recursion and generating functions
- **Uniform multijections**

Good reasons for other proofs - But not today

Lots of generalizations - But not today

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

Multijective proof:

- $\binom{2n}{n}$ balanced strings of parenthesis: e.g)())()((
- of these C_n are well formed: e.g. ()(()())
- Find an $(n+1)$ to 1 map!

$n = 5$: Catalan sequence: ()((()()))

Find inverse image:

()((()())) ⇒ ()((()())) ⇒)(((()())) ⇒)(((()()))

()((()())) ⇒ ()((()())) ⇒ ())(()())(⇒ ())(()())(

()((()())) ⇒ ()((()())) ⇒ ()))()()((⇒ ()))()()((

()((()())) ⇒ ()((()())) ⇒ ())))(()((⇒ ())))(()((

()((()())) ⇒ ()((()())) ⇒ ()))())(((⇒ ()))())(((

the forward map: iteratively remove () until)))(((then flip these.

Multijective proof that well formed balanced parenthesis are counted by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

$n = 3$

- Partition $\binom{6}{3} = 20$ balanced strings
- Into $C_3 = \frac{1}{3+1}\binom{6}{3} = \frac{1}{4} \cdot 20 = 5$
- Parts of size $3 + 1 = 4$
- Each with exactly 1 Catalan object

((()))	(()())	(())()	()(())	()()()
)(())(	)()()(	)()(()	)((())	)(()()
))()((	))(()(	))((()	())()(	())(()
)))(((	)()))((	(()))(	()))((	()())(

By ~~Me~~ Rubenstein (1994)

Multijective proof that 012…n dominated n-multisets are counted by $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\left(\!\binom{n+1}{n}\!\right)$.

$n = 3$

- Partition $\left(\!\binom{4}{3}\!\right) = 20$ multisets from $\{0, 1, 2, 3\}$
- Into $C_3 = \frac{1}{3+1}\left(\!\binom{4}{3}\!\right) = \frac{1}{4} \cdot 20 = 5$
- Parts of size $3 + 1 = 4$
- Each with exactly 1 Catalan object

000	001	002	011	012
111	112	113	122	123
222	223	220	233	230
333	330	331	300	301

Add 0,1,2,3 mod 4

One version of the Chung-Feller-MacMahon proof

$n = 3$: balanced binary string sequence: 110010

Find inverse image: append a 1 and write down all cyclic shifts starting with 1

$$\begin{array}{ccccccccccccc}
1 & 1 & 1 & 0 & 0 & 1 & 0 & & & & & \\
 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & & & & \\
 & & 1 & 0 & 0 & 1 & 0 & 1 & 1 & & & \\
 & & & & & 1 & 0 & 1 & 1 & 1 & 0 & 0
\end{array}$$

Removing the lead 1 from these we get the inverse image of 110010 to be
$110010, 100101, 001011, 011100.$

Cycle Lemma of Dvortzky and Motzkin

Multijective proof that ballot lists are counted by $C_n = \frac{1}{n+1}\binom{2n}{n}$.

$n = 3$

- Partition $\binom{6}{3} = 20$ binary strings with 3 ones and 3 zeros
- Into $C_3 = \frac{1}{3+1}\left(\binom{4}{3}\right) = \frac{1}{4} \cdot 20 = 5$
- Parts of size $3 + 1 = 4$ (using cycle lemma)
- Each with exactly 1 Catalan object

111000	110100	110010	101100	101010
110001	101001	100101	011001	010101
100011	010011	001011	100110	010110
000111	001110	011100	001101	011010

Cycle Lemma of Dvoretzky and Motzkin

Rubenstein:

((()))	(()())	(())()	()(())	()()()
)(())(	)()()(	)()(()	)((())	)(()()
))()((	))(()(	))((()	())()(	())(()
)))(((	)())((	(()))(	()))((	()())(

Chung-Feller-MacMahon:

000	001	002	011	012
111	112	113	122	123
222	223	022	233	023
333	033	133	003	013

Cycle Lemma Devoretzky and Motzkin:

111000	110100	110010	101100	101010
110001	101001	100101	011001	010101
100011	010011	001011	100110	010110
000111	001110	011100	001101	011010

Are these the ‘same’ partition?

Rubenstein:

000	001	002	011	012
113	123	122	111	112
233	223	222	023	022
333	133	003	033	013

Chung-Feller-MacMahon:

000	001	002	011	012
111	112	113	122	123
222	223	022	233	023
333	033	133	003	013

Cycle Lemma Devoretzky and Motzkin:

000	001	002	011	012
003	013	023	113	123
033	133	233	022	122
333	222	111	223	112

Not the same?

Rubenstein:

111000	110100	110010	101100	101010
011001	010101	010110	011100	011010
001011	001101	001110	100101	100110
000111	010011	110001	100011	101001

Chung-Feller-MacMahon:

111000	110100	110010	101100	101010
011100	011010	011001	010110	010101
001110	001101	100110	001101	100101
000111	100011	010011	110001	101001

Cycle Lemma Devoretzky and Motzkin:

111000	110100	110010	101100	101010
110001	101001	100101	011001	010101
100011	010011	001011	100110	010110
000111	001110	011100	001101	011010

Same? 2nd and 3rd are complements!

$$\begin{array}{ccccc} & A & \overset{\sigma}{\longleftrightarrow} & B & \\ \alpha & \big\downarrow & & \big\downarrow & \beta \\ & \mathcal{A} & \underset{\pi}{\longleftrightarrow} & \mathcal{B} & \end{array}$$

General framework: α, β multijective proofs for Catalan numbers; σ, π bijections

> *α, β are* Translations *with respect to σ, π if the partitions of A and B map to each other.*

Informally:

- α, β are *translations* if for ‘the’ natural/obvious σ, π
- α, β are *different* if obvious π forces non-obvious σ

$$\begin{array}{ccccc} & & \sigma & & \\ \alpha & A & \longleftrightarrow & B & \beta \\ & \downarrow & & \downarrow & \\ & \mathcal{A} & \longleftrightarrow & \mathcal{B} & \\ & & \pi & & \end{array}$$

- Cycle Lemma and CFM are different under the ‘standard’ bijections
- But if σ is complementation followed by standard bijection then they are translations
- Rubenstein seems to be different from cycle lemma and CFM
- Maybe an interesting ‘new’ σ can be found.

$$
\begin{array}{ccccc}
 & A & \overset{\sigma}{\longleftrightarrow} & B & \\
\alpha & \downarrow & & \downarrow & \beta \\
 & \mathcal{A} & \underset{\pi}{\longleftrightarrow} & \mathcal{B} &
\end{array}
$$

Large number of new exercises/projects

- For given α and 'obvious' σ, π is β interesting/new?
- Find interesting translations of known multijections in new Catalan families
- Find new and 'different' multijections
- Determine if known multijections are 'different'

Translate Cycle Lemma to multiset version

122255 Good lists dominated by 001234

If smallest is 0 $\Rightarrow$ n & shift If not 0 $\Rightarrow$ -1 from all

122255
011144
111445
000334
003345
033455
334555
223444
112333
001122
012223

122255

$n = 5$, size 6 multisets from $\{0, 1, 2, 3, 4, 5\}$

rules partition $\left(\!\binom{6}{6}\!\right)$ multisets

into classes of size $2 \cdot 5 + 1 = 11$

with exactly one good multiset

partition $\left(\!\binom{n+1}{n+1}\!\right)$ into size $2n + 1$ classes

$$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\left(\!\binom{n+1}{n}\!\right) = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{2n+1}\left(\!\binom{n+1}{n+1}\!\right)$$

$$\begin{array}{ccccc} & A & \overset{\sigma}{\longleftrightarrow} & B & \\ \alpha & \Big\downarrow & & \Big\downarrow & \beta \\ & \mathcal{A} & \underset{\pi}{\longleftrightarrow} & \mathcal{B} & \end{array}$$

Large number of new exercises/projects

- For given α and 'obvious' σ, π is β interesting/new?
- Find interesting translations of known multijections in new Catalan families
- Find new and 'different' multijections
- Determine if known multijections are 'different'
- Look at partition 'statistics' for multijections
- Some Catalan families do not have an obvious $\binom{2n}{n}$ find multijections for these