1. Let $W$ be a set of men and $M$ a set of women (with the same number of men and women, $|W| = |M| = n$) and $E$ a set of pairs $(w, m)$ with $w \in W$ and $m \in M$.

If there are subsets $R \subseteq W$ and $S \subseteq M$ such that $|R| + |S| < n$ and every pair in $E$ contains at least one member of $R \cup S$ (that is, for each $(w, m) \in E$ either $w \in R$ or $m \in S$ or both), then there is no matching of the men and women with each pair from $E$. The marriage theorem shows that the converse also holds: if there is no matching of the men and women then there are $R$ and $S$ as described in the previous sentence.

Another condition is as follows: If there is a set $T$ of women who 'like' strictly less than $|T|$ men then there is no matching of the men and women. More formally, if there is $T \subseteq W$ such that $|\{(w, m) \in E \text{ for some } w \in T\}| < |T|$ then there is no matching of the men and women. Use the marriage theorem to prove that the converse also holds: if there is no matching of men and women then there is a set $T$ as described in the previous sentence.

If there is no matching then by the marriage theorem there are subsets $R \subseteq W$ and $S \subseteq M$ such that $|R| + |S| < n$ and for each $(w, m) \in E$ either $w \in R$ or $m \in S$ or both. So, if $(w, m) \in E$ and $w \in W - R$ then $m \in S$. Thus $\{(w, m) \in E \text{ for some } w \in W - R\} \subseteq S$. Then, using $|S| < n - |R|$ and $|W - R| = n - |R|$ we have $|\{(w, m) \in E \text{ for some } w \in W - R\}| \leq |S| < n - |R| = |W - R|$ and so $T = W - R$ give the desired set.

2. Prove by induction that the Fibonacci numbers satisfy the following formula:

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$

We prove that the formula is correct using mathematical induction. Since $F_0 = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^0 + \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^0 = \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} = 0$ and $F_1 = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^1 + \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^1 = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2} \right) = \frac{1}{\sqrt{5}} \cdot 1$ the formula holds for $n = 0$ and $n = 1$. For $n \geq 2$, by induction

$$F_n = F_{n-1} + F_{n-2}$$

$$= \left[ \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n-1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n-1} \right] + \left[ \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n-2} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n-2} \right]$$

$$= \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^2 \left( \frac{1 + \sqrt{5}}{2} \right)^{n-2} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^2 \left( \frac{1 - \sqrt{5}}{2} \right)^{n-2}$$

$$= \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

here we have also used $\frac{1 + \sqrt{5}}{2} + 1 = \frac{3 + \sqrt{5}}{2} = \frac{6 + 2\sqrt{5}}{4} = \frac{1 + 2\sqrt{5} + 5}{4} = \left( \frac{1 + \sqrt{5}}{2} \right)^2$ and similarly $\frac{1 - \sqrt{5}}{2} + 1 = \left( \frac{1 - \sqrt{5}}{2} \right)^2$. Hence by induction the formula holds for all $n = 0, 1, \ldots$. 
3. Prove by induction that \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \).

When \( n = 1 \) we have \( \sum_{i=1}^{1} i^2 = 1^2 = \frac{1(1+1)(2\cdot1+1)}{6} \) so the formula holds for \( n = 1 \). By induction we may assume \( \sum_{i=1}^{n-1} i^2 = \frac{(n-1)(n-1+1)(2(n-1)+1)}{6} = \frac{(n-1)(n)(2n-1)}{6} \). Then \( \sum_{i=1}^{n} i^2 = \frac{(\sum_{i=1}^{n-1} i^2) + n^2}{n(n+1)(2n+1)} = \frac{(n-1)(n)(2n-1)}{6} + \frac{6n^2}{6} = \frac{n(2n^2-3n+1+6n)}{6} = \frac{n(2n^2+3n+1)}{6} \). Hence, by induction the formula holds for all \( n = 1, 2, \ldots \).