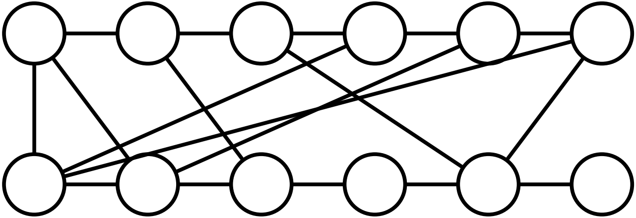
Min-max Theorems for the k-Path Partition Problem

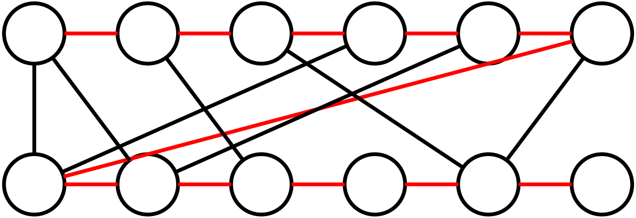
Breeanne Baker The Citidel
Garth Isaak Lehigh University

Cumberland Conference May 2014

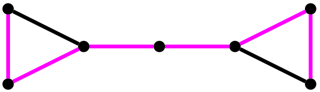
Hamiltonian Path



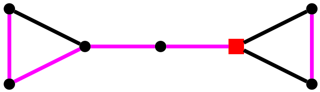
Hamiltonian Path



1-HP (Hamiltonian Path) problem:
Hamiltonian path with a specified end?

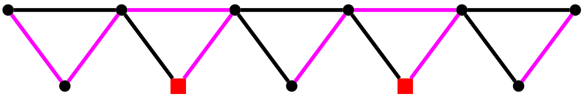


Hamiltonian path

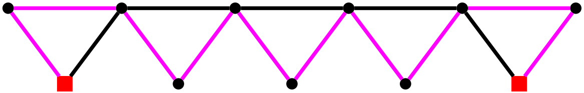


No 1-Hamiltonian path

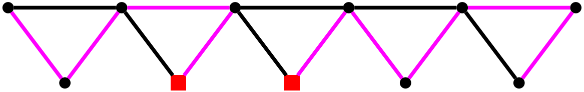
k-PP (Path Partition/Path Cover) problem:
Minimum path partition with k specified ends



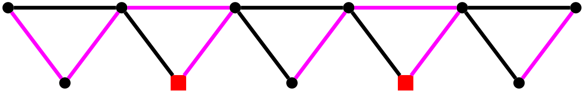
3 paths to partition with specified ends



1 path to partition with specified ends



2 paths to partition with specified ends



3 paths to partition with specified ends

(Regular) Path Partition

- Efficient algorithms and min-max theorems for Co-comparability graphs (threshold, cographs, interval, ...)
- Efficient algorithms but no nice min-max theorem for block graphs (Hamiltonian path even for distance hereditary)

(with specified ends) k-Path Partition

- Efficient algorithms for
  - Unit interval (Asdre, Nikopoulos 2009; Mertzios, Unger 2010)
  - Cographs (includes threshold) (Hung; Asdre, Nikopoulos 2006)
  - Block graphs (from regular partition)
  - Interval (1-HP only) (Asdre, Nikopoulos 2009)
  - Distance hereditary (2-HP only, from regular partition)
- Our goal - min-max theorems/certificate/certifying algorithm for 'nice' classes

(Regular) Path Partition

- Efficient algorithms and min-max theorems for Co-comparability graphs (threshold, cographs, interval, ...)
- Efficient algorithms but no nice min-max theorem for block graphs (Hamiltonian path even for distance hereditary)

(with specified ends) *k*-Path Partition

- Efficient algorithms for
  - Unit interval (Asdre, Nikopoulos 2009; Mertzios, Unger 2010)
  - Cographs (includes threshold) (Hung; Asdre, Nikopoulos 2006)
  - Block graphs (from regular partition)
  - Interval (1-HP only) (Asdre, Nikopoulos 2009)
  - Distance hereditary (2-HP only, from regular partition)
- Our goal - min-max theorems/certificate/certifying algorithm for ‘nice’ classes

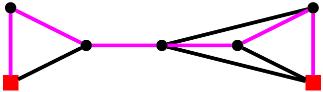
(Regular) Path Partition

- Efficient algorithms and min-max theorems for Co-comparability graphs (threshold, cographs, interval, ...)
- Efficient algorithms but no nice min-max theorem for block graphs (Hamiltonian path even for distance hereditary)

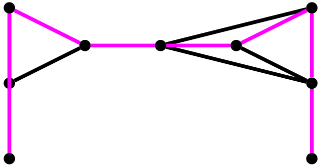
(with specified ends) k-Path Partition

- Efficient algorithms for
  - Unit interval (Asdre, Nikopoulos 2009; Mertzios, Unger 2010)
  - Cographs (includes threshold) (Hung; Asdre, Nikopoulos 2006)
  - Block graphs (from regular partition)
  - Interval (1-HP only) (Asdre, Nikopoulos 2009)
  - Distance hereditary (2-HP only, from regular partition)
- Our goal - min-max theorems/certificate/certifying algorithm for 'nice' classes

k-Path Partition $' =' $ (regular) PP with pendants



2-HP

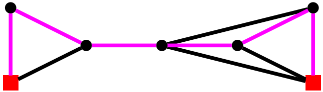


Regular HP

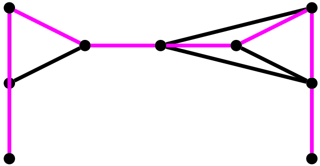
So just add pendants and translate known results:

- works for trees
- Fails for block graphs - no certificate for regular PP
- Fails for other classes - Adding pendant $\Rightarrow$ out of class

k-Path Partition $' =' $ (regular) PP with pendants



2-HP

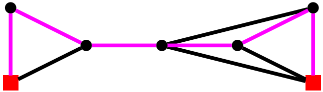


Regular HP

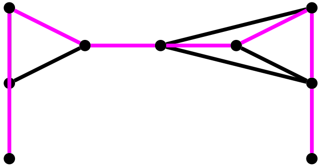
So just add pendants and translate known results:

- works for trees
- Fails for block graphs - no certificate for regular PP
- Fails for other classes - Adding pendant $\Rightarrow$ out of class

k-Path Partition $'='$ (regular) PP with pendants



2-HP

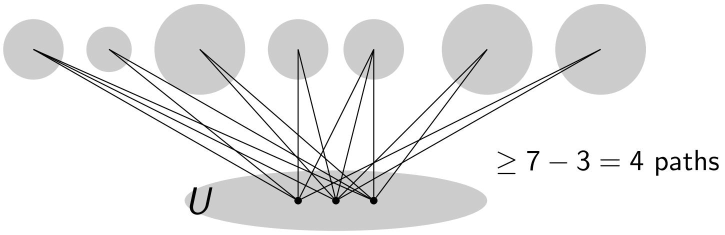


Regular HP

So just add pendants and translate known results:

- works for trees
- Fails for block graphs - no certificate for regular PP
- Fails for other classes - Adding pendant $\Rightarrow$ out of class

Certificate for (Regular) Path Partition

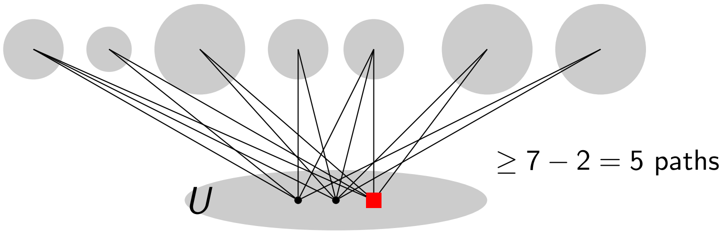


Theorem

If G is (interval graph, threshold graph, ...) then
Min Path Partition = Max $C(G - U) - |U|$

Would like something similar for k-Path Partition

Certificate (gen 1) for Path Partition with terminals T

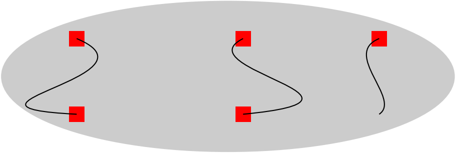


Theorem

If G is a tree with terminals T then
Min $|T|$-Path Partition = Max $C(G - U) - |U - T|$

prove directly via induction or
use pendant construction and certificate for regular partitions

Trivial Certificate for Path Partition with terminals T



need $\geq \lceil \frac{5}{2} \rceil = 3$ paths

Theorem

If G is 2-connected unit interval graph with terminals T then

Min $|T|$-Path Partition $= \left\lceil \frac{|T|}{2} \right\rceil$

or $+1$ when ....

$k = 2$ case Mertzios and Unger 2010

Note: 2-connected unit interval $\Rightarrow$ Hamiltonian cycle;
connected unit interval $\Rightarrow$ Hamiltonian path

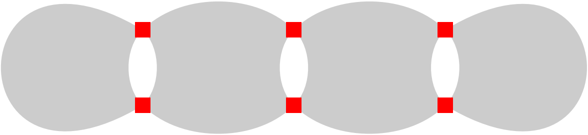
Theorem

If G is 2-connected unit interval graph with terminals T then

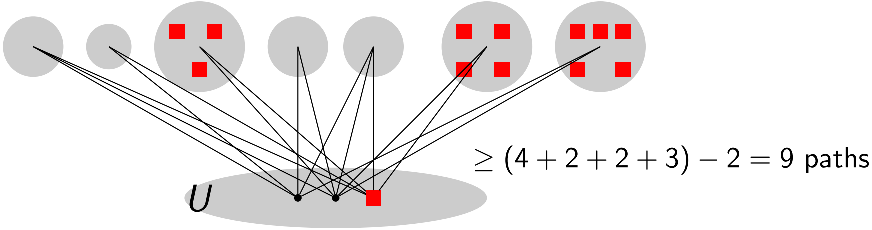
Min $|T|$-Path Partition $= \left\lceil \frac{|T|}{2} \right\rceil$

or $+1$ when (see below)

Unit interval = intersection graph of unit intervals



Certificate (gen 2) for Path Partition with terminals T



Theorem

If G is a threshold graph with terminals T then

Min $|T|$-Path Partition $=$ Max $\left(\sum \left\lceil \frac{|T_i|}{2} \right\rceil\right) + |R| - |U - T|$

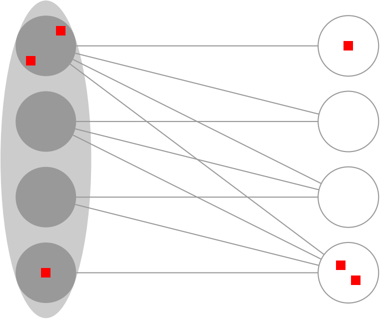
or $+1$ when ....

Theorem

If G is a threshold graph with terminals T then

$$\text{Min } |T|\text{-Path Partition} = \text{Max } \left(\sum \left\lceil \frac{|T_i|}{2} \right\rceil\right) + |R| - |U - T|$$

or $+1$ when ....



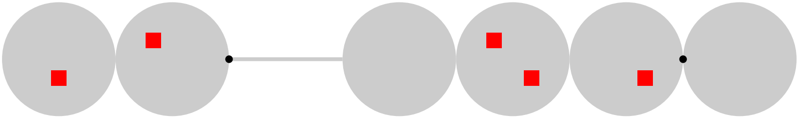
Trees with terminals T:
Min $|T|$-Path Partition $=$ Max $C(G-U)-|U-T|$

2-connected unit interval graphs with terminals T:
Min $|T|$-Path Partition $= \left\lceil \frac{|T|}{2} \right\rceil$
or $+1$ when ....

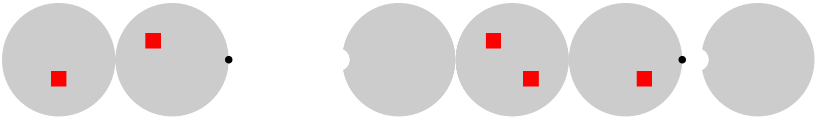
Threshold graphs with terminals T:
Min $|T|$-Path Partition $=$ Max $\left(\sum \left\lceil \frac{|T_i|}{2} \right\rceil \right) + |R| - |U-T|$
or $+1$ when ....

Block graphs, unit interval graphs with cut vertices, terminals T:
complicated min $=$ max ....

Linear Block Graphs (and unit interval with cut vertices)



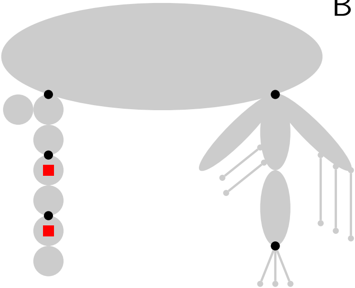
Impose left right ordering and use ‘forced ends’
Decompose along carefully selected cut vertices
i.e., those vertices do not pass through in left/right algorithm



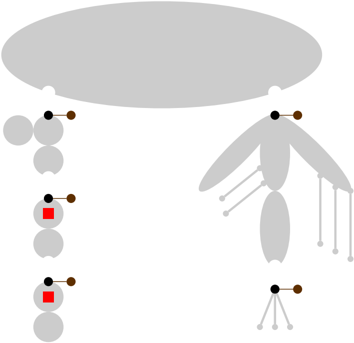
Count the number of ‘right’ ends in each part

General block graphs: Similar idea with ‘tops of paths’ and parity of components ‘below’ special cut vertices

Block Graphs



Decompose on certain cut vertices
count using 'forced ends' and ...



Trees with terminals T:
Min $|T|$-Path Partition $=$ Max $C(G-U)-|U-T|$

2-connected unit interval graphs with terminals T:
Min $|T|$-Path Partition $= \left\lceil \frac{|T|}{2} \right\rceil$
or $+1$ when ....

Threshold graphs with terminals T:
Min $|T|$-Path Partition $=$ Max $\left(\sum \left\lceil \frac{|T_i|}{2} \right\rceil \right) + |R| - |U-T|$
or $+1$ when ....

Block graphs, unit interval graphs with cut vertices, terminals T:
complicated min $=$ max ....