Star-Critical Ramsey Numbers

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Abstract

The *graph Ramsey number* $R(G, H)$ is the smallest integer r such that every 2-coloring of the edges of K_r contains either a red copy of G or a blue copy of H. We find the largest star that can be removed from K_r such that the underlying graph is still forced to have a red G or a blue H. Thus, we introduce the *star-critical Ramsey number* $r_*(G, H)$ as the smallest integer k such that every 2-coloring of the edges of $K_r - K_{1,r-1-k}$ contains either a red copy of G or a blue copy of H. We find the star-critical Ramsey number for trees versus complete graphs, multiple copies of K_2 and K_3, and paths versus a 4-cycle. In addition to finding the star-critical Ramsey numbers, the critical graphs are classified for $R(T_n, K_m)$, $R(nK_2, mK_2)$ and $R(P_n, C_4)$.

Key words: graph Ramsey number, Critical graph

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1. Introduction

A 2-*coloring* of the edges of G is a labeling $c : E(G) \to [2]$, where [2] is the set of possible labels or colors, say red or blue. For the purposes of this paper, when a graph is referred to as having a specific subgraph it is assumed that the graph has a 2-coloring. The *graph Ramsey number* $R(G, H)$ is the smallest integer n such that every 2-coloring of the edges of K_n contains either a red copy of G or a blue copy of H. This definition also implies that there exists a *critical graph*, that is, a 2-coloring of the edges of K_{n-1} that does not contain a red copy of G or a blue copy of H. Therefore, every 2-coloring of the edges of K_n contains either a red G or a blue H and there exists a coloring of K_{n-1} with neither a red G nor a blue H. These facts propose a question.

For known Ramsey numbers, $R(G, H) = n$, and a 2-coloring of the graph $K_{n-1} + \{v\}$, if we add colored edges individually from v to vertices of K_{n-1},

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then at what point must the graph have a red G or a blue H? Alternatively, what is the largest star that can be removed from K_n so that the underlying graph is still forced to have either a red G or a blue H?

The vertex set and edge set of a graph G will be denoted as $V(G)$ and $E(G)$, respectively. The graph $G+H$ or $G+\{v\}$ is the *disjoint union* of G and a graph H or G and a vertex v. The graph $G-H$ is the subgraph of G resulting from the deletion of the edges of H where H is a subgraph of G. The deletion of a vertex is $G-\{v\}$. The graph $G \cup H$ is obtained by adding the edges of H to G where $V(H) \subseteq V(G)$. When discussing the graph $G \cup H$ in subsequent definitions and theorems, the vertices of H will be clearly stated as a subset of vertices of G. The graph $G \vee H$ is *join* of G and H obtained by adding the edges $\{xy : x \in V(G) \text{ and } y \in V(H)\}$ to $G \cup H$. For any graph G, the disjoint union of m copies of G will be denoted as mG.

Definition 1.1. The graph $K_{n-1} \sqcup K_{1,k}$ is the union of K_{n-1} and $K_{1,k}$ such that v is the vertex of $K_{1,k}$ with degree k and the k vertices adjacent to v are vertices of K_{n-1}.

Definition 1.2. The *star-critical Ramsey number* $r_*(G,H)$ is the smallest integer k such that every 2-coloring of the edges of $K_{n-1} \sqcup K_{1,k}$ contains either a red copy of G or a blue copy of H.

In the language of arrowing, the above definition can be restated as the following: The star-critical Ramsey number $r_*(G,H)$ is the smallest integer k such that $K_{n-1} \sqcup K_{1,k} \longrightarrow (G,H)$. Also, note that the graph $K_{n-1} \sqcup K_{1,k}$ is the graph $K_n - K_{1,n-1-k}$. We will not be using the arrow notation as it is cumbersome to write for our purposes. However, it is beneficial to use the "$\sqcup$" notation to emphasize the size of the star as the number of edges added to K_{n-1}.

In most cases throughout this paper, we will first classify the critical graphs of $R(G,H)=n$ and then use the classification to find the star-critical Ramsey number. We will refer to the critical graphs as having a *(G,H)-free coloring*, that is a 2-coloring of K_{n-1} that avoids a red G and a blue H. The description of the (G,H)-free colorings will be as follows. Let $G=(V,E)$ be a graph whose edges are colored red or blue such that E^ρ denotes the red edge set and E^β denotes the blue edge set. Then the graph $G=(V,E^\rho \cup E^\beta)$ has red subgraph $G^\rho = (V,E^\rho)$ and blue subgraph $G^\beta = (V,E^\beta)$.

When finding Ramsey numbers for 2-colored graphs, it is customary to deal with a complement of a graph and discuss edges and non-edges instead of red and blue edges. For example, in proving $R(T_m,K_n)=(m-1)(n-1)+1$ [3], we can consider subgraphs of $K_{(m-1)(n-1)+1}$ that either contain T_m, a tree on m vertices, or an independent set of size n. We cannot adapt this terminology since the resulting graph forced to have a red G or a blue H contains a star that is not necessarily adjacent to every vertex.

Aside from the star-critical Ramsey number, there are other Ramsey numbers concerning edges and subgraphs. The size Ramsey number $\hat{r}(G,H)$, defined by Erdös, Faudree, Rousseau and Schelp in [7], is the smallest integer m such that there exists a graph with m edges and every 2-coloring of this graph must contain a red G or a blue H. Later in [6], Erdös and Faudree also defined size Ramsey functions: Let $r = R(G,H)$. The lower size Ramsey number $l(G,H)$ is the smallest integer l such that there exists a graph L with l edges that is a subgraph of K_r and every 2-coloring of L must contain a red G or a blue H. The upper size Ramsey number $u(G,H)$ is the smallest integer such that if a subgraph of K_r has at least $u(G,H)$ edges, then it must contain a red G or a blue H. Thus, every 2-coloring of every subgraph of K_r with m edges must contain a red G or a blue H for $m \geq u(G,H)$ and there exists a 2-coloring of every subgraph of K_r with m edges that does not contain a red G or a blue H for $m < l(G,H)$. In comparison to the star-critical Ramsey number, every 2-coloring of $K_{r-1} \sqcup K_{1,r_*(G,H)}$ must contain a red G or a blue H and there is a 2-coloring of $K_{r-1} \sqcup K_{1,r_*(G,H)-1}$ without a red G or a blue H. Therefore, $l(G,H) \leq \binom{r-1}{2} + r_*(G,H) \leq u(G,H)$.

In the following sections, we will prove

$$r_*(T_n, K_m) = (n-1)(m-2)+1, \text{ for any tree on } n \text{ vertices} \quad (1)$$
$$r_*(nK_2, mK_2) = m, \text{ for } n \geq m \geq 1 \quad (2)$$
$$r_*(nK_3, mK_3) = 3n+2m-1, \text{ for } n \geq m \geq 1 \text{ and } n \geq 2 \quad (3)$$
$$r_*(P_n, C_4) = 3, \text{ for } n \geq 3. \quad (4)$$

and we will characterize all critical graphs for the Ramsey numbers corresponding to the above star-critical Ramsey numbers except for $R(nK_3, mK_3)$. For the critical graphs of $R(T_n, K_m)$ and $R(nK_2, mK_2)$, we show that the graph that established the lower bound for the Ramsey number is in fact unique. We present a class of critical graphs for $R(P_n, C_4)$ that consists of all (P_n, C_4)-free colorings of $K_{R(P_n,C_4)-1}$.

Before proceeding into the proofs, it is worth noting two facts. First, the star-critical Ramsey number varies greatly. For example, $r_*(mK_3, mK_3) = R(mK_3, mK_3) - 1$ which requires all edges between v and $V(K_{R(mK_3,mK_3)-1})$ to be added in the graph $K_{R(mK_3,mK_3)-1} + \{v\}$ to force m monochromatic triangles. In contrast, $r_*(P_n, C_4) = 3$ requiring just three edges between v and $V(K_{R(P_n,C_4)-1})$ in the graph $K_{R(P_n,C_4)-1} + \{v\}$ to force either a red P_n or a blue C_4. Secondly, the only case where the Ramsey number is unknown and the star-critical Ramsey number is known is the Ramsey number of two complete graphs, $R(K_n, K_m)$. If $R(K_n, K_m) = s$, then $r_*(K_n, K_m) = s - 1$ requiring all edges between v and $V(K_{s-1})$ to be present in the graph $K_{s-1} + \{v\}$. Let w be any vertex of a (K_n, K_m)-free coloring of K_{s-1}. If we add a vertex v with the same red and blue adjacencies as w, then the new graph is a (K_n, K_m)-free coloring of $K_s - \{vw\}$. This idea was first observed by Chvátal as noted in [7].

For the figures in this paper, red edges are solid and blue edges are dashed. A red clique of size n is denoted K_n^ρ and a blue clique of size n is denoted K_n^β. If all the edges between two cliques are red (or blue), a thick solid (or dashed) line is drawn between the cliques.

2. Trees versus Complete Graphs

Theorem 2.1. *For any tree on n vertices, $R(T_n, K_m) = (n-1)(m-1)+1$. [3]*

In order to determine the star-critical Ramsey number $r_*(T_n, K_m)$, first we will show the critical graph that established the lower bound of the Ramsey number is the unique (T_n, K_m)-free coloring of $K_{R(T_n,K_m)-1}$ using the following well-known proposition for trees.

Proposition 2.2. *If T is a tree with k edges and G is a simple graph with $\delta(G) \geq k$, then T is a subgraph of G. [10]*

Definition 2.3. For given n and m with $n, m \geq 2$, let $r = R(T_n, K_m) = (n-1)(m-1)+1$. Define the graph G_1 to be the complete graph K_{r-1} with a 2-coloring of the edges such that

$G_1^\rho = (m-1)K_{n-1}$, and

G_1^β is the complete $(m-1)$-partite graph with partite sets of size $n-1$.

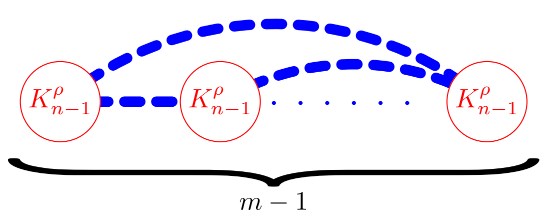

Figure 1: G_1, Critical graph for $R(T_n, K_m)$.

The graph G_1 in Definition 2.3 is a (T_n, K_m)-free coloring of K_{r-1}. There does not exist a blue K_m since the blue subgraph is an $(m-1)$-partite graph. In the red subgraph, each component has size $n-1$ and so there is no red tree on n vertices.

Proposition 2.4. *For given n and m with $n, m \geq 2$, let $r = R(T_n, K_m) = (n-1)(m-1)+1$. If c is a (T_n, K_m)-free coloring of K_{r-1}, then the resulting graph must be the graph G_1 described in Definition 2.3.*

Proof. We will proceed by using induction on m with base case $m = 2$. Since $R(T_n, K_2) = n$, the critical graph has $n-1$ vertices. If there are no blue edges, then the graph must be a red $(n-1)$-clique which does not contain a tree on n vertices. Therefore, the unique (T_n, K_2)-free coloring of K_{n-1} is the graph G_1

in Definition 2.3 with $m = 2$.

Let c be a (T_n, K_m)-free coloring of K_{r-1}. If the blue degree of each vertex is at most $(n-1)(m-2)-1$, then the red degree of each vertex is at least $[(n-1)(m-1)-1]-[(n-1)(m-2)-1] = n-1$. By Proposition 2.2, there is a red tree with $n-1$ edges. Hence, there is a red tree on n vertices.

Thus, there is a vertex v with blue degree $(n-1)(m-2)$. Let H be the subgraph induced by the blue neighbors of v. Then, by induction, H has either a red T_n, a blue K_{m-1}, or the structure of the graph in Definition 2.3. Clearly, H does not have a red T_n. If H has a blue K_{m-1}, then this blue K_{m-1} and v form a blue K_m. Hence, H must be the graph such that $H^\rho = (m-2)K_{n-1}$ and H^β is an $(m-2)$-partite graph with partite sets of size $n-1$. Since every vertex h in H belongs to a red K_{n-1} containing a red T_{n-1}, a red edge vh creates a red T_n with v as a leaf. This implies that v has a blue edge to every vertex of H. Similarly, every vertex of $K_{r-1} - H$ has a blue edge to every vertex of H. If there is a blue edge vw for $v, w \in V(K_{r-1} - H)$, then the graph has a blue K_m, namely v, w and a vertex from each of the $m-2$ partite sets, since both v and w have blue edges to all the vertices of $K_{r-1} - H$. Therefore, among the $n-1$ vertices of $K_{r-1} - H$, there cannot be any more blue edges and the vertices outside of H are a red K_{n-1}. Thus, the graph is G_1 in Definition 2.3. ∎

To find the star-critical Ramsey number $r_*(T_n, K_m)$, consider the (T_n, K_m)-free coloring of K_{r-1} described above in Definition 2.3 and a vertex v. If v is adjacent to $(n-1)(m-2)+1$ vertices of K_{r-1}, then there does not exist a (T_n, K_m)-free coloring of $K_{r-1} \sqcup K_{1,(n-1)(m-2)+1}$.

Theorem 2.5. *For any tree on n vertices, $r_*(T_n, K_m) = (n-1)(m-2)+1$.*

Proof. Let $r = (n-1)(m-1)+1 = R(T_n, K_m)$. For the lower bound, a (T_n, K_m)-free coloring of $K_{r-1} \sqcup K_{1,(n-1)(m-2)}$ is the unique coloring of K_{r-1} as in Proposition 2.4 and a vertex v with all blue edges adjacent to every vertex in $(m-2)$ copies of K_{n-1}. Hence, $r_*(T_n, K_m) \geq (n-1)(m-2)+1$.

For the upper bound, consider the graph K_{r-1} and a vertex v. By Proposition 2.4, a (T_n, K_m)-free coloring of K_{r-1} must have the structure of the critical graph. A red edge adjacent to v and a vertex of K_{r-1} produces a red T_n since any vertex in K_{r-1} belongs to a red clique of size $n-1$. Thus, all edges adjacent to v must be blue and $(n-1)(m-2)+1$ edges adjacent to v forces v to have a blue edge to each copy of K_{n-1}. If v has a blue edge to each of the $(m-1)$ copies of K_{n-1}, then v along with an edge to each K_{n-1} yields a blue K_m. ∎

3. Multiple copies of graphs

3.1. Multiple copies of K_2

Theorem 3.1. *For $n \geq m \geq 1$, $R(nK_2, mK_2) = 2n+m-1$. [4, 5, 9]*

In determining the star-critical Ramsey number $r_*(nK_2, mK_2)$, we will show that if $n > m$, then the critical graph of the Ramsey number is unique, and if $n = m$, then there are two (mK_2, mK_2)-free colorings of $K_{R(mK_2,mK_2)-1}$.

We include the following easy lemma that will be referenced in both the proof of the critical graphs and the proof of the star-critical Ramsey number.

Lemma 3.2. *The graphs $K_{2m-1} \sqcup K_{1,1}$ and $K_{2m-1} + K_2$ both contain mK_2.*

Proof. The graph K_{2m-1} has $\lfloor \frac{2m-1}{2} \rfloor = m - 1$ disjoint edges. If a vertex v is adjacent to any vertex of K_{2m-1}, then there are m disjoint edges in the graph. Also, if there is an edge disjoint from K_{2m-1}, there are m disjoint edges in the graph. ■

Definition 3.3. For given n and m with $n \geq m \geq 1$, let $r = R(nK_2, mK_2) = 2n + m - 1$ and c be a 2-coloring of K_{r-1}. The graphs G_1 and G_2 defined below are colorings of K_{r-1}.

$$\text{If } n \geq m, \quad G_1 : \quad \begin{aligned} G_1^\rho &= K_{2n-1} + (m-1)K_1 \\ G_1^\beta &= K_{m-1} \vee (2n-1)K_1 \end{aligned}$$

$$\text{If } n = m, \quad G_2 : \quad \begin{aligned} G_2^\rho &= K_{m-1} \vee (2m-1)K_1 \\ G_2^\beta &= K_{2m-1} + (m-1)K_1 \end{aligned}$$

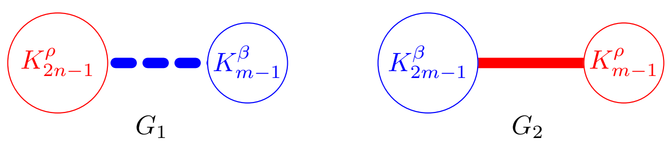

Figure 2: Critical graphs of $R(nK_2, mK_2)$.

The graphs in Definition 3.3 are (nK_2, mK_2)-free colorings of K_{r-1}. The graph G_1 does not contain n disjoint red edges since G_1^ρ has at most $\lfloor \frac{2n-1}{2} \rfloor$ disjoint red edges. Also, G_1 contains at most $m-1$ disjoint blue edges with one endpoint of each edge in the red clique. Similarly, the graph G_2 does not contain m disjoint monochromatic edges. Note that when $n = m$ switching the red and blue edges gives two (mK_2, mK_2)-free colorings of K_{r-1}. This is not the case when $n > m$. If we switch the colors of the graph G_1, then the blue subgraph is K_{2n-1} which has m disjoint blue edges.

Proposition 3.4. *For given n and m with $n \geq m \geq 1$, let $r = R(nK_2, mK_2) = 2n + m - 1$. If c is an (nK_2, mK_2)-free coloring of K_{r-1}, then if $n > m$, the resulting graph must be the graph G_1 described in Definition 3.3, and if $n = m$, the resulting graph must be either the graph G_1 or the graph G_2 described in Definition 3.3.*

Proof. For given n and m, let $r = R(nK_2, mK_2) = 2n + m - 1$ and c be an (nK_2, mK_2)-free coloring of K_{r-1}. We will proceed by induction on $n+m$. For the base case with $m = 1$ and any $n \geq 1$, $R(nK_2, K_2) = 2n$ and the critical graph on $2n-1$ vertices cannot have any blue edges. Hence, the graph is a red K_{2n-1} except in the case of $n = m = 1$ the graph is single vertex which can be thought of as either a red 1-clique or a blue 1-clique.

The graph K_{r-1} has a vertex v with both a red edge and a blue edge adjacent to vertices a and b, respectively. Note that if such a vertex v did not exist then the either every vertex has all red edges or all blue edges and the graph would either be entirely a red or blue clique which is not an (nK_2, mK_2)-free coloring for $m \geq 2$. Now, let H be the induced subgraph of $K_{r-1} - \{v, a, b\}$. Since $R((n-1)K_2, (m-1)K_2) = 2n + m - 4$ and H consists of $2n + m - 5$ vertices, the graph H must be one of the two following graphs by induction.

$$H_1 : H_1^\rho = K_{2n-3} + (m-2)K_1, H_1^\beta = K_{m-2} \vee (2n-3)K_1, \text{if } n \geq m.$$
$$H_2 : H_2^\rho = K_{m-2} \vee (2m-3)K_1, H_2^\beta = K_{2m-3} + (m-2)K_1, \text{if } n = m.$$

If $n > m$, then H is the graph H_1. Let A be the red K_{2n-3} and B be the blue K_{m-2}. The graph H_1 has $n-2$ disjoint red edges contained in A and $m-2$ disjoint blue edges where each edge has an endpoint in A and B. Note that the entire graph has $n-1$ disjoint red edges and $m-1$ disjoint blue edges since the edges va and vb are disjoint from H_1. This implies that a cannot have any blue edges to A and b cannot have any red edges to B. Therefore, $A \cup \{a\}$ is a red K_{2n-2} and $B \cup \{b\}$ is a blue K_{m-1}. Since there are $n-1$ disjoint red edges in K_{2n-2}, v must have all blue edges to B. If ab and va_i are both blue for any a_i in $V(A)$, then these blue edges along with the $m-2$ disjoint blue edges in H create a blue mK_2. Similarly, if ab and va_i are both red for any a_i in $V(A)$, then these red edges along with the $n-2$ disjoint red edges in H create a red nK_2. Thus, ab and va_i must have distinct colors. If ab is red and va_i is blue, then b must have all red edges to A and the graph contains a red nK_2. Hence, ab is blue, v has all red edges to A and $A \cup \{v, a\}$ is a red K_{2n-1}. By Lemma 3.2, there cannot be any more red edges and the resulting graph is G_1 in Definition 3.3.

If $n = m$, there is symmetry and H could be either H_1 or H_2. If the graph is H_1, then the same proof as above holds and the resulting graph is G_1 in Definition 3.3. Note that H_2 is the graph H_1 with the colors reversed. If the graph is H_2, then the same proof as above follows with red and blue interchanged and the resulting graph is G_2 in Definition 3.3. ■

Theorem 3.5. *For $n \geq m \geq 1$, $r_*(nK_2, mK_2) = m$.*

Proof. An (nK_2, mK_2)-free coloring of $K_{2n+m-2} \sqcup K_{1,m-1}$ is the graph G_1 as in Definition 3.3 and a vertex v with all blue edges to the blue K_{m-1} in the coloring of K_{2n+m-2}. Hence, $r_*(nK_2, mK_2) \geq m$.

Let $n > m$ and consider the graph K_{2n+m-2} and a vertex v. By Proposition 3.4, an (nK_2, mK_2)-free coloring of K_{2n+m-2} must have the structure of the graph G_1 in Definition 3.3, i.e. a red subgraph of K_{2n-1}. If the red subgraph is K_{2n-1}, then v cannot have any adjacent red edges by Lemma 3.2. Thus, all edges adjacent to v must be blue and m edges adjacent to v forces v to have a blue edge to the red K_{2n-1}. If v has a blue edge to the red K_{2n-1}, then there are m disjoint blue edges.

Let $n = m$ and consider the graph K_{3m-2} and a vertex v. By Proposition 3.4, an (mK_2, mK_2)-free coloring of K_{3m-2} must have the structure of the critical graphs G_1 or G_2, i.e. a blue subgraph of $K_{m-1} \vee (2m-1)K_1$ or K_{2m-1}. If the blue subgraph is $K_{m-1} \vee (2m-1)K_1$, then the red subgraph is K_{2m-1} and the same proof as above holds. If the blue subgraph is K_{2m-1}, v cannot have any adjacent blue edges by Lemma 3.2. Thus, all edges adjacent to v must be red and m edges adjacent to v forces v to have a red edge to the blue K_{2m-1} yielding m disjoint red edges. ■

3.2. Multiple copies of K_3

Theorem 3.6. *For $n \geq m \geq 1$ and $n \geq 2$, $R(nK_3, mK_3) = 3n + 2m$. [2]*

The graph G_1 defined below is similar to the critical graph for $R(mK_3, mK_3)$ constructed by Burr, Erdös, and Spencer in [2].

Definition 3.7. For given n and m with $n \geq m \geq 1$ and $n \geq 2$, let $r = R(nK_3, mK_3) = 3n + 2m$. Define the graph G_1 to be the complete graph K_{r-1} with a 2-coloring of the edges such that

$$G_1^{\rho} = K_{3n-1} + K_{1,2m-1}, \text{ and}$$
$$G_1^{\beta} = (3n-1)K_1 \vee (K_{2m-1} \cup K_1).$$

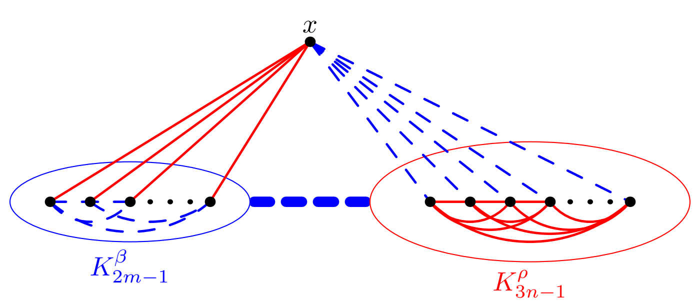

Figure 3: G_1, Critical graph for $R(nK_3, mK_3)$.

The graph G_1 in Definition 3.7 does not have n disjoint red triangles since every red triangle would be contained in the red clique K_{3n-1} and there are not enough vertices. Similarly, the coloring does not contain m disjoint blue triangles since every blue triangle would need to use at least 2 vertices of the blue clique K_{2m-1} and there are not enough vertices.

For the star-critical Ramsey number $r_*(nK_3, mK_3)$, we will construct a critical graph on K_{3n+2m} minus an edge using the above construction and Chvátal's observation, mentioned in the introduction, for the Ramsey number for a pair of complete graphs. This will imply that the star must be adjacent to every vertex of $K_{3n+2m-1}$. Hence, there does not exist an intermediate graph between $K_{3n+2m-1}$ and K_{3n+2m} with a 2-coloring that has n disjoint red triangles or m disjoint blue triangles.

Theorem 3.8. *For $n \geq m \geq 1$ and $n \geq 2$, $r_*(nK_3, mK_3) = 3n + 2m - 1$.*

Proof. Consider the graph G_1 in Definition 3.7. It consists of two cliques, K_{2m-1} and K_{3n-1}, and a vertex in G_1^ρ of degree $2m-1$, say x. Add a vertex x' with the same red and blue adjacencies as the vertex x. No new red or blue triangles have been formed and the only non-edge is xx'. ■

Note that the classification of critical graphs of $R(nK_3, mK_3)$ is not needed in the proof of $r_*(nK_3, mK_3)$. In fact, the critical graph in Definition 3.7 is not unique. Let G be the disjoint union of a blue clique K_{2m-1}, a red clique K_{3m-1}, and a vertex x with red edges to the blue clique and blue edges to the red clique. If the edges between the cliques are all blue, then this graph is the critical graph in Definition 3.7. Another critical graph can be created by allowing both red and blue edges between the cliques. However, the red edges between the cliques must be restricted in the following matter. The vertices of the blue K_{2m-1} can have red degree at most two; each vertex has a red edge to x and possibly another red edge to a vertex in the red K_{3n-1}. If v is a vertex of the blue K_{2m-1} of red degree three, then v has two red edges to distinct vertices of a red clique forming a red triangle. We conjecture that this class of graphs consists of all (nK_3, mK_3)-free colorings of $K_{3n+2m-1}$.

4. Paths versus cycles

A path P_n and cycle C_n on n vertices will be denoted as $p_1p_2\ldots p_n$ and $(c_1c_2\ldots c_n)$, respectively.

Theorem 4.1. *For all $n \geq 3$, $R(P_n, C_4) = n + 1$. [1]*

In order to determine the star-critical Ramsey number $r_*(P_n, C_4)$, we will classify the critical graphs of the Ramsey number.

Definition 4.2. For a given $n \geq 3$, let $r = R(P_n, C_4) = n + 1$. Define the class of graphs $\mathcal{G}$ to consist of the graphs G_i for $i = 0, \ldots, \lfloor \frac{n-1}{2} \rfloor$ such that

$$G_i^\rho = (K_{n-1} - iK_2) + K_1$$
$$G_i^\beta = K_{1,n-1} \cup iK_2.$$

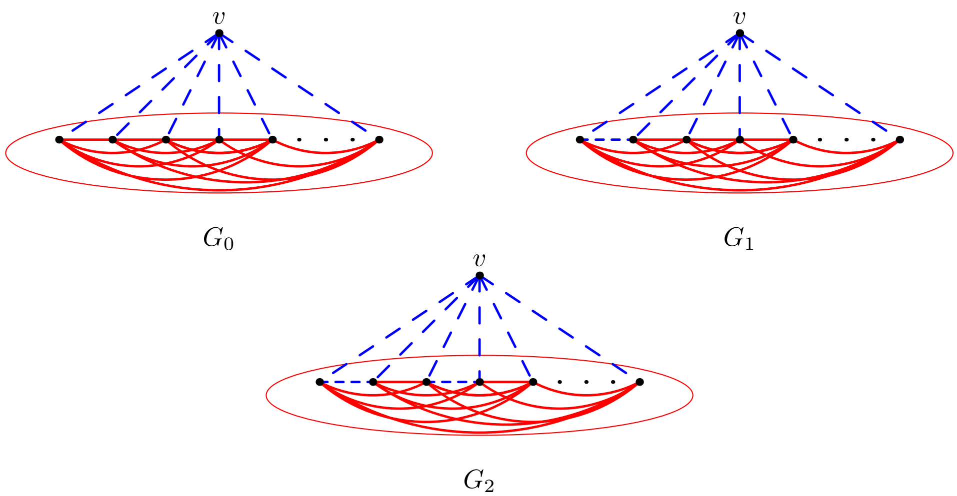

Figure 4: Critical graphs of $R(P_n, C_4)$ with $i = 0, 1, 2$.

The graphs G_i do not contain a red path on n vertices since the red subgraph is at most K_{n-1}. Clearly, the blue subgraph of G_0 does not contain a blue C_4 since it is a star. For $i = 1, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor$, the blue subgraph of G_i has circumference 3 since it is a star with the central vertex joined to i disjoint K_2 and hence is C_4-free.

Proposition 4.3. *For a given $n \geq 3$, let $r = R(P_n, C_4) = n + 1$. If c is a (P_n, C_4)-free coloring of K_{r-1}, then the resulting graph must belong to the class of graphs $\mathcal{G}$ in Definition 4.2.*

Proof. Let c be a (P_n, C_4)-free coloring of K_{r-1}. Note that for $n = 3$, the only (P_3, C_4)-free colorings of K_3 are G_0 and G_1. We will now proceed for $n \geq 4$. By Theorem 4.1, $R(P_{n-1}, C_4) = n$. Since $r - 1 = n$, the graph must have a red P_{n-1}, namely $p_1 p_2 \ldots p_{n-1}$. Let v be the vertex not on the red P_{n-1}. Clearly, v must have blue edges to the endpoints of the red path. Suppose v has a red edge to p_k for some $k \in \{2, 3, \ldots, n-2\}$. Note that v cannot have red edges to adjacent vertices on the path and so vp_{k-1} and vp_{k+1} must be blue. Also, $p_1 p_{n-1}$ must be blue or else $vp_k \ldots p_1 p_{n-1} \ldots p_{k+1}$ is a red P_n. (See Figure 5.) If $p_{k-1} p_{n-1}$ is blue, then $(vp_1 p_{n-1} p_{k-1})$ is a blue C_4. If $p_{k-1} p_{n-1}$ is red, then $p_1 \ldots p_{k-1} p_{n-1} \ldots p_k v$ is a red P_n. Note that the previous two statements hold for $k = 2$ by symmetry with $k = n - 2$. Therefore, v must have all blue edges to the red P_{n-1}.

Suppose that p_k for some k has blue degree three, that is, p_k has a blue edge to v and two other vertices on the red P_{n-1}, say p_j and p_l. But, both vp_j and vp_l are blue and so $(vp_j p_k p_l)$ is a blue C_4. Therefore, each p_k has blue degree at most two and can have at most one blue edge to another vertex on the the path. The vertices p_k are either a red clique or a red clique minus a matching.

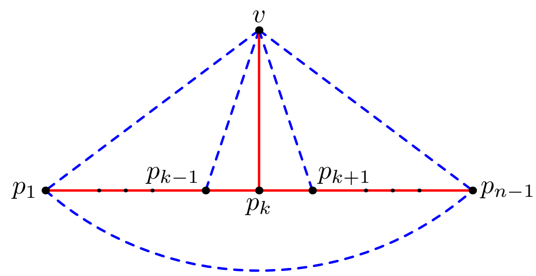

Figure 5: A red edge from v to the path forces either a red P_n or a blue C_4.

Hence, the red subgraph is $K_{n-1} - iK_2$ for some $i = 0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$. ■

Theorem 4.4. *For all* $n \geq 3$, $r_*(P_n, C_4) = 3$.

Proof. A (P_n, C_4)-free coloring of $K_n \sqcup K_{1,2}$ is the graph G_i as in Definition 4.2 and a vertex x with a red edge to v and a blue edge to any vertex in $G - v$. Hence, $r_*(P_n, C_4) \geq 3$.

Consider the graph K_n and a vertex x. By Proposition 4.3, a (P_n, C_4)-free coloring of K_n must have the structure of a graph G_i in Definition 4.2. A red edge from x to a vertex in $G - v$ yields a red P_n. Two blue edges from x to vertices in $G - v$ yield a blue C_4. Therefore, three edges from x force either a red P_n or a blue C_4. ■

The star-critical Ramsey numbers in this paper are a portion of the first author's dissertation under the advisement of the second author [8]. Further results have been obtained including cycles C_n versus complete graphs of size three and four, wheels versus K_3 and paths P_n versus P_m. Within these results, we classify the critical graphs of $R(C_n, K_4)$ and $R(P_n, P_m)$.

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