Choose Multichoose

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Notation

$\binom{n}{k}$ = number of k element sets from $[n]$
$\binom{[n]}{k}$ = collection of k element sets from $[n]$

$\left(\!\binom{n}{k}\!\right)$ = number of k element multisets from $[n]$
$\left(\!\binom{[n]}{k}\!\right)$ = collection of k element sets from $[n]$

Will avoid using $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$ (later)

Choose Triangle

Recall ‘Pascal’s Triangle’ 1650

Kayyam’s Triangle 1000; Yang Hui’s Triangle 1350, ...

Display: left justified (left) rather than the typical way (right)

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
  1   7  21  35  35  21   7   1
```

Mulitchoose Array

Shift columns of Choose Triangle up or use multichoose recursion

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Mulitchoose Array

Shift columns of Choose Triangle up or use multichoose recursion

1	0	0	0	0	0	0	0	0
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	

Mulitchoose Array

Shift columns of Choose Triangle up or use multichoose recursion

1	0	0	0	0	0	0	0	0
1	1	1						
1	2	3						
1	3	6	1					
1	4	10	4	1				
1	5	15	10	5	1			
1	6	21	20	15	6	1		
1	7		35	35	21	7	1	

Mulitchoose Array

Shift columns of Choose Triangle up or use multichoose recursion

1	0	0	0	0	0	0	0	0
1	1	1	1					
1	2	3	4					
1	3	6	10					
1	4	10	20	1				
1	5	15	35	5	1			
1	6	21		15	6	1		
1	7			35	21	7	1	

Mulitchoose Array

Shift columns of Choose Triangle up or use multichoose recursion

1	0	0	0	0	0	0	0	0
1	1	1	1	1				
1	2	3	4	5				
1	3	6	10	15				
1	4	10	20	35				
1	5	15	35		1			
1	6	21			6	1		
1	7				21	7	1	

Mulitchoose Array

Shift columns of Choose Triangle up or use multichoose recursion

1	0	0	0	0	0	0	0	0
1	1	1	1	1	1			
1	2	3	4	5	6			
1	3	6	10	15	21			
1	4	10	20	35				
1	5	15	35					
1	6	21				1		
1	7					7	1	

Mulitchoose Array

Shift columns of Choose Triangle up or use multichoose recursion

1	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1		
1	2	3	4	5	6	7		
1	3	6	10	15	21			
1	4	10	20	35				
1	5	15	35					
1	6	21						
1	7						1	

Mulitchoose Array

Shift columns of Choose Triangle up or use multichoose recursion

1	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	
1	3	6	10	15	21		
1	4	10	20	35			
1	5	15	35				
1	6	21					
1	7						

Mulitchoose Array

Shift columns of Choose Triangle up or use multichoose recursion

n \ k	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	10	15	21	28
4	1	4	10	20	35	56	84
5	1	5	15	35	70	126	210
6	1	6	21	56	126	252	462

Choose Triangle

Hockey Stick Formula

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

[Left] $\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r}$

[*Right*] $\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{n+r}{n} = \binom{n+r+1}{n+1}$

Mulitchoose Array
Hockey Stick Formula

1	0	0	0	0	0	0
1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210
1	6	21	56	126	252	462

1	0	0	0	0	0	0
1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210
1	6	21	56	126	252	462

[Left]

$$\sum_{i=0}^{k} \left(\!\binom{m}{i}\!\right) = \left(\!\binom{m}{0}\!\right) + \left(\!\binom{m}{1}\!\right) + \left(\!\binom{m}{2}\!\right) + \cdots + \left(\!\binom{m}{k}\!\right) = \left(\!\binom{m+1}{k}\!\right)$$

Condition on number of $(m+1)$'s

[Right]

$$\sum_{j=1}^{m} \left(\!\binom{j}{k}\!\right) = \left(\!\binom{1}{k}\!\right) + \left(\!\binom{2}{k}\!\right) + \left(\!\binom{3}{k}\!\right) + \cdots + \left(\!\binom{m}{k}\!\right) = \left(\!\binom{m}{k+1}\!\right)$$

Condition on largest element

Choose Triangle

Anti-diagonal sums are Fibonacci numbers

1	=	1								
1	=	1	1							
2	=	1	2	1						
3	=	1	3	3	1					
5	=	1	4	6	4	1				
8	=	1	5	10	10	5	1			
13	=	1	6	15	20	15	6	1		
21	=	1	7	21	35	35	21	7	1	
34	=	1	8	28	56	70	56	28	8	1

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{n-k}{k-1} + \cdots$$

$F_n =$ number of 1,2 strings with sum $n-1$

Condition on # 2's

Choose Triangle

Shallow diagonal sums are Fibonacci numbers

1									=	1
1	1								=	1
1	2	1							=	2
1	3	3	1						=	3
1	4	6	4	1					=	5
1	5	10	10	5	1				=	8
1	6	15	20	15	6	1			=	13
1	7	21	35	35	21	7	1		=	21
1	8	28	56	70	56	28	8	1	=	34

$F_n = \binom{n-1}{n-1} + \binom{n-2}{n-3} + \binom{n-3}{n-5} + \cdots + \binom{n-k}{n-2k+1} + \cdots$

$F_n =$ number of 1,2 strings with sum $n-1$

Condition on # 1's

Recall:

The Fibonacci numbers F_n (for $n \geq 2$) count each of the following:
(Also Pingali 200 BCE)

A: The number of strings of 1's and 2's with sum $n-1$.

B: The number of strings of odd positive integers with sum n.

C: The number of strings of integers greater than 1 with sum $n+1$.

Illustrate with $n = 6$, where $F_6 = 8$.

A	B	C
11111	111111	7
2111	3111	25
1211	1311	34
1121	1131	43
1112	1113	52
221	51	223
212	33	232
122	15	322

Multichoose Array

Steep anti-diagonal sums are Fibonacci numbers

1	0	0	0	0	0	0
1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210
1	6	21	56	126	252	462
1						

$F_n = \left(\!\binom{n}{0}\!\right) + \left(\!\binom{n-2}{1}\!\right) + \left(\!\binom{n-4}{2}\!\right) + \cdots + \left(\!\binom{1}{\frac{n}{2}}\!\right)$ [n even]

$F_n =$ number of strings of odd positive integers with sum n

Condition on string length

$$13 = F_7 = \left(\!\!\binom{7}{0}\!\!\right) + \left(\!\!\binom{5}{1}\!\!\right) + \left(\!\!\binom{3}{2}\!\!\right) + \left(\!\!\binom{1}{3}\!\!\right)$$

e.g., String length 3, Sum 7:

$\underline{*\qquad}\quad \underline{*\qquad}\quad \underline{*\qquad}\qquad **\,,\,**$

3 bins, 7 *'s - odd number of *'s/bin

331, 313, 133, 511, 151, 115

Multichoose Array

Shallow anti-diagonal sums are Fibonacci numbers

1	0	0	0	0	0	0
1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210
1	6	21	56	126	252	462

$$\left(\!\!\binom{1}{n-1}\!\!\right) + \left(\!\!\binom{2}{n-3}\!\!\right) + \left(\!\!\binom{3}{n-5}\!\!\right) + \cdots + \left(\!\!\binom{\frac{n+1}{2}}{0}\!\!\right) \text{ [} n \text{ odd]}$$

$F_n =$ number of strings of integers greater than 1 with sum $n + 1$

Condition on string length

$$13 = F_7 = \left(\!\binom{1}{6}\!\right) + \left(\!\binom{2}{4}\!\right) + \left(\!\binom{3}{2}\!\right) + \left(\!\binom{4}{0}\!\right)$$

e.g., String length 3, Sum 8:

** ______ ** ______ ** ______ * *

3 bins, 8 *'s - fill with at least 2 *'s/bin

224, 242, 422, 233, 323, 332

Catalan Numbers

$$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\left(\!\!\binom{n+1}{n}\!\!\right)$$

Catalan counts we will refer to:

Ballot Lists: n 0 's; n 1's, # 0's $\geq$ # 1's in initial segments

Parentheses: n (, n) , well formed pairs

(n)-Multisets from $\{0, 1, \ldots, n\}$: $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n$ with $a_i \leq i - 1$

Proof versions:

- Reflection: count bad lists and substract
- Recursion and generating functions
- Partitions (n-multijections?)

Choose not multichoose

Catalan sequence: (()())(())

(()())(())	(()())(())	(()())(())	(()())(())	(()())(())
))(()((())	)()))(((())	)()()(((())	(()()))(((	(()()))()(

Map between each Catalan string and n ‘bad’ strings

$\Rightarrow \frac{1}{n+1}\binom{2n}{n}$

$$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\left(\!\!\binom{n+1}{n}\!\!\right)$$

$$= \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{2n+1}\left(\!\!\binom{n+1}{n+1}\!\!\right)$$

Ballot Lists: $n+1$ 0 's; n 1's, # 0's $>$ # 1's in initial segments

(n+1)-Multisets from $\{0, 1, \ldots, n\}$: $0 \le a_0 \le a_1 \le \cdots \le a_n \le n$ with $a_0 = 0$ and $a_i \le i-1$ for $i \ge 1$

i.e., Add a leading 0

Cycle Lemma (Dvoretzky and Motzkin 1947):
Write Ballot List cyclically and cut $2n+1$ places

Translate Cycle Lemma to multiset version

122255 Good lists dominated by 001234

If smallest is $0 \Rightarrow n$ If not -1 from all

122255
011144
111445
000334
003345
033455
334555
223444
112333
001122
012223

122255

$n = 5$, size 6 multisets from $\{0, 1, 2, 3, 4, 5\}$

rules partition $\left(\!\!\binom{6}{6}\!\!\right)$ multisets

into classes of size $2 \cdot 5 + 1 = 11$

with exactly one good multiset

partition $\left(\!\!\binom{n+1}{n+1}\!\!\right)$ into size $2n + 1$ classes

$$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\left(\!\!\binom{n+1}{n}\!\!\right) = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{2n+1}\left(\!\!\binom{n+1}{n+1}\!\!\right)$$

$\left(\!\!\binom{[n]}{k}\!\!\right)$ with $\#\, i \le c_i$

$=$ number of solutions to $x_1 + x_2 + \cdots + x_n = k$ such that $x_i < c_i$

Via generating functions or inclusion-exclusion

Choose version:

$$\sum_{S \subseteq [n]} (-1)^{|S|} \binom{k - c(S) + n - 1}{n - 1}$$

Multichoose version:

$$\sum_{S \subseteq [n]} (-1)^{|S|} \left(\!\!\binom{n}{k - c(S)}\!\!\right)$$

$c(S) = \sum_{i \in S} c_i$

Vandermonde's Identity (1789) (Chu Shih-Chieh 1303)

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r-1}\binom{m}{1} + \binom{n}{r}\binom{m}{0}$$

Multiset version:

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n}{1}\left(\!\!\binom{1}{k-1}\!\!\right) + \binom{n}{2}\left(\!\!\binom{2}{k-2}\!\!\right) + \cdots \binom{n}{t}\left(\!\!\binom{t}{k-t}\!\!\right) + \cdots$$

Condition on size of underlying set

Implied bijection

Poker Deck

$2\clubsuit, 2\diamondsuit, 2\heartsuit, 2\spadesuit, 2\clubsuit, 2\diamondsuit, \cdots, J\diamondsuit, \cdots, A\spadesuit$

4 Suits $\clubsuit, \diamondsuit, \heartsuit, \spadesuit$

13 Ranks: $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$

Poker Hand: 5 card subset of 52 cards

Full house - 5 card hand with 3 of one rank 2 of another

e.g., $7\clubsuit, 7\diamondsuit, 7\heartsuit, J\clubsuit, J\diamondsuit$

What is the probability of a full house poker hand?

Full house - 5 card hand with 3 of one rank 2 of another

e.g., $7\clubsuit, 7\diamondsuit, 7\heartsuit, J\clubsuit, J\diamondsuit$

What is the probability of a full house poker hand?

$$\frac{13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}}{\binom{52}{5}}$$

What is ______________________________*?*

$$\frac{13 \cdot \left(\!\!\binom{4}{3}\!\!\right) \cdot 12 \cdot \left(\!\!\binom{4}{2}\!\!\right)}{\left(\!\!\binom{52}{5}\!\!\right)}$$

What is the probability of a full house in multiset poker?
(Every 5 card multiset hand is equally likely)

$$\frac{13 \cdot \left(\!\binom{4}{3}\!\right) \cdot 12 \cdot \left(\!\binom{4}{2}\!\right)}{\left(\!\binom{52}{5}\!\right)}$$

What is the probability of a full house

If we use 5 decks?

$$\frac{13 \cdot \left(\!\binom{4}{3}\!\right) \cdot 12 \cdot \left(\!\binom{4}{2}\!\right)}{\left(\!\binom{52}{5}\!\right)}$$

What is the probability of a full house
If we use 5 decks?

NO! hands are not equally likely

$$\frac{13 \cdot \left(\!\!\binom{4}{3}\!\!\right) \cdot 12 \cdot \left(\!\!\binom{4}{2}\!\!\right)}{\left(\!\!\binom{52}{5}\!\!\right)}$$

What is the probability of a full house

If we deal with replacement?

$$\frac{13 \cdot \left(\!\!\binom{4}{3}\!\!\right) \cdot 12 \cdot \left(\!\!\binom{4}{2}\!\!\right)}{\left(\!\!\binom{52}{5}\!\!\right)}$$

What is the probability of a full house
If we deal with replacement?

NO! hands are not equally likely

$$\frac{13 \cdot \left(\!\!\binom{4}{3}\!\!\right) \cdot 12 \cdot \left(\!\!\binom{4}{2}\!\!\right)}{\left(\!\!\binom{52}{5}\!\!\right)}$$

Question

How can we play multiset poker with a $56 = 52 + 5 - 1$ *card deck?*

Question

How can we play multiset poker with a $56 = 52 + 5 - 1$ card deck?

Add 4 cards (knights) to deck:

$C\clubsuit, C\diamondsuit, C\heartsuit, C\spadesuit$

And apply a bijection between $\binom{[56]}{5}$ and $\left(\!\binom{[52]}{5}\!\right)$

The standard bijection: $\left(\!\!\binom{[6]}{5}\!\!\right) \Leftrightarrow \binom{[6+5-1]}{5}$

<table>
<tr><th>multiset</th><th>set</th><th>stars and bars</th></tr>
<tr><td>$1, 2, 3, 4, 5$</td><td>$1, 3, 5, 7, 9$</td><td>*|*|*|*|*|</td></tr>
<tr><td>$2, 3, 4, 4, 4$</td><td>$2, 4, 6, 7, 8$</td><td>|*|*|***||</td></tr>
<tr><td>$1, 1, 2, 3, 6$</td><td>$1, 2, 4, 6, 10$</td><td>**|*|*|||*</td></tr>
<tr><td>$3, 3, 3, 3, 3$</td><td>$3, 4, 5, 6, 7$</td><td>||*****|||</td></tr>
<tr><td>$2, 3, 4, 5, 6$</td><td>$2, 4, 6, 8, 10$</td><td>|*|*|*|*|*</td></tr>
<tr><td>$1, 1, 3, 3, 6$</td><td>$1, 2, 5, 6, 10$</td><td>**||**|||*</td></tr>
</table>

The standard bijection

Dealt hand		multiset hand
$3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit$	$\Longleftrightarrow$	$3\clubsuit 3\clubsuit 3\clubsuit 3\clubsuit 3\clubsuit$
$3\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$	$\Longleftrightarrow$	$3\clubsuit 3\clubsuit 4\heartsuit 10\heartsuit 10\heartsuit$
$3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit C\clubsuit$	$\Longleftrightarrow$	$3\diamondsuit 4\spadesuit 10\spadesuit 10\spadesuit A\clubsuit$
$3\diamondsuit 5\clubsuit J\heartsuit C\diamondsuit C\spadesuit$	$\Longleftrightarrow$	$3\diamondsuit 4\spadesuit J\clubsuit A\heartsuit A\spadesuit$
$3\diamondsuit 5\clubsuit J\heartsuit C\diamondsuit C\heartsuit$	$\Longleftrightarrow$	$3\diamondsuit 4\spadesuit J\clubsuit A\heartsuit A\heartsuit$

$\approx$ 2.6 million of $\approx$ 3.8 million multiset hands are regular poker hands

None map to themselves!

The Knight's bijection

$3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit$	$\Longleftrightarrow$	$3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit$
$3\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$	$\Longleftrightarrow$	$3\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$
$C\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$	$\Longleftrightarrow$	$3\diamondsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$
$3\diamondsuit C\diamondsuit 5\clubsuit C\spadesuit J\heartsuit$	$\Longleftrightarrow$	$3\diamondsuit 5\clubsuit 5\clubsuit J\heartsuit J\heartsuit$
$3\diamondsuit C\diamondsuit C\heartsuit 5\clubsuit J\heartsuit$	$\Longleftrightarrow$	$3\diamondsuit 5\clubsuit 5\clubsuit 5\clubsuit J\heartsuit$

- A hand with no knight maps to itself
- Place knights in their location $C\clubsuit$, $C\diamondsuit$, $C\heartsuit$, $C\spadesuit$ $=$ 1,2,3,4 left to right
- Place remaining cards in open spaces in order
- Knights take value of first regular card to their right

Knight's bijection $C : \binom{[n+k-1]}{k} \Leftrightarrow \left(\!\binom{[n]}{k}\!\right)$

- For $S \in \binom{[n+k-1]}{k}$, let $T = S \cap \{n+1, n+2, \ldots, n+k-1\}$
- $|T| = t$ and $R = S \cap [n] = S - T$ with $|R| = k - t$
- Write $R = a_1 < a_2 < \cdots < a_{k-t}$ and $T = n + b_1 < n + b_2 < \cdots < n + b_t$
- $T' = \{b_1, b_2, \ldots, b_t\} \subset \binom{[k-1]}{t}$ is a t element set from $[k-1]$
- Use the standard bijection B to map $T' = \{b_1, b_2, \ldots, b_t\}$ to a t element multiset from $[(k-1) - t + 1] = [k - t]$
- Use these as indices of repeated elements from R.
- In particular $B(T') = \{b_i - i + 1 | i = 1, 2, \ldots, t\}$.
- Then let $R' = \{a_{b_i - i+1} | i = 1, 2, \ldots, t\}$
- The image of S under the knight's bijection is then $C(S) = R \cup R'$.

Knight's bijection $C : \binom{[n+k-1]}{k} \Leftrightarrow \left(\!\binom{[n]}{k}\!\right)$

- Any set avoiding knights maps to itself
- Place knights in their location
- Place regular elements in order in open spots
- Knights take value of first regular element to their right

Knight's bijection $C : \binom{[n+k-1]}{k} \Leftrightarrow \left(\!\binom{[n]}{k}\!\right)$

- Stars and bars bijection with 'extra' elements as stars and 'regular' elements as bars

Knight's Bijection $\left(\!\binom{5}{7}\!\right) = \binom{11}{7}$

$$\{1, 3, 4\} \cup \{C_1, C_2, C_4, C_6\} \subseteq \{1, 2, 3, 4, 5\} \cup \{C_1, C_2, \ldots, C_6\}$$

$$\underline{\quad}\ \ \underline{\quad}\ \ \underline{\ 1\ }\ \ \underline{\quad}\ \ \underline{\ 3\ }\ \ \underline{\quad}\ \ \underline{\ 4\ }$$

Knight's Bijection $\left(\!\binom{5}{7}\!\right) = \binom{11}{7}$

$$\{1, 3, 4\} \cup \{C_1, C_2, C_4, C_6\} \subseteq \{1, 2, 3, 4, 5\} \cup \{C_1, C_2, \ldots, C_6\}$$

$$\underline{\ *\ }\quad \underline{\ *\ }\quad \underline{\ 1\ }\quad \underline{\ *\ }\quad \underline{\ 3\ }\quad \underline{\ *\ }\quad \underline{\ 4\ }$$

$$1113344 \in \left(\!\!\binom{[5]}{7}\!\!\right)$$

Playing poker with Knight's bijection

- No 'numerical' computations needed
- 'Normal' hands are themselves
- No 2 players can get the same card
- At most 4 instances of duplicated cards
-
- High card 'beats' one pair

General Poker Games

3 'Deals'

- *Multiple Decks (t decks)*
- *Multiset bijection*
- *Dealing with replacement*

- r ranks
- s suits
- hand size h

limit as $t \to \infty$ multideck is dealing with replacement

Notation for general poker

$$\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle$$

13 ranks, 4 suits, hand size 5: one 3 of a kind, one pair

$$\langle 0^{11}, 1^{0}, 2^{1}, 3^{1} \rangle$$

5 ranks, 7 suits, hand size 9: two 3 of a kind, one pair

$$\langle 0^{1}, 1^{1}, 2^{1}, 3^{2} \rangle$$

$r = \sum p_i$ and $h = \sum i \cdot p_i$

Notation for general poker

$\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle$

Regular 13 rank poker:
$\langle 0^{11}, 1^1, 2^0, 3^0, 4^1 \rangle$ is 4 of a kind
$\langle 0^{11}, 1^0, 2^1, 3^1 \rangle$ is full house
$\langle 0^{10}, 1^2, 2^0, 3^1 \rangle$ is 3 of a kind
$\langle 0^{10}, 1^1, 2^2 \rangle$ is 2 pair
$\langle 0^9, 1^5 \rangle$ is high card

With $r = 17$ ranks and hand size $h = 9$
$\langle 0^{14}, 1^0, 2^0, 3^3 \rangle$ is three 3 of a kinds
$\langle 0^{14}, 1^1, 2^0, 3^0, 4^2 \rangle$ is two 4 of a kinds

Observe that full house and 4 of a kind have same exponents as do 2 pair and 3 of a kind
$r = \sum p_i$ and $h = \sum i \cdot p_i$

Rank Selection: Independent of suits and deal type

Fact

For a poker hand of type $\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle$*, the number of ways to pick the ranks is the multinomial coefficient*

$$N_{ra} = \binom{r}{p_0, p_1, p_2, \ldots} = \frac{r!}{p_0!p_1!\cdots p_h!}$$

4 of a kind and full house:
$N_{ra}(\langle 0^{11}, 1^1, 2^0, 3^0, 4^1 \rangle) = N_{ra}(\langle 0^{11}, 1^0, 2^1, 3^1 \rangle)$

$$= \binom{13}{11, 1, 1} = \frac{13!}{11!1!1!} = 13 \cdot 12$$

3 of a kind and 2 pair:
$N_{ra}(\langle 0^{10}, 1^2, 2^0, 3^1) = N_{ra}(\langle 0^{10}, 1^1, 2^2 \rangle)$

$$= \binom{13}{10, 2, 1} = \frac{13!}{10!2!1!} = \frac{13 \cdot 12 \cdot 11}{2}$$

two four of a kind with $r = 17$ ranks and hand size $h = 9$:

$$N_{ra}(\langle 0^{14}, 1^1, 2^0, 3^0, 4^2 \rangle) = \binom{17}{14, 2, 1} = \frac{17!}{14!2!1!}$$

Definition

The number of poker hands of type λ is

$$N(\lambda) = N_{ra}(\lambda) \cdot N_{su}(\lambda)$$

 # ways to pick the ranks $\cdot$ # ways to pick the suits
Suit selection $N_{ra}(\lambda)$ does depend on deal type

Definition

The number of poker hands of type λ is

$$N(\lambda) = N_{ra}(\lambda) \cdot N_{su}(\lambda)$$

 # ways to pick the ranks · # ways to pick the suits
Suit selection $N_{ra}(\lambda)$ does depend on deal type

Fact

For a poker hand of type $\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle$, the number of ways to pick the suits is

- *(t decks):* $N_{su}^{md}(\lambda) = \prod \binom{st}{i}^{p_i}$
- *(multiset):* $N_{su}^{ms}(\lambda) = \prod \left(\!\binom{s}{i}\!\right)^{p_i}$
- *(dealing with replacement):* $N_s^r(\lambda) = \binom{h}{\lambda} \cdot s^h$

Regular poker full house probabilities (including full house flush)

$\langle 0^{11}, 1^0, 2^1, 3^1 \rangle$

5 decks:

$$\frac{\binom{13}{11,1,1} \cdot \binom{20}{3}\binom{20}{2}}{\binom{260}{5}}$$

multiset:

$$\frac{\binom{13}{11,1,1} \cdot \left(\!\binom{4}{3}\!\right) \left(\!\binom{4}{2}\!\right)}{\left(\!\binom{52}{5}\!\right)}$$

Dealing with replacement:

$$\frac{\binom{13}{11,1,1} \cdot \binom{5}{3,2} \cdot 4^5}{52^5}$$

17 ranks, 3 suits, 9 card hands

two 4 of a kind (including flushes), $\langle 0^{14}, 1^1, 2^0, 3^0, 4^2 \rangle$

2 decks:

$$\frac{\binom{17}{14,2,1} \cdot \binom{6}{4}^2 \binom{6}{1}}{\binom{102}{9}}$$

multiset:

$$\frac{\binom{17}{14,2,1} \cdot \left(\!\!\binom{3}{4}\!\!\right)^2 \left(\!\!\binom{3}{1}\!\!\right)}{\left(\!\!\binom{51}{9}\!\!\right)}$$

Dealing with replacement:

$$\frac{\binom{17}{14,2,1} \cdot \binom{9}{4,4,1} \cdot 3^9}{51^9}$$

$\lambda = \langle 0^{p_0}, 1^{p_1}, 2^{p_2}, \ldots, \rangle$

t decks

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \prod \binom{st}{i}^{p_i}}{\binom{rst}{h}}$$

Multiset

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \prod \left(\!\!\binom{s}{i}\!\!\right)^{p_i}}{\left(\!\!\binom{rs}{h}\!\!\right)}$$

Dealing with replacement

$$\frac{\binom{r}{p_0, p_1, p_2, \ldots} \cdot \binom{h}{(0!)^{p_0}\,(1!)^{p_1}\,(2!)^{p_2}\,(3!)^{p_3}\ldots} \cdot s^h}{(rs)^h}$$

Multiset vs. regular probabilities (as percents %)

	multiset	regular
Straight flush	.001	.001
5 kind flush	.001	0
4 kind flush	.016	0
full house flush	.016	0
5 kind	.02	0
3 kind flush	.09	0
2 pair flush	.09	0
flush	.13	.20
straight	.27	.39
pair flush	.30	0
4 kind	.56	.02
full house	.80	.14
3 kind	7.10	2.87
2 pair	8.90	4.75
High card	34.10	49.68
1 pair	47.62	42.3

regular poker hands $\binom{52}{5} = 2,598,960$

multiset poker hands $\left(\!\binom{52}{5}\!\right) = \binom{56}{5} = 3,819,816$

- A hand with no knight maps to itself
- Place knights in their location $C\clubsuit$, $C\diamondsuit$, $C\heartsuit$, $C\spadesuit$ $=$ 1,2,3,4 left to right
- Place remaining cards in open spaces in order
- Knights take value of first regular card to their right

The Knight's bijection

$$3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit \iff 3\clubsuit 3\diamondsuit 3\heartsuit 3\spadesuit 4\clubsuit$$

$$3\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit \iff 3\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$$

$$C\clubsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit \iff 3\diamondsuit 3\diamondsuit 5\clubsuit J\diamondsuit J\heartsuit$$

$$3\diamondsuit C\diamondsuit 5\clubsuit C\spadesuit J\heartsuit \iff 3\diamondsuit 5\clubsuit 5\clubsuit J\heartsuit J\heartsuit$$

$$3\diamondsuit C\diamondsuit C\heartsuit 5\clubsuit J\heartsuit \iff 3\diamondsuit 5\clubsuit 5\clubsuit 5\clubsuit J\heartsuit$$