Mathematics of Round Robin Tournaments

Garth Isaak

Lehigh University

Math Club

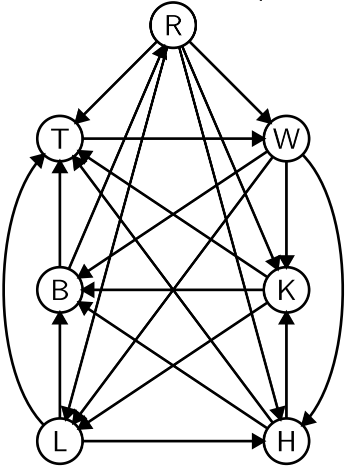
Several times each year the math club meets for lectures by guest speakers or Bethel math professors. Twice this year famous mathematicians came to speak following banquets. The club provides additional resources for students interested in learning more about different aspects of the math field.

Photos by Tom Claassen and Eldon Esau

1st row: Ronald Quiring; 2nd row: Arnold Wedel, Alan Deckert; 3rd row: Garth Isaac, Amy Deckert; Rannie Goering, Steve Goering; 4th row: Dave King, John Thiesen, Al Dobbendick, Ron Headings.

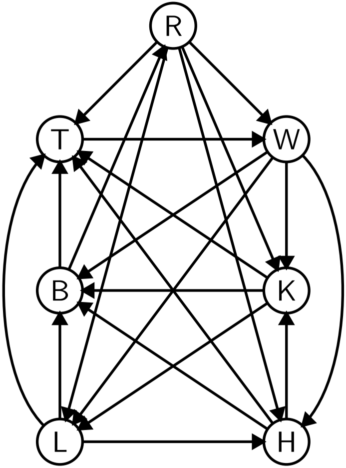
Round Robin Tournaments

Seven players: Rempel, Wedel, Krehbiel, Helrich, Lehman, Brenneman, Thimm compete in a round robin tournament



Score sequence
(5,4,3,3,3,2,1)
records number of wins

What questions might a mathematician (or ...) ask?



How do we rank players?

What scores are possible?

Structures in tournaments?

Applications?

- We need tools from basic mathematics
- Linear algebra (Bethel Fall 1979)
- Combinatorics and Graph Theory (winter 1983)

Elementary Linear Algebra Final Exam December 12, 1979

1. Solve the system of equations
$$\begin{cases} 2x - y + z = 1 \\ 3x + y - 3z = 2 \\ 9x \quad\quad - 4z = 5 \end{cases}$$

a) by writing as $AX = B$ and multiplying both sides by A^{-1}.
b) by Cramer's rule.
c) by Gaussian elimination.

2. Solve the system
$$\frac{dx}{dt} = 2x - 4y - 2z$$
$$\frac{dy}{dt} = -x + 2y - z$$
$$\frac{dz}{dt} = -2x + 2y - z$$

subject to the initial conditions $x(0) = 1$, $y(0) = 2$, $z(0) = -1$.

3. Consider a simple model of an economy in which there are three "goods"; steel, food and labor. The production of each good consumes a part of what was produced the year before, and the economist's question is whether (and at what rate) the economy can expand. Suppose a new unit of steel requires .4 unit of existing steel and .5 unit of labor, a unit of food requires .1 unit of food and .7 unit of labor, and producing (or maintaining) a unit of labor needs .8 unit of food and .1 unit each of steel and labor. Write a matrix equation $V_0 = AV_1$ relating the inputs s_0, f_0, l_0 to the outputs s_1, f_1, and l_1. Explain why ... Show that

5. Find the inverse of the matrix $\begin{bmatrix} \cdots & \cdots & \cdots \\ -1 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix}$.

to solve the systems

$$\begin{aligned} x + y + z &= 5 \\ -x + 3y + 2z &= 2 \\ 2x + y + z &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} x + y + z &= 8 \\ -x + 3y + 2z &= -2 \\ 2x + y + z &= 0. \end{aligned}$$

6. Suppose A, B and C are $n \times n$ matrices and $ABC =$

(a) Express A^{-1} in terms of B and C.

(b) What can you say about C^{-1}?

(c) Can you express B^{-1} in terms of A and C?

7. Prove that S is a subspace of a vector space V if and only if S is nonempty and $r\alpha + s\beta \in S$ for all $\alpha, \beta \in S$ and real numbers r, s.

(8) You may substitute this question, arising from last night's Club talk, for any question above for full credit, or do it fo[r]
Define the arithmetic mean, geometric mean and harmonic
two numbers, represent them on a semicircle, and show
line segments you used have the desired lengths.

Fall 1979 Elementary Linear Algebra - Bethel College
Professor Richard Rempel

- Mark Carpenter - Ph.D. Mech Eng, Carnegie Mellon; NASA
- Allen Daubendiek - Ph.D. EE, Florida(?); engineer
- Amy Deckert, Civil engineer
- William Ewy, engineer at HP Hong Kong
- Mark D. Friesen, software consultant
- John D. Harder, masters in statistics WSU/KSU; Ford
- Ron Headings, MBA Indiana; senior manager P&G research,
- Garth Isaak, Ph.D. Operations Research, Rutgers; Lehigh U
- J. David King, office 101st airborne
- Don Klippenstien, MD, Minnesota; Moffit Cancer center
- Richard Neufeld, nurse anesthesiologist
- Layne Reusser, MD, KSU; cardiovascular consultants, Wichita
- William Schmidt, Ph.D. Computer Science, Iowa; IBM
- John Thiesen, MS history WSU; Bethel

ACCK INTERTERM COURSE

Title: COMBINATORICS AND GRAPH THEORY

Description: Directed graphs, trees, circuits, paths, basic combinatorics, generating functions. Emphasis on applications and on use of computer problem solutions.

Credit: 3 credit hours towards mathematics or computer science.

Time and place: At McPherson College, Harnly Hall, Room 208, 1:30-4:30 daily January 3-27. First meeting 1:30, Jan. 3.

rincipal instructor: Richard Rempel, Bethel College(283-2500).

rerequisites: Some calculus, some linear algebra (at least familiarity with matrices and their use in solving systems of linear equations), and a computer programming language.

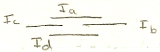
ext: APPLIED COMBINATORICS, by Alan Tucker, Wiley, 1980. (On sale at the McPherson College Bookstore).

opics:
- Counting methods
- Elementary graph concepts
- Graph models
- Graph circuits and graph coloring
- Trees and searching
- [illegible]
- Games with graphs
- Generating functions
- Recurrence relations
- Inclusion-exclusion
- Polya's enumeration formula

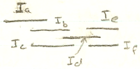
ading: Based on homeworks and two exams.

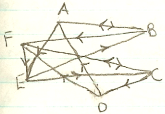
ecial Feature: Alan Tucker of SUNY at Stony Brook, the author of our text, will be a guest lecturer in the course for two days (dates and other details still not worked out).

can be done as



can be done as -

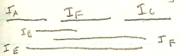




draw graph with arrows indicating seeing another at the library!

note that D saw F and A and B saw F and A but F and A didn't see each other so weren't in library at same time this creates a contradiction in times

Use ~~indifference~~ competition graph because seeing each other in the library is not transitive.



I_D ------ cannot be placed without a contradiction since I_D must overlap I_F, I_C, I_A but not I_B or I_E so removing just one of I_B or I_E would not alleviate the inconsistancy

Therefore we conclude that D is the thief

1

ELA 8-29

ex. 1200 cal/day

x	y	z
bread ~~meat~~	fruit	meat ~~brea~~
100 cal	50 cal	300 cal
10¢	15¢	75¢

\$2.50 4 x servings as bread or fruit as meat

$100x + 50y + 300z = 1200$

$10x + 15y + 75z = 2.50$

$x + y - 4z = 0$

* system of 3 simultaneous linear eq - in 3 unknowns

* " m " " " N "

a_{11} 1st - 1st equation 2nd - unknow

$a_{11}x_1 + a_{12}x_2 + a_{13} + \dots + a_{1N}x_N = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \dots + a_{2N}x_N \quad b_2$

$\vdots$

$a_{m1} + a_{m2}x_2 + a_{m3}x_3 \dots + a_{mN}x_N = b_m$

→ * inconsistent - No solution

* all $b_m = 0$ → homogeneous → never inconsistent $(0, \dots 0)$ always sln — none - trivial sln

4 blue books

assign - read 1.1; 1.2 solve ↑

8-31

‘Review’ a little linear algebra
this will be the hardest part, I promise

Do these have nonnegative solutions?

$$\begin{aligned} x + y + 2z &= 3 \\ 5x + 8y + 13z &= 21 \\ x - y + z &= 0 \end{aligned} \qquad \begin{aligned} x + y + 2z &= 13 \\ 5x + 8y + 13z &= 21 \\ x - 3y - 3z &= 1 \end{aligned}$$

Do these have nonnegative solutions?

$$\begin{aligned} x + y + 2z &= 3 \\ 5x + 8y + 13z &= 21 \\ x - y + z &= 0 \end{aligned} \qquad \begin{aligned} x + y + 2z &= 13 \\ 5x + 8y + 13z &= 21 \\ x - 3y - 3z &= 1 \end{aligned}$$

$x = 0,\ y = z = 1$

yes

no

Why not?

This system has no nonnegative solution

$$\begin{aligned} x + y + 2z &= 13 \\ 5x + 8y + 13z &= 21 \\ x - 3y - 3z &= 1 \end{aligned}$$

This system has no nonnegative solution

$$\begin{array}{rrcrcrcl}
-2 & x & + & y & + & 2z & = & 13 \\
1 & 5x & + & 8y & + & 13z & = & 21 \\
2 & x & - & 3y & - & 3z & = & 1
\end{array}$$

Multiply equations by (-2), 1, 2 respectively

This system has no nonnegative solution

$$\begin{array}{rrrrrrrl}
-2 & x & + & y & + & 2z & = & 13 \\
1 & 5x & + & 8y & + & 13z & = & 21 \\
2 & x & - & 3y & - & 3z & = & 1
\end{array}$$

Multiply equations by (-2), 1, 2 respectively
Add resulting equations

$$\begin{array}{rrrrrrl}
-2x & - & 2y & - & 4z & = & -26 \\
5x & + & 8y & + & 13z & = & 21 \\
2x & - & 6y & - & 6z & = & 2
\end{array}$$

This system has no nonnegative solution

$$\begin{array}{rrcrcrcr} -2 & x & + & y & + & 2z & = & 13 \\ 1 & 5x & + & 8y & + & 13z & = & 21 \\ 2 & x & - & 3y & - & 3z & = & 1 \end{array}$$

Multiply equations by (-2), 1, 2 respectively
Add resulting equations

$$\begin{array}{rcrcrcr} -2x & - & 2y & - & 4z & = & -26 \\ 5x & + & 8y & + & 13z & = & 21 \\ 2x & - & 6y & - & 6z & = & 2 \end{array}$$

Result is

$$5x + 0y + 3z = -3$$

Every solution has at least one of x, y, z negative

Farkas' Lemma

Either a linear system has a nonnegative solution

OR

There are multipliers showing inconsistency

$$\begin{array}{rrcl} -2 & x + y + 2z & = & 13 \\ 1 & 5x + 8y + 13z & = & 21 \end{array} \Rightarrow 3x + 6y + 9z = -5$$

also called Lagrange multipliers or Shadow prices

This hints at Linear programming - significant use in applications
Major LP advances reported in Wall Street Journal, NY Times etc
including when I was in linear algebra at Bethel

- start with score sequences
- quick look at tournament structures
- quick look at rankings
- return to score sequences

Is the following a possible score sequence for a tournament with 12 players?

$$10, 10, 9, 8, 8, 7, 5, 2, 2, 2, 2, 1$$

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Simplest algorithm: test *ALL* possible tournaments

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Simplest algorithm: test *ALL* possible tournaments

- Can we check if everyone in the room checks cases by hand for an hour? NO
- Can we check if everyone in the room uses a computer for an hour? NO
- Can we check using the world's fastest supercomputer for an hour? YES
- But we need the full hour

Is the following a possible score sequence for a tournament with 12 players?

$$10, 10, 9, 8, 8, 7, 5, 2, 2, 2, 2, 1$$

Simplest algorithm: test *ALL* possible tournaments

- There are $2^{66} \approx 7 \times 10^{19}$ possible tournaments
- Fastest computer cluster runs at 10 Petaflops: 10×10^{15} operations per second
- We can check a size 12 problem in about an hour with this supercomputer.

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- We can check a size 12 problem in about an hour with this supercomputer.
- What is the largest size we can check with 1,000 of these supercomputers?
- 12 (its not a typo) but with 10,000 supercomputers we can check size 13

Is the following sequence of 25 numbers a score sequence?

22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3

Is the following sequence of 25 numbers a score sequence?

$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

Try checking *ALL* possible tournaments?

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Try checking *ALL* possible tournaments?

UNIVERSE-ALL computer:

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UNIVERSE-ALL computer:

All of the atoms in the known universe checking a billion tournaments per second

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Still not done checking all possibilities for this instance

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Try checking *ALL* possible tournaments?

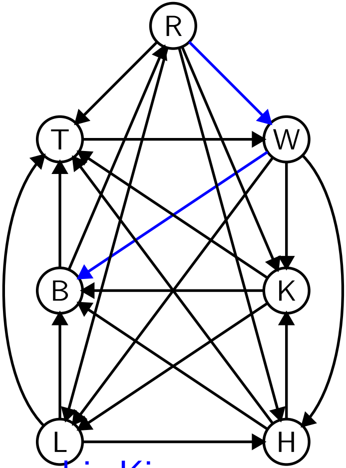
UNIVERSE-ALL computer:

All of the atoms in the known universe checking a billion tournaments per second

Still not done checking all possibilities for this instance

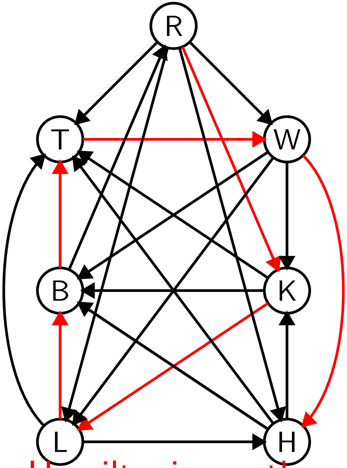
Use mathematical tools to make the check faster

Quick look at some structures in tournaments



Rempel is King

beats all in 2 steps



Hamiltonian path

path through all players

application - Travelling Salesperson problem

These exercises involve no numbers or equations

Exercise

Prove that every tournament has a King

King - beats every other player or beats someone who beats them

Exercise

Prove that every Tournament has a Hamiltonian path

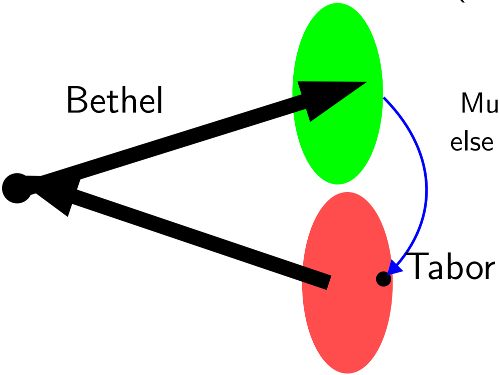
Hamiltonian path - sequence Rempel beats Wedel beats Kreibhel ... including each player exactly once

Exercise

Which Bethel Chemistry Professor did Garth attend High School with in Pennsylvania?

For those of you who prefer not to prove any theorems on a Saturday morning

Team with the most wins (Bethel) is a king



Must have some blue arc
else Tabor wins more than Bethel

Exercise

Every tournament either dominant player or at least three Kings
Prove it

What is the best way to rank players in a tournament?
Depends on what 'best' means (need to define terms)

- Rank based on score sequence (number of wins)
- Rank to minimize the number of upsets
- Rank on 'iterated scores'

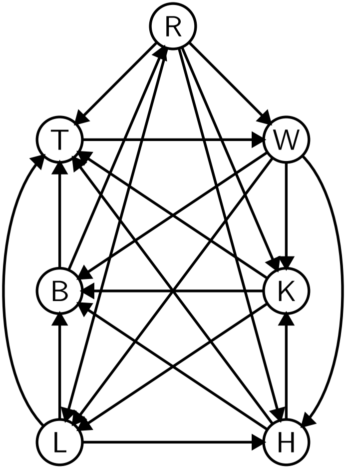
Iterated scores: K, H, L all have 3 wins

K beats: $3 + 2 + 1 = 6$

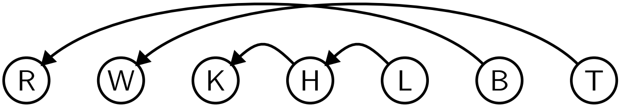
H beats: $3 + 2 + 1 = 6$

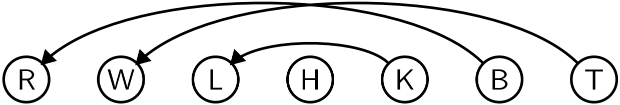
L beats: $5 + 2 + 1 = 8$

- First iterated scores: $(14, 11, 6, 6, 8, 6, 4)$
- Normalize (so they sum to 1) and repeat ...
- $\Rightarrow$ Eigenvector
- Google's Page Rank similar



Ranking to minimize upsets





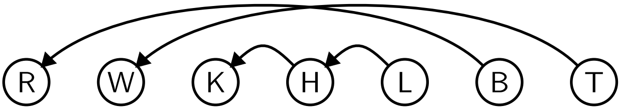
Can be different from ranking by wins

Independently 'discovered'

- feedback arc sets (electrical engineering)
- linear ordering problem (comparing economic sectors)
- acyclic subgraph problem mathematics

Ranking to minimize upsets

How bad can it get?



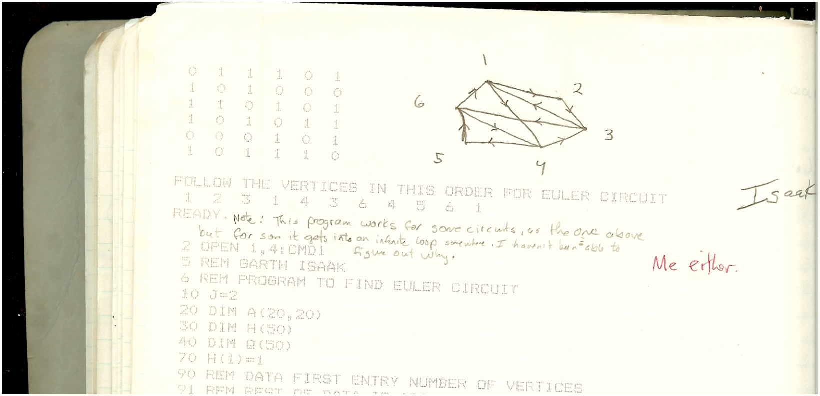
- Can ranking 100 players get 35 all wrong?
- YES
- We don't know for 99 players
- linear programming bound: $3n - 1 - \lfloor \log_2 n \rfloor$ for n players ranked wrong
- Similar bad things with ranking by scores
- Current research by graduate student Matt Prudente and former student Darren Narayan

Two problems

Is a list of numbers a score sequence?

Can we rank a tournament with at most k upsets

- One is 'easy'
  nice necessary and sufficient conditions
  - fast algorithm
  - nice certificate when answer is 'no'
- One is 'hard'
  - 'Probably' no fast algorithms
  - 'Probably' no nice theorems exist
  - Not 'we are too dumb to find them'
    but in our model of mathematics and computation
    they can not exist




```
 0  1  1  1  0  1
 1  0  1  0  0  0
 1  1  0  1  0  1
 1  0  1  0  1  1
 0  0  0  1  0  1
 1  0  1  1  1  0
FOLLOW THE VERTICES IN THIS ORDER FOR EULER CIRCUIT
 1  2  3  1  4  3  6  4  5  6  1
READY.
```

Note: This program works for some circuits, as the one above but for some it gets into an infinite loop somewhere. I haven't been able to figure out why.

```
2 OPEN 1,4:CMD1
5 REM GARTH ISAAK
6 REM PROGRAM TO FIND EULER CIRCUIT
10 J=2
20 DIM A(20,20)
30 DIM H(50)
40 DIM Q(50)
70 H(1)=1
90 REM DATA FIRST ENTRY NUMBER OF VERTICES
91 REM REST OF DATA
```

$1 Million dollar question

(One of 7 Clay millenium problems)

Is $P = NP$?

- P = polynomial (fast efficient algorithm)
- NP = nondeterministic Turing machine polynomial
- NP - can verify a solution from ‘god’ efficiently
- NP-complete: at least as ‘hard’ as other NP problems
- To show $P = NP$ find a fast algorithm NP-complete problem e.g. for Travelling Salesperson Problem
- To show $P \neq NP$ prove that no efficient algorithm exists for NP-complete problems

- Ranking to minimize upsets is NP-complete
  - ‘Probably’ no nice theorems can exist
  - If $P \neq NP$ *and* problem is *NP*-complete no such theorem can exist
  - Problems like these arise in applications. Use linear programming to get bounds and approximations
- Recognizing score sequences has nice theorems and fast algorithms

Is the following sequence of 25 numbers a score sequence?

$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

- Want necessary and sufficient conditions
  - We heard about necessary and sufficient conditions in yesterday's talk
- Can certify 'YES' by producing a tournament with these scores
- Want 'nice' certificate is answer is NO

For 7 players there are $\frac{7(7-1)}{2} = 21$ games in a round robin tournament

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Which of the following are score sequences for a tournament with 7 players?

$(7, 5, 4\frac{1}{3}, 4, 2\frac{3}{7}, 0, -2)$

$(5, 4, 3, 3, 3, 1, 0)$

$(3, 3, 3, 3, 3, 3, 3)$

$(6, 6, 4, 2, 1, 1, 1)$

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Which of the following are score sequences for a tournament with 7 players?

$(7, 5, 4\frac{1}{3}, 4, 2\frac{3}{7}, 0, -2)$ NO - Scores must be non-negative integers

$(5, 4, 3, 3, 3, 1, 0)$

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$(5, 4, 3, 3, 3, 1, 0)$ NO - Total number of wins must be $21 = \frac{7 \cdot 6}{2}$

$(3, 3, 3, 3, 3, 3, 3)$

$(6, 6, 4, 2, 1, 1, 1)$

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$(5, 4, 3, 3, 3, 1, 0)$ NO - Total number of wins must be $21 = \frac{7 \cdot 6}{2}$

$(3, 3, 3, 3, 3, 3, 3)$

$(6, 6, 4, 2, 1, 1, 1)$ NO - two teams cannot win all of their games

For 7 players there are $\dfrac{7(7-1)}{2} = 21$ games in a round robin tournament

Which of the following are score sequences for a tournament with 7 players?

$(7, 5, 4\frac{1}{3}, 4, 2\frac{3}{7}, 0, -2)$ NO - Scores must be non-negative integers

$(5, 4, 3, 3, 3, 1, 0)$ NO - Total number of wins must be $21 = \frac{7 \cdot 6}{2}$

$(3, 3, 3, 3, 3, 3, 3)$ YES

$(6, 6, 4, 2, 1, 1, 1)$ NO - two teams cannot win all of their games

- Can the two lowest scores both be 0?
- Can the three lowest scores be $1, 1, 0$?
- Can the four lowest scores be $2, 1, 1, 1$?

- Can the two lowest scores both be 0?
  NO - they play each other
- Can the three lowest scores be $1, 1, 0$?
  NO - they play 3 games
- Can the four lowest scores be $2, 1, 1, 1$?
  NO - they play 6 games
- $\vdots$
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- $\vdots$
- Seven teams play $21 = \frac{7(7-1)}{2}$ games so the lowest seven scores must sum to at least 21
- $\vdots$
- k teams play $\frac{k(k-1)}{2}$ games so they have at least that many wins

Landau (1951) considered tournaments in the context of pecking order in poultry populations

Necessary condition

The number of wins for any set of teams must be as large as the number of games played between those teams

Landau's Theorem:

This necessary condition is also sufficient

That is: Not a score sequence $\Rightarrow$ there is a set of teams violating these obvious conditions

The sequence
$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$
can be checked by hand in a few minutes. It is not a score sequence

The sequence
$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$
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$22, 22, 20, 20, 20, 20, 19, 19, 18, 16, 16, 13, 13, 10, 8, 6, 6, 6, 5, 4, 4, 4, 3, 3, 3$

Not a score sequence

Last 10 teams have 44 wins in $45 = \frac{10 \cdot 9}{2}$ games

Landau's Theorem:
A sequence $(s_1, s_2, \ldots, s_n)$ of non-negative integers is a score sequence of a round-robin tournament if and only if

$$\sum_{i \in I} s_i \geq \binom{|I|}{2} \text{ for any } I \subseteq \{1, 2, \ldots, n\}$$

$$\text{with equality when } I = \{1, 2, \ldots, n\}$$

There are many proofs: by induction ...

Try modeling score sequences using equations

Shadow prices (Lagrange multipliers) plus ... yield a proof

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What if we allow ties?

What if the score is 3 points for a win, 1 for a tie and 0 for a loss (world cup soccer scoring)?

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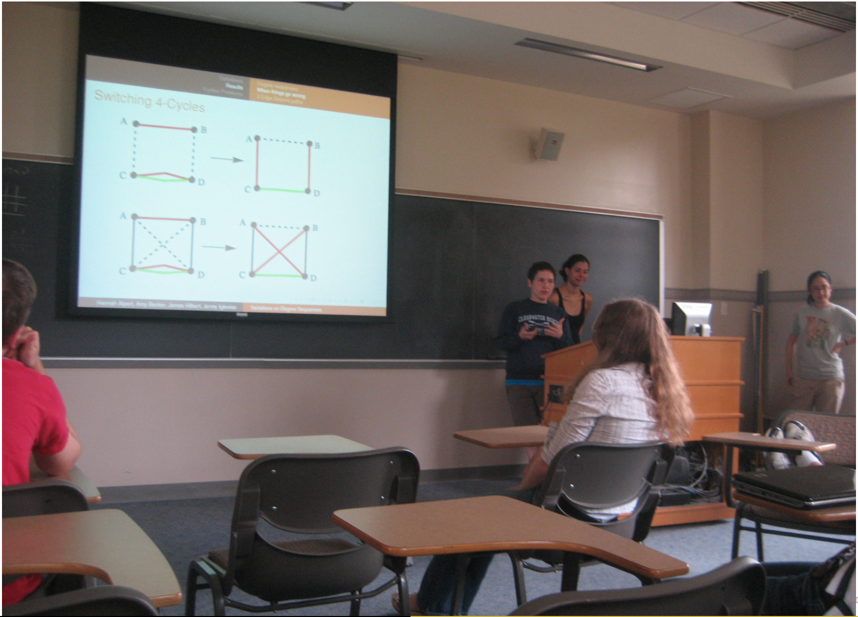
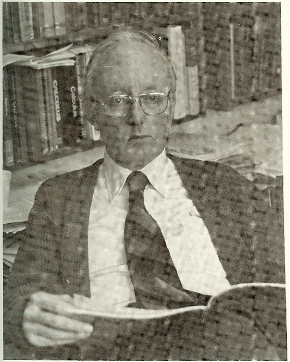
What if we allow ties?
This problem is not solved
Summer 2010 REU, Amy, Hannah, Jenny used a discrete tomography problem to show its NP-hard for boy/girl version

What if the score is 3 points for a win, 1 for a tie and 0 for a loss (world cup soccer scoring)?
This problem is not solved


Switching 4-Cycles
A
B
C
D
A
B
C
D
A
B
C
D
A
B
C
D

1



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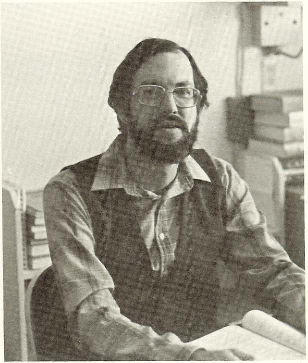

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1. Ron Headings helps Herman Bubbert with his homework.
2. Arnold Wedel, Math Professor.
3. Richard Rempel, Math Professor.

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