Interval Orders with Length Bounds

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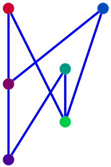
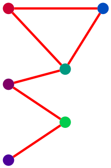
Intervals

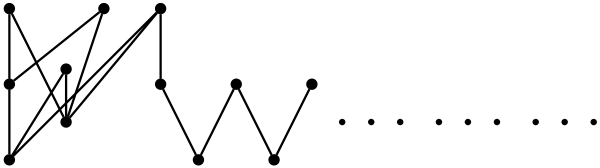
$\Rightarrow$



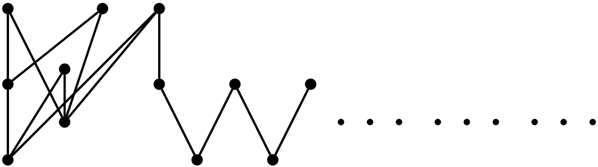
Interval Order



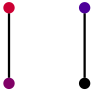
Interval Graph



Interval Order??



Interval Order?? $\Leftrightarrow$



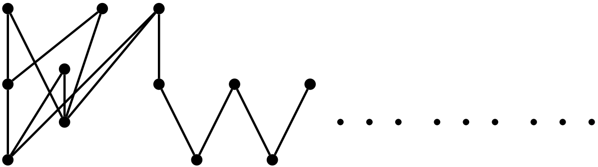
NO 2+2

Theorem (Wiener 1914)

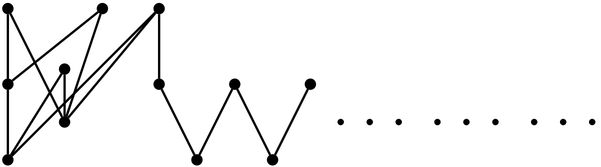
Interval Order $\Leftrightarrow$ *NO* $2+2$

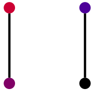
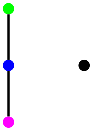
R. Duncan Luce Example (1956) from Econometrica

- Add one grain of sugar at a time
- Cannot distinguish between consecutive cups
- Can distinguish first and last
- Indifference is not transitive!
- Intervals with unit length (semiorders)



Unit Interval Order??



Unit Interval Order?? $\Leftrightarrow$  & 

NO 2+2 NO 3+1

Theorem (Scott-Suppes 1956)

Unit Interval Order $\Leftrightarrow$ *NO* $2+2$ *& NO* $3+1$

Does a given order have a representation
subject to given length bounds?

- Is there an efficient algorithm?
  Certifying algorithm?
- Forbidden suborder characterization?
- Graphs? Comparability invariant if
  all lower bounds equal

Assume interval order: no $2+2$

Theorem (Fishburn 1983,84)

Lengths between 1 and n (integer)

$$\Longleftrightarrow$$

NO 1 + (n+2)

e.g., lengths between 1 and 4 $\Leftrightarrow$ no $1+6$

Assume interval order: no $2+2$

Theorem (Fishburn 1983,84)

Lengths between a and b (integers)

$$\Leftrightarrow$$

NO forbidden picycle

e.g., Finite list of forbidden suborders but ....

For lengths between 2 and 3:
NO $x \sim^2 \prec^4 y$ or $x \sim\prec^2\sim^2\prec^3 y$ or $x \sim\prec^3\sim^2\prec^2 y$

NO finite list for irrational

Theorem (I 1990)

Lengths between a and b AND integer endpoints

$$\Leftrightarrow$$

NO nonpositive cycle in related digraph

Finite list of forbidden suborders for lengths 0 to n

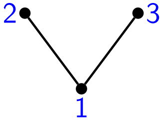
Infinite list of forbidden suborders for lengths 1 to n

Model Interval Order using Inequalities

$l_1 \le r_1 \qquad r_1 < l_2 \qquad \textit{not } r_2 < l_3$

$l_2 \le r_2 \qquad r_1 < l_3 \qquad \textit{not } r_3 < l_2$

$l_3 \le r_3$



Rewrite as:

$$
\begin{array}{cccccccl}
 & r_1 & -l_2 & & & & & < 0 \\
 & r_1 & & & -l_3 & & & < 0 \\
 & & & -r_2 & +l_3 & & & \le 0 \\
 & & l_2 & & & & -r_3 & \le 0 \\
l_1 & -r_1 & & & & & & \le 0 \\
 & & l_2 & -r_2 & & & & \le 0 \\
 & & & & l_3 & & -r_3 & \le 0
\end{array}
$$

We can represent an an order with intervals

$\Leftrightarrow$

Particular system of inequalities has a solution

Extends to:

- Constraints on interval length
- Minimize number of distinct endpoints
- Minimize ‘support’ length
  (if all lengths non-trivial)
- Partial information on ordering
- .....

Lemma (Farkas' Lemma 1906)

A system of inequalities has a solution
$\Leftrightarrow$ *it is not inconsistent*

$Ax \leq b$ *has a solution*
or $yA = 0,\ y \geq 0,\ yb < 0$ *has solution*

Theorem (LP duality)

$$\max\{cx | Ax \leq b\} = \min\{yb | yA \geq c, y \geq 0\}$$