Distinguishing colorings of Cartesian products of complete graphs

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Abstract

We determine the values of s and t for which there is a coloring of the edges of the complete bipartite graph $K_{s,t}$ which admits only the identity automorphism. In particular this allows us to determine the distinguishing number of the Cartesian product of complete graphs.

The distinguishing number of a graph is the minimum number of colors needed to label the vertices so that the only color preserving automorphism is the identity. The distinguishing number was introduced by Albertson and Collins in [2] and a number of papers on this topic have been written recently. In this paper we determine values of c, s, t for which the Cartesian product of complete graphs of sizes s and t have an identity c coloring. In particular this allows us to determine the distinguishing number of the Cartesian product of complete graphs. For $s \leq t$, the distinguishing number of the Cartesian product of complete graphs on s and t vertices, $D(K_s \square K_t)$ is either $\lceil (t+1)^{1/s} \rceil$ or $\lceil (t+1)^{1/s} \rceil + 1$ and it is the smaller value for large enough t. In almost all cases it can be determined directly which value holds. In a few remaining cases the value can be determined by a simple recursion.

Our original version of this paper [3] was motivated by a problem of Harary and titled 'Edge colored complete bipartite graphs with trivial automorphism groups'. We later discovered the connection to the distinguishing number. The current version has a final added section making the connection to distinguishing numbers. Thus the rest of paper, except the final section where we make the connection to distinguishing numbers is the original version written in terms of identity edge colorings of complete bipartite graphs.

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Harary and Jacobson [7] examined the minimum number of edges that need to be oriented so that the resulting mixed graph has the trivial automorphism group and determined some values of s and t for which this number exists for the complete bipartite graph $K_{s,t}$. These are values for which there is a mixed graph resulting from orienting some of the edges with only the trivial automorphism. Such an orientation is called an identity orientation. Harary and Ranjan [8] determined further bounds on when $K_{s,t}$ has an identity orientation. They showed that $K_{s,t}$ does not have an identity orientation for $t \leq \lfloor \log_3(s-1) \rfloor$ or $t \geq 3^s - \lfloor \log_3(s-1) \rfloor$ and that it does have an identity orientation for $\lceil \sqrt{2s} - 3/2 \rceil \leq t \leq 3^s - \lceil \sqrt{2s} - 3/2 \rceil$. In addition they determined exact values when $2 \leq s \leq 17$. We will show that the first bound is nearly correct.

Observe that a partial orientation of a complete bipartite graph with parts X and Y has three types of edges: unoriented, oriented from X to Y, and oriented from Y to X. We can more generally think of coloring the edges with some number c of colors. The case $c = 1$ is trivial so we will assume throughout that $c \geq 2$. Automorphisms map vertices to other vertices in the same part except possibly when $s = t$. So the partial orientation case corresponds to the case $c = 3$ except possibly when $s = t$. An identity orientation exists whenever $s = t$ and we will observe that an identity coloring also exists when $s = t$ except when $s = t = 1$. Thus except for $s = t = 1$ our results using $c = 3$ will correspond to results for the identity orientations in [8]. A (color preserving) automorphism is a bijection from the vertex set to itself with the color of the edge between two vertices the same as the color of the edge between their images. An *identity coloring* is a coloring with only the trivial automorphism.

Our main result is

Theorem 1 *Let $c \geq 2$ and $s \geq 1$ be given integers. When $s \geq 2$ let $x = \lfloor \log_c(s-1) \rfloor$. Then $K_{s,t}$ has an identity c-edge coloring if and only if exactly one of the following holds:*
(i) $s = 1$ and $2 \leq t \leq c$
(ii) $2 \leq s \leq c$ and $1 \leq t \leq c^s - 1$ except for $c = s = t = 2$
(iii) $s > c$ and $s \leq c^{1+x} - \lfloor \log_c x \rfloor - 2$ and $x + 1 \leq t \leq c^s - x - 1$
(iv) $s > c$ and $s \geq c^{1+x} - \lfloor \log_c x \rfloor$ and $x + 2 \leq t \leq c^s - x - 2$
(v) $s > c$ and $s = c^{1+x} - \lfloor \log_c x \rfloor - 1$ and $x+2 \leq t \leq c^s - x - 2$ or $t = x+1, c^s - x - 1$ and $K_{x+1,s}$ has an identity c-edge coloring, except for the case $c = 2$ and $s = t = 3$.

Observe that we can determine if there is an identity coloring directly from s and t except in case (v) when $s = c^{1+x} - \lfloor \log_c x \rfloor - 1$ and $t = x + 1$ or $t = c^s - x - 1$. In this situation we let $s' = x + 1$ and $t' = s$ and check the conditions for s' and t'. This needs to be repeated at most $\log_c^*(s-1)$ times where $\log_c^*(s-1)$ is the iterated logarithm base c.

We are considering the problem of identity orientations examined by Harary and Ranjan [8] in a more general setting and will use a different notation, however, many of our results and proofs are direct extensions of those in [8].

A coloring with c colors of the edges of a complete bipartite graph $K_{s,t}$ having parts X of size s and Y of size t corresponds to a t by s matrix with entries from $\{0, 1, \ldots, c-1\}$. The i, j entry of the matrix is k whenever the edge between the i^{th} vertex in Y and the j^{th} vertex in X has color k. We will call this the bipartite adjacency matrix (the usual case being that of general bipartite graph, which can be thought of as a two coloring, edges and non-edges, of a complete bipartite graph). For edge colored complete bipartite graphs, the parts X and Y map to themselves if $|X| \neq |Y|$. In this case, if A is the bipartite adjacency matrix, then an automorphism corresponds to selecting permutation matrices P_Y and P_X such that $A = P_Y A P_X$. If $|X| = |Y|$ then we also have automorphisms of the form $A = P_Y A^T P_X$. These will play a role in our results only for certain small cases. We will discuss these exceptions and then be able to consider only identity colorings of the form where the only solution to $A = P_Y A P_X$ has both P_Y and P_X identity matrices. Throughout we will assume that permutation matrices are of the correct size for multiplication without stating the size explicitly. We will use the term identity coloring to refer to both the edge colored $K_{s,t}$ and to the corresponding adjacency matrix A.

Results

We will prove our main theorem by proving a series of lemmas which will be stated in terms of the matrix perspective described above.

For a given c and s we will call any c^s by s matrix with rows corresponding to the c^s distinct c-ary s-tuples *full*. Any two full matrices of the same size differ only by a permutation of their rows. If a c-ary t by s matrix A has distinct rows, then its complement A^* is 'the' $c^s - t$ by s matrix with rows consisting of the c-ary strings of length s that are not rows of A. (The ordering of the rows of A^* will not matter for our purposes.)

For any matrix with entries from $\{0, 1, \ldots, c-1\}$ the *degree* of a column is a c-tuple $(x_0, x_1, \ldots, x_{c-1})$ with x_i equal to the number of entries that are i in the column. Note that $\sum_{i=0}^{c-1} x_i$ equals the number of rows of A. Thinking of the bipartite adjacency matrix of as corresponding to a two edge coloring of a complete bipartite graph then the degree of vertices in X would be x_1 in the degree of A. Note that the degree of a vertex and its image in an automorphism are the same.

The following basic facts, which can easily be checked, will be used.

Fact 2 *Let A be the adjacency matrix of a c-edge colored complete bipartite graph then:*
(i) If A is full then so are P_1A and AP_2 for permutation matrices P_1, P_2 of appropriate sizes.
(ii) If there are two identical rows in A then A is not an identity coloring.
(iii) If A is not square and if the columns of A have distinct degrees and the rows are distinct then A is an identity coloring. If A is square, has distinct rows, distinct column degrees and the multiset of column degrees is different from the multiset of row degrees then A is an identity coloring.
(iv) A is an identity coloring if and only if A^T is.

We first consider the cases when $|X| = |Y|$.

Lemma 3 *Let s be a non-negative integer. $K_{s,s}$ has an identity c-edge coloring except when $s = 1$ (for any c) and when $c = 2$ and $s = 2$ or $s = 3$.*

Proof: When $s = 1$ the graph consists of a single colored edge. Switching the vertices is a non-trivial automorphism.

It is straightforward to check that every 2 coloring of $K_{2,2}$ has a non-trivial automorphism. (Note that when there is exactly one edge of one of the colors, a non-trivial automorphism must map X to Y and vice-versa.) It is straightforward, but tedious to check that every 2 coloring of $K_{3,3}$ has a non-trivial automorphism.

For $c \geq 3$ the adjacency matrices $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity colorings by Fact 2 (iii).

For $c \geq 2$ and $s \geq 4$ start with a matrix with entries 1 above the main diagonal and 0 elsewhere and then replace the 4 by 4 matrix of the first 4 rows and columns with $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This is an identity coloring by Fact 2 (iii). □

Note that for identity orientations when $s = t = 1$ the orientation consisting of a single edge is an identity orientation. This is the only case for complete bipartite graphs where the existence of an identity orientation and identity 3-edge coloring are not the same.

The next lemma is the c colors version of Lemma 1 of [7].

Lemma 4 *For any integers $c \geq 2$ and $s \geq 2$, $K_{s,t}$ does not have an identity c-edge coloring for $t \geq c^s$.*

Proof: If at least two rows of the corresponding adjacency matrix are the same apply Fact 2 (ii). If not, then $t = c^s$ and the corresponding adjacency matrix is full. For any non-trivial permutation matrix P_2 (which exist when $s \geq 2$), AP_2 is also full by Fact 2 (i). Thus, for some permutation matrix P_1 we have $P_1AP_2 = A$. □

The $c = 3$ version of the next lemma is used implicitly several times in [8].

Lemma 5 *Let A be the adjacency matrix of a c-edge colored complete bipartite graph $K_{s,t}$. If $s \neq t$ and A has distinct rows, then A is an identity coloring if and only if its complement A^* is an identity coloring. In the case that $t = s$ or $t = c^s - s$, the same holds if we exclude automorphisms that switch the parts.*

Proof: Assume that A is not an identity coloring (for $s = t$ assume that there is a non-trivial automorphism that maps each part to itself). Since $s \neq t$, for some permutation matrices, P_1, P_2, at least one of which is not the identity, we have $A = P_1AP_2$ and thus $P_1^TA = AP_2$. If P_2 is the identity, then $P_1A = A$ and P_1 is also the identity. Thus P_2 is not the identity.

The block matrix $\begin{bmatrix} A \\ A^* \end{bmatrix}$ is full. By Fact 2(i) so is $\begin{bmatrix} A \\ A^* \end{bmatrix} P_2 = \begin{bmatrix} P_1^TA \\ A^*P_2 \end{bmatrix}$. Thus A^*P_2 and A^* have the same rows. Hence for some permutation matrix P_3 we have $A^* = P_3A^*P_2$ and thus A^* is not an identity coloring. □

Corollary 6 *For $s \neq t$, $s \geq 2$ and $t \leq c^s$, $K_{s,t}$ has an identity c-edge coloring if and only if K_{s,c^s-t} has an identity c-edge coloring.*

Proof: Let A be an identity coloring. By Fact 2 (ii) the rows are distinct and Lemma 5 applies. □

Applying Lemmas 4 and 5 yield the following c color version of Corollary 7 in [8].

Lemma 7 *Let $s \geq 2$. If $t \leq \lfloor \log_c(s-1) \rfloor$ or $t \geq c^s - \lfloor \log_c(s-1) \rfloor$ then $K_{s,t}$ does not have an identity c-edge coloring.*

Proof: The cases $t \geq c^s$ are covered by Lemma 4. By Corollary 6 it is enough to consider $t \leq \lfloor \log_c(s-1) \rfloor$ for the remaining cases. If A is an identity coloring in these cases, then A has distinct rows and, by Lemma 5, A^* is an identity coloring. By Fact

2 (iv), the transpose of A^* is an identity coloring with $s' = t \leq \lfloor \log_c(s-1) \rfloor$ rows and $t' = s$ columns. Then $c^{s'} < s = t'$ contradicting Lemma 4. $\square$

The next lemma is the c color version of Theorem 13 in [8] and the proof is similar.

Lemma 8 *Let $s \geq 1$. Let r be the smallest integer such that $\binom{r+c-1}{r} \geq s$. For $r \leq t \leq c^s - r$ there exists a c-ary t by s matrix A with distinct rows and distinct column degrees. Furthermore, A is an identity coloring except possibly when A is square.*

Proof: The furthermore follows immediately from Fact 2 (iii).

If A has distinct rows and distinct column sums then so does its complement A^*. Thus it is enough to prove the theorem for $r \leq t \leq c^s/2$.

We will use the color set $\{0, 1, \ldots, c-1\}$. For $u \leq v$ let $B_{u,v}$ be the u by v matrix with the i^{th} row consisting of all zeros except for a 1 in column i for $1 \leq i < u$ and row u having the first $u-1$ entries 0 and the remaining entries 1. So when $u = 1$ the matrix has one row of all 1's and when $u = v$ it is the identity matrix. Note that each $B_{u,v}$ and its complement $B^*_{u,v}$ has constant column degrees, has distinct rows and has at least one 1 in each column.

Observe that when $s = 1$, $r = 0$. For $1 \leq t \leq c$ make the i^{th} row i. (The case $r = 0$ can be considered to be true as there is only one column.) When $s = 2$, $r = 1$. Consider the $c^2 \times 2$ matrix with rows specified as follows: each i in $\{1, 2, \ldots, c^2\}$ can be written uniquely as $i = jc + k$ for some $j \in \{0, 1, \ldots, c-1\}$ and $k \in \{1, 2, \ldots, c\}$. Let row i be $\begin{bmatrix} j & c-k \end{bmatrix}$. For $1 \leq t \leq c^2 - 1$ taking the first t rows of this matrix gives the needed matrix as can easily be checked.

For $s \geq 3$ use induction on s. Let r' be the smallest integer such that $\binom{r'+c-1}{r'} \geq s-1$. Note that $r' = r$ or $r' = r-1$. For $c = 2$ we have $r = s-1$. For $c \geq 3$, by the choice of r we have $s > \binom{(r-1)+c-1}{c-1} \geq \binom{r-1+2}{2}$ which implies that $r < \sqrt{2s}$.

For $r \leq t \leq c^s/2$ we will consider several cases. Note that $c^s/2 \leq (c-1)c^{s-1}$ for $c \geq 2$. For $c = 2$ cases 1 and 2 suffice.

Case 1: $r \leq t \leq c^{s-1} - r$. Since $r' \leq r$, by induction there exists a t by $s-1$ matrix with distinct rows and distinct column degrees. The $s-1$ column degrees each satisfy $x_0 + x_1 + \cdots + x_{c-1} = t$ where $t \geq r$. So there exists a solution $x_0^* + x_1^* + \cdots + x_{c-1}^* = t$ distinct from any of the degrees. Add a new column s with x_i^* entries equal to i. The rows are still distinct and the new column degree is also distinct from the first $s-1$.

Case 2: $(a+1)c^{s-1} - r < t \leq (a+1)c^{s-1}$ for some non-negative integer a. Let $u = t - ((a+1)c^{s-1} - r)$ so $1 \leq u \leq r$.

For $c = 2$: Then $a = 0$ since we need only consider $t \leq 2^s/2$. The cases $s = 2$ and $s = 3$ are easily checked. Consider $s > 3$. Let D be the $s-1$ by $s-1$ matrix with zeros above the main diagonal and ones elsewhere. Then D^* has distinct rows, distinct column degrees and at least $2^{s-2} - (s-2) \geq 2$ zeros in each column. Take the rows of $B_{u,s-1}$ with a last column of 0's added along with the rows of D^* with a last column of 1's added. The result has distinct rows. The last column has u zeros and the other columns have at least 2 zeros from D^* and $u-1$ zeros from $B_{u,s-1}$ for a total of more than u. Thus the last column degree is distinct from the others. Since also the column degrees of D^* are distinct and those of $B_{u,s-1}$ are constant we get distinct column degrees.

For $c > 2$: Let A' be a solution with $s-1$ columns and $t = c^{s-1} - r$ rows, which exists by induction. Note that $a \leq c-2$ since if $a \geq c-1$ then $t > ((c-1)+1)c^{s-1} - r \geq c^s - \sqrt{2s} \geq c^s/2$.

If $a = 0$ take A' with a last column of 0's added and $B_{u,s-1}$ with a last column of 2's added. Each part has distinct rows and the last entries for the rows differ for the different parts so the rows are distinct. On the first $s-1$ columns $B_{u,s-1}$ has constant degree and A' has distinct degrees so the first $s-1$ column degrees are distinct. The last column has no 1's and every other column does since $B_{u,s-1}$ does. Thus the last column has degree distinct from the others.

Now assume that $a \geq 1$. Let D be the ac^{s-1} by s matrix with rows all c-ary s-tuples with last entry from $\{2, 3, \ldots, a+1\}$. D has constant degree on the first $s-1$ columns and distinct rows. Take D, $B_{u,s-1}$ with a last column of 0's added and A' with a last column of 1's added. Each part has distinct rows and the last entries for the rows differ for the different parts so the rows are distinct. On the first $s-1$ columns D and $B_{u,s-1}$ with the appended column have constant degree and A' with the appended column has distinct degrees so the first $s-1$ column degrees are distinct. The number of 0's on each of the first $s-1$ columns is at least $(u-1) + ac^{s-2}$ from the 0's in $B_{u,s-1}$ and D respectively. This is strictly greater than u as $c \geq 3$, $s \geq 3$ and $a \geq 1$. The last column has u 0's. Thus the last column has degree distinct from the others.

Case 3: $(a+1)c^{s-1} < t \leq (a+1)c^{s-1} + r$ for some non-negative integer a. From the remarks above we can also assume that $c \geq 3$. Let $u = t - (a+1)c^{s-1}$ so $1 < u \leq r$. Note that $a \leq c-3$ since if $a \geq c-2$ then $t > (c-1)c^{s-1} \geq c^s/2$. Since $r < \sqrt{2s}$ one can then check that $u + r \leq 2r \leq c^{s-1} - r$. Thus, by induction there is a solution A' with $s-1$ columns and $u+r$ rows. Let D be the ac^{s-1} by s matrix with rows all c-ary s-tuples with last entry from $\{2, 3, \ldots, a+1\}$. In the case that $a = 0$, D will be empty. D has constant degree on the first $s-1$ columns and distinct rows. Take D, $B_{r,s-1}$ with a last column of 0's added and A' with a last column of $(c-1)$'s added. Since $a \leq c-3$, color $(c-1)$ is not used on D. Hence, each part has distinct rows and the last entries for the rows differ for the different parts so the rows are distinct. The last column has no 1's and every other column does since $B_{u,s-1}$ does. Thus the last column has degree distinct from the others.

Case 4: $(a+1)c^{s-1}+r < t < (a+2)c^{s-1}-r$ for some non-negative integer a. Let $u = t-(a+1)c^{s-1}$ so that $r < u < c^{s-1}-r$. Note that $a \leq c-3$ since if $a \geq c-2$ then $t > (c-1)c^{s-1}+r \geq c^s - c^{s-1} \geq c^s/2$. Let A' be a solution with $s-1$ columns and u rows which exists by induction. Let D be the $(a+1)c^{s-1}$ by s matrix with rows all c-ary s-tuples with last entry from $\{2,3,\ldots,a+2\}$. D has constant degree on the first $s-1$ columns and distinct rows and has c^{s-1} 1's in each of these columns. Since $s \geq 2$ and $c \geq$ there is at least one 1 in each of the first $s-1$ columns. Take D and A' with a last column of 0's added. Each part has distinct rows and the last entries for the rows differ for the different parts so the rows are distinct. On the first $s-1$ columns D has constant degree and A' has distinct degrees so the first $s-1$ column degrees are distinct. The last column has no 1's and every other column does since D does on these columns. Thus the last column has degree distinct from the others. $\square$

Observe that this lemma is best possible in the sense for $t < r$ no such set could exist. If it did then we would have $s > \binom{t+c-1}{t}$ distinct solutions to $x_0+x_1+\cdots+x_{c-1}=t$, a contradiction. By complementation no such set exists for $t > c^s-r+1$.

Proof of Theorem 1:

For (i) in the theorem, the case $s=t=1$ is noted in lemma 3. If $t > c$ then two edges have the same color and by Fact 2 (ii) the coloring is not an identity coloring. For $2 \leq t \leq c$ assigning different colors to the edges gives an identity coloring by Fact 2 (iii).

For the remaining cases we use induction on s.

For $s \leq c$ we have $r=1$ in Lemma 8 and (ii) follows except when $s=t$. The $s=t$ cases are covered by Lemma 3. The $s \leq c$ cases will also be the basis for the induction.

Lemma 7 covers the cases $t \geq c^s-x$ and $t \leq x$. Lemma 8 covers the cases $s \leq t \leq c^s-s$ since $r \leq s$ always. For the remaining cases it is enough to consider $x+1 \leq t < s$ by Corollary 6.

In each case we will let $s'=t$ and $t'=s$ and use fact 2 (iv), that $K_{s,t}$ has an identity coloring if $K_{s',t'}$ does. Then since $s'=t<s$ we can inductively check $K_{s',t'}$.

For $x+2 \leq t < s$ we need to show that $K_{s',t'}$ has an identity coloring. Note that $c^{s'} \geq c^{x+2} = (c)c^{1+\lfloor \log_c(s-1)\rfloor} \geq c(s-1)$. When $s' \leq c$ we have $c^{s'}-1 \geq c(s-1)-1 \geq s = t'$ and we get an identity coloring. For $s' > c$ we have $c^{s'} - \lfloor \log_c(s'-1)\rfloor - 2 \geq c(s-1) - \lfloor \log_c(s'-1)\rfloor - 2 \geq s = t'$ and again get an identity coloring.

The case $s' = t = x + 1$ remains. When $s = c^{x+1} - \lfloor \log_c x \rfloor - 1$ statement (v) is that we look at $K_{s',t'}$. We need to show that $K_{s',t'}$ does not have an identity coloring when (iv) $s \geq c^{x+1} - \lfloor \log_c x \rfloor$ and it does when (iii) $s \leq c^{x+1} - \lfloor \log_c x \rfloor - 2$.

First note that if $s' = x+1 \leq c$ then $\lfloor \log_c x \rfloor \leq \lfloor \log_c (c-1) \rfloor = 0$. So (iv) only occurs if $s = c^{1+x}$. Then $t' = s = c^{s'} > c^{s'} - 1$ and by (ii) $K_{s',t'}$ does not have an identity coloring. (The case $s' = t = 1$ is already covered by (i).) When $s' = x + 1 \leq c$ and (iii) occurs we have $t' = s \leq c^{s'} - 2$ and we have an identity coloring.

Now assume that $s' = x+1 > c$. If (iv) then $t' = s \geq c^{x+1} - \lfloor \log_c x \rfloor = c^{s'} - \lfloor \log_c (s' - 1) \rfloor$ and hence by Lemma 7 $K_{s',t'}$ does not have an identity coloring. If (iii) then $t' = s \leq c^{x+1} - \lfloor \log_c x \rfloor - 2 = c^{s'} - \lfloor \log_c (s'-1) \rfloor - 2$ and hence by induction $K_{s',t'}$ has an identity coloring. $\square$

While our Theorem gives an exact answer for determining if $K_{s,t}$ has an identity coloring in nearly all cases (a recursive check is required in (v)) we give here a few specific examples of using Corollary 6 directly to determine if there is an identity coloring for illustration. We will take $c = 3$. The conclusions in [8] state that $K_{s,t}$ has an identity coloring if and only if $1 \leq t \leq 3^s - 1$ when $s \in \{2, 3\}$, if and only if $2 \leq t \leq 3^s - 2$ for $s \in \{4, 5, \ldots, 8\}$, and if and only if $3 \leq t \leq 3^s - 3$ for $s \in \{9, 10, \ldots, 17\}$. Since $K_{3,t}$ has an identity coloring for $t \in \{1, 2, \ldots, 26\}$ we get, for example, that $K_{26,3}$ has an identity coloring. In a similar manner we can conclude that $K_{s,t}$ has an identity coloring if and only if $3 \leq t \leq 3^s - 3$ for $s \in \{9, 10, \ldots, 26\}$. The facts that $K_{4,79}$ has an identity coloring and that $K_{3,79}$ does not have an identity coloring and other similar cases show us that $K_{79,t}$ has an identity coloring if and only if $4 \leq t \leq 3^{79} - 4$.

Distinguishing Numbers

Distinguishing numbers of Cartesian products have been investigated in [1], [4], [5] and [6]. The main result in [4], which appears to have been done independently while our paper was under submission, is similar to our Corollary 9. For most sizes, this result and ours are the same. However there are a few differences which we will clarify after the statement of the corollary.

Recall that line graphs of a complete bipartite graphs are Cartesian products of complete graphs. That is, $L(K_{s,t}) = K_s \square K_t$. Thus our results correspond to vertex colorings of $K_s \square K_t$. Our automorphisms are on the vertex set of the bipartite graphs $K_{s,t}$ so we need to observe that they do indeed correspond to automorphisms of $K_s \square K_t$. This follows directly from the following result of Imrich and Miller, cited in [1]. Theorem: If G is connected and $G = H_1 \square H_2 \square \cdots \square H_r$ is its prime decomposition, then every automorphism of G is generated by the automorphisms of the factors and the transpositions of isomorphic factors. As the cases $s = t$ are easily dealt with we can directly translate our results on coloring $L(K_{s,t})$ to coloring $K_s \Box K_t$.

To determine the distinguishing number of $K_s \Box K_t$ we need to determine the smallest c in Theorem 1 for which $K_{s,t}$ has an identity coloring. When $s = 1$ we have $K_1 \Box K_t = K_t$ and we see from part (i) that $c = t$. This corresponds to the known result that $D(K_t) = t$. From the upper bounds on t we see that c should be approximately $\lceil (t+1)^{1/s} \rceil$. If $c < \lceil (t+1)^{1/s} \rceil$ then $t > c^s - 1$ and Theorem 1 tells us that there is no identity coloring. If $c = \lceil (t+1)^{1/s} \rceil + 1$ then $t + 1 \leq (c-1)^s$ and hence $t \leq c^s - sc^{s-1} - 1 \leq c^s - x - 2$ for $x = \lfloor \log_c(s-1) \rfloor$ and Theorem 1 tells us that there is an identity coloring. So the distinguishing number is $\lceil (t+1)^{1/s} \rceil$ or $\lceil (t+1)^{1/s} \rceil + 1$. In particular, for large t relative to s, (for example $t \geq s^s$) we get that the distinguishing number is $\lceil (t+1)^{1/s} \rceil$.

With the details of Theorem 1 we get the following. Observe that in all but one case we determine the distinguishing number immediately. In the remaining case we determine it from a recursion that is similar to that of Theorem 1. In particular it is repeated only when we need to determine if $D(K_{x+1,s}) = c$. So c stays fixed in the recursive computations and thus number of steps in the recursion is at most iterated logarithm $\log_c^*(s-1)$.

Corollary 9 *For $2 \leq s \leq t$ let $c = \lceil (t+1)^{1/s} \rceil$. So $t \leq c^s - 1$. Then $D(K_s \Box K_t)$ equals c or $c+1$. When $t \geq s^s$ the value is c.*

In particular, letting $x = \lfloor \log_c(s-1) \rfloor$ we have:
(i) $D(K_s \Box K_t) = c$ for $s \leq t \leq c^s - x - 2$ except for the case $D(K_2 \Box K_2) = 3$.
(ii) $D(K_s \Box K_t) = c + 1$ for $c^s - x \leq t \leq c^s - 1$.
(iii) $D(K_s \Box K_t) = c$ for $t = c^s - x - 1$ and $s \leq c^{1+x} - \lfloor \log_c x \rfloor - 2$.
(iv) $D(K_s \Box K_t) = c + 1$ for $t = c^s - x - 1$ and $s \geq c^{1+x} - \lfloor \log_c x \rfloor$.
(v) When $t = c^s - x - 1$ and $s = c^{1+x} - \lfloor \log_c x \rfloor - 1$ then $D(K_s \Box K_t) = c$ if $D(K_{x+1} \Box K_s) \leq c$ and $D(K_s \Box K_t) = c + 1$ if $D(K_{x+1} \Box K_s) \geq c + 1$

Finally, we note some of the differences between our result and that of [4], using notation of Corollary 9. The main theorem of [4] as stated is parts (*i*) and (*ii*) of Corollary 9 and a proposition is part (*iv*). Another proposition relates to parts (*iii*) and (*v*) but covers slightly different sizes.

What follows holds except for some very small sizes s, t, c. For a given s and c let $x = \lfloor \log_c(s-1) \rfloor$. That is, x is such that $c^x < s \leq c^{x+1}$. For each s and c, whenever $t = c^s - x - 1 = c^s - \lfloor \log_c(s-1) \rfloor - 1$, the distinguishing number $D(K_{s,t})$ is either c or $c+1$. It is c if $t < c^s - \lfloor \log_c(s-1) \rfloor - 1$ (part (*i*)) and $c+1$ if $t > c^s - \lfloor \log_c(s-1) \rfloor - 1$ (part (*ii*)). When $t = c^s - \lfloor \log_c(s-1) \rfloor - 1$, it is c if $c^x < s \leq c^{x+1} - \lfloor \log_c x \rfloor - 2$

(part (iii)) and $c+1$ if $c^{x+1} - \lfloor \log_c x \rfloor \le s \le c^{x+1}$ (part (iv)). Only one value, $s = c^{x+1} - \lfloor \log_c x \rfloor - 1$ requires the recursion as in part (v).

As noted above, the main theorem in [4] is the same as parts (i) and (ii) and one of the propositions is the same as part (iv). Another proposition, applies the recursion once yielding a slightly different version of part (iii), showing that when $t = c^s - \lfloor \log_c(s-1) \rfloor - 1$, the distinguishing number is c if $c^x < s \le c^{x+1} - x$. Thus when $x = 1$, [4] determines $D(K_{c^2-1} \Box K_{c^{c^2-1}-2}) = c$ while our Corollary requires recursion. When $x > 1$, our Corollary determines $D(K_s \Box K_{c^s-x-1}) = c$ for $c^x < s \le c^{x+1} - \lfloor \log_c x \rfloor - 2$ while [4] requires recursion for some of these values, namely $c^{x+1} - x + 1 \le s \le c^{x+1} - \lfloor \log_c x \rfloor - 2$. Both papers require recursion to determine the distinguishing number when $s = c^{x+1} - \lfloor \log_c x \rfloor - 1$. That is, when $x > 1$, we do not know of a non-recursive condition to decide if $D(K_{c^{x+1} - \lfloor \log_c x \rfloor - 1} \Box K_{c^s-x-1})$ is c or $c+1$. However in all other cases the distinguishing number can be determined directly.

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