A LOWER BOUND FOR HIGHER TOPOLOGICAL COMPLEXITY OF REAL PROJECTIVE SPACE

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Abstract. We obtain an explicit formula for the best lower bound for the higher topological complexity, $\mathrm{TC}_k(RP^n)$, of real projective space implied by mod 2 cohomology.

1. Main theorem

The notion of higher topological complexity, $\mathrm{TC}_k(X)$, of a topological space X was introduced in [2]. It can be thought of as one less than the minimal number of rules required to tell how to move consecutively between any k specified points of X. In [1], the study of $\mathrm{TC}_k(P^n)$ was initiated, where P^n denotes real projective space. Using $\mathbb{Z}_2$ coefficients for all cohomology groups, define $\mathrm{zcl}_k(X)$ to be the maximal number of elements in $\ker(\Delta^* : H^*(X)^{\otimes k} \to H^*(X))$ with nonzero product. It is standard that

$$\mathrm{TC}_k(X) \geqslant \mathrm{zcl}_k(X).$$

In [1], it was shown that

$$\mathrm{zcl}_k(P^n) = \max\{a_1 + \cdots + a_{k-1} : (x_1 + x_k)^{a_1} \cdots (x_{k-1} + x_k)^{a_{k-1}} \neq 0\}$$

in $\mathbb{Z}_2[x_1, \ldots, x_k]/(x_1^{n+1}, \ldots, x_k^{n+1})$. In Theorem 1.2 we give an explicit formula for $\mathrm{zcl}_k(P^n)$, and hence a lower bound for $\mathrm{TC}_k(P^n)$.

Our main theorem, 1.2, requires some specialized notation.

Definition 1.1. *If $n = \sum \varepsilon_j 2^j$ with $\varepsilon_j \in \{0, 1\}$ (so the numbers ε_j form the binary expansion of n), let*

$$Z_i(n) = \sum_{j=0}^{i} (1 - \varepsilon_j) 2^j,$$

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and let

$$S(n) = \{i : \varepsilon_i = \varepsilon_{i-1} = 1 \text{ and } \varepsilon_{i+1} = 0\}.$$

Thus $Z_i(n)$ is the sum of the 2-powers $\leqslant 2^i$ which correspond to the 0's in the binary expansion of n. Note that $Z_i(n) = 2^{i+1} - 1 - (n \bmod 2^{i+1})$. The i's in $S(n)$ are those that begin a sequence of two or more consecutive 1's in the binary expansion of n. Also, $\nu(n) = \max\{t : 2^t \text{ divides } n\}$.

Theorem 1.2. *For $n \geqslant 0$ and $k \geqslant 3$,*

$$\mathrm{zcl}_k(P^n) = kn - \max\{2^{\nu(n+1)} - 1, 2^{i+1} - 1 - k \cdot Z_i(n) : i \in S(n)\}. \quad (1.3)$$

It was shown in [1] that, if $2^e \leqslant n < 2^{e+1}$, then $\mathrm{zcl}_2(P^n) = 2^{e+1} - 1$, which follows immediately from our Theorem 1.6.

In Table 1, we tabulate $\mathrm{zcl}_k(P^n)$ for $1 \leqslant n \leqslant 17$ and $2 \leqslant k \leqslant 8$.

Table 1. Values of $\mathrm{zcl}_k(n)$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\mathrm{zcl}_2(n)$	1	3	3	7	7	7	7	15	15	15	15	15	15	15	15	31	31
$\mathrm{zcl}_3(n)$	2	6	6	12	14	14	14	24	26	30	30	30	30	30	30	48	50
$\mathrm{zcl}_4(n)$	3	8	9	16	19	21	21	32	35	40	41	45	45	45	45	64	67
$\mathrm{zcl}_5(n)$	4	10	12	20	24	28	28	40	44	50	52	60	60	60	60	80	84
$\mathrm{zcl}_6(n)$	5	12	15	24	29	35	35	48	53	60	63	72	75	75	75	96	101
$\mathrm{zcl}_7(n)$	6	14	18	28	34	42	42	56	62	70	74	84	90	90	90	112	118
$\mathrm{zcl}_8(n)$	7	16	21	32	39	48	49	64	71	80	85	96	103	105	105	128	135

The smallest value of n for which two values of i are significant in (1.3) is $n = 102 = 2^6 + 2^5 + 2^2 + 2^1$. With $i = 2$, we have $7 - k$ in the max, while with $i = 6$, we have $127 - 25k$. Hence

$$\mathrm{zcl}_k(P^{102}) = 102k - \begin{cases} 127 - 25k & 2 \leqslant k \leqslant 5 \\ 7 - k & 5 \leqslant k \leqslant 7 \\ 0 & 7 \leqslant k. \end{cases}$$

For all k and n, $\mathrm{TC}_k(P^n) \leqslant kn$ for dimensional reasons ([1, Prop 2.2]). Thus we obtain a sharp result $\mathrm{TC}_k(P^n) = kn$ whenever $\mathrm{zcl}_k(P^n) = kn$. Corollary 3.4 tells exactly when this is true. Here is a simply-stated partial result.

Proposition 1.4. *If n is even, then $\mathrm{TC}_k(P^n) = kn$ for $k \geqslant 2^{\ell+1} - 1$, where ℓ is the length of the longest string of consecutive 1's in the binary expansion of n.*

Proof. We use Theorem 1.2. We need to show that if $i \in S(n)$ begins a string of j 1's with $j \leqslant \ell$, then $2^{i+1} - 1 \leqslant (2^{\ell+1} - 1)Z_i(n)$. If $j < \ell$, then $Z_i(n) \geqslant 2^{i-j} + 1$, and the desired inequality reduces to $2^{i+1} + 2^{i-j} \leqslant 2^{\ell+1+i-j} + 2^{\ell+1}$, which is satisfied since $2^{\ell+1+i-j}$ is strictly greater than both 2^{i+1} and 2^{i-j}.

If $j = \ell$, then
$$Z_i(n) \geqslant 1 + \sum_{\alpha} 2^{i+1-\alpha(\ell+1)},$$
where α ranges over all positive integers such that $i + 1 - \alpha(\ell + 1) > 0$. This reflects the fact that the binary expansion of n has a 0 starting in the $2^{i-\ell}$ position and at least every $\ell + 1$ positions back from there, and also a 0 at the end since n is even. The desired inequality follows easily from this. ■

Theorem 1.2 shows that $\mathrm{zcl}_k(P^n) < kn$ when n is odd. In the next proposition, we give complete information about when $\mathrm{zcl}_k(n) = kn$ if $k = 3$ or 4.

Proposition 1.5. *If $k = 3$ or 4, then $\mathrm{zcl}_k(P^n) = kn$ if and only if n is even and the binary expansion of n has no consecutive 1's.*

Proposition 1.5 follows easily from Theorem 1.2 and the fact that if $i \in S(n)$, then $Z_i(n) \leqslant 2^{i-1} - 1$.

The following recursive formula for $\mathrm{zcl}_k(P^n)$, which is interesting in its own right, is central to the proof of Theorem 1.2. It will be proved in Section 2.

Theorem 1.6. *Let $n = 2^e + d$ with $0 \leqslant d < 2^e$, and $k \geqslant 2$. If $z_k(n) = \mathrm{zcl}_k(P^n)$, then*
$$z_k(n) = \min(z_k(d) + k2^e, (k-1)(2^{e+1} - 1)), \text{ with } z_k(0) = 0.$$
Equivalently, if $g_k(n) = kn - \mathrm{zcl}_k(P^n)$, then
$$g_k(n) = \max(g_k(d), kn - (k-1)(2^{e+1} - 1)), \text{ with } g_k(0) = 0. \quad (1.7)$$

We now use Theorem 1.6 to prove Theorem 1.2.

Proof of Theorem 1.2. We will prove that $g_k(n)$ of Theorem 1.6 satisfies
$$g_k(n) = \max\{2^{\nu(n+1)} - 1, 2^{i+1} - 1 - kZ_i(n) : i \in S(n)\} \quad (1.8)$$

if $k \geqslant 3$, which is clearly equivalent to Theorem 1.2. The proof is by induction, using the recursive formula (1.7) for $g_k(n)$. Let $n = 2^e + d$ with $0 \leqslant d < 2^e$.

Case 1: $d = 0$. Then $n = 2^e$ and by (1.7) we have $g_k(n) = \max(0, k2^e - (k-1)(2^{e+1} - 1))$. If $e = 0$, this equals 1, while if $e > 0$, it equals 0, since $k \geqslant 3$. These agree with the claimed answer $2^{\nu(n+1)} - 1$, since $S(2^e) = \varnothing$.

Case 2: $0 < d < 2^{e-1}$. Here $\nu(n+1) = \nu(d+1)$, $S(n) = S(d)$, and $Z_i(n) = Z_i(d)$ for any $i \in S(d)$. Substituting (1.8) with n replaced by d into (1.7), we obtain

$$g_k(n) = \max\{2^{\nu(n+1)}-1, 2^{i+1}-1-kZ_i(n) : i \in S(n), kn-(k-1)(2^{e+1}-1)\}.$$

We will be done once we show that $kn - (k-1)(2^{e+1} - 1)$ is $\leqslant$ one of the other entries, and so may be omitted. If i is the largest element of $S(n)$, we will show that $kn - (k-1)(2^{e+1} - 1) \leqslant 2^{i+1} - 1 - kZ_i(n)$, i.e.,

$$kn_i \leqslant (k-1)(2^{e+1} - 2^{i+1}), \tag{1.9}$$

where $n_i = n - (2^{i+1} - 1 - Z_i(n))$ is the sum of the 2-powers in n which are greater than 2^i. The largest of these is 2^e, and no two consecutive values of i appear in this sum, hence $n_i \leqslant \sum 2^j$, taken over $j \equiv e$ (2) and $i + 2 \leqslant j \leqslant e$. If $k = 3$, (1.9) is true because the above description of n_i implies that $3n_i \leqslant 2(2^{e+1} - 2^{i+1})$, while for larger k, it is true since $\frac{k}{k-1} < \frac{3}{2}$. If $S(n)$ is empty, then $kn - (k-1)(2^{e+1} - 1) \leqslant 2^{\nu(n+1)} - 1$ by a similar argument, since $n \leqslant 2^e + 2^{e-2} + 2^{e-4} + \cdots$, so $3n \leqslant 2(2^{e+1} - 1)$, and values of $k > 3$ follow as before.

Case 3: $d \geqslant 2^{e-1}$. If $e - 1 \in S(d)$, then it is replaced by e in $S(n)$, while other elements of $S(d)$ form the rest of $S(n)$. If $e - 1 \notin S(d)$, then $S(n) = S(d) \cup \{e\}$. If $i \in S(n) - \{e\}$, then $Z_i(n) = Z_i(d)$, so its contribution to the set of elements whose max equals $g_k(n)$ is $2^{i+1} - 1 - kZ_i(n)$, as desired. For $i = e$, the claimed term is $2^{e+1} - 1 - kZ_e(n) = kn - (k-1)(2^{e+1} - 1)$, which is present by the induction from (1.7). If $e-1 \in S(d)$, then the $i = e-1$ term in the max for $g_k(d)$ is $2^e - 1 - kZ_i(n)$ and contributes to $g_k(n)$ less than the term described in the preceding sentence, and hence cannot contribute to the max. The $2^{\nu(n+1)} - 1$ term is obtained from the induction since $\nu(n+1) = \nu(d+1)$. ■

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2. Recursive formulas

In this section, we prove Theorem 1.6 and the following variant.

Theorem 2.1. *Let $n = 2^e + d$ with $0 \leqslant d < 2^e$, and $k \geqslant 2$. If $h_k(n) = \mathrm{zcl}_k(P^n) - (k-1)n$, then*

$$h_k(n) = \min(h_k(d) + 2^e, (k-1)(2^{e+1} - 1 - n)), \text{ with } h_k(0) = 0. \quad (2.2)$$

Proof of Theorems 1.6 and 2.1. It is elementary to check that the formulas for z_k, g_k, and h_k are equivalent to one another. We prove (2.2). We first look for nonzero monomials in $(x_1 + x_k)^{a_1} \cdots (x_{k-1} + x_k)^{a_{k-1}}$ of the form $x_1^n \cdots x_{k-1}^n x_k^\ell$ with $\ell \leqslant n$. Letting $a_i = n + b_i$, the analogue of $h_k(n)$ for such monomials is given by

$$\widetilde{h}_k(n) = \max\{\sum_{i=1}^{k-1} b_i : \tbinom{n+b_1}{n} \cdots \tbinom{n+b_{k-1}}{n} \text{ is odd and } \sum_{i=1}^{k-1} b_i \leqslant n\}, \quad (2.3)$$

since $\sum b_i$ is the exponent of x_k. We will begin by proving

$$\widetilde{h}_k(n) = \min(\widetilde{h}_k(d) + 2^e, (k-1)(2^{e+1} - 1 - n)). \quad (2.4)$$

For a nonzero integer m, let $Z(m)$ (resp. $P(m)$) denote the set of 2-powers corresponding to the 0's (resp. 1's) in the binary expansion of m, with $Z(0) = P(0) = \varnothing$. By Lucas's Theorem, $\tbinom{n+b_i}{n}$ is odd iff $P(b_i) \subset Z(n)$. Note that the integers $Z_i(n)$ considered earlier are sums of elements of subsets of $Z(n)$.

For a multiset S, let $\|S\|$ denote the sum of its elements, and let

$$\phi(S, n) = \max\{\|T\| \leqslant n : T \subset S\}.$$

Note that $\|Z(n)\| = 2^{\lg(n)+1} - 1 - n$, where $\lg(n) = \lfloor \log_2(n) \rfloor$, ($\lg(0) = -1$). Let $Z(n)^j$ denote the multiset consisting of j copies of $Z(n)$, and let

$$m_j(n) = \phi(Z(n)^j, n).$$

Then, from (2.3), we obtain the key equation $\widetilde{h}_k(n) = m_{k-1}(n)$. Thus (2.4) follows from Lemma 2.5 below.

Lemma 2.5. *If $n = 2^e + d$ with $0 \leqslant d < 2^e$, and $j \geqslant 1$, then*

$$m_j(n) = \min(m_j(d) + 2^e, j(2^{e+1} - 1 - n)).$$

Proof. The result is clear if $j = 1$ since $2^{e+1} - 1 - n < 2^e$, so we assume $j \geqslant 2$. Let $S \subset Z(d)^j$ satisfy $\|S\| = m_j(d)$.

First assume $d < 2^{e-1}$. Then $2^{e-1} \in Z(n)$. Let $T = S \cup \{2^{e-1}, 2^{e-1}\}$. No other subset of $Z(n)^j$ can have larger sum than T which is $\leqslant n$ due to maximality of $\|S\|$ and the fact that the 2-powers in $Z(n)^j - Z(d)^j$ are larger than those in $Z(d)^j$. Thus $m_j(n) = m_j(d) + 2^e$ in this case, and this is $\leqslant j(2^{e+1} - 1 - n) = \|Z(n)^j\|$.

If, on the other hand, $d \geqslant 2^{e-1}$, then $Z(d)^j = Z(n)^j$. If $\|Z(n)^j - S\| < 2^e$, then let $T = Z(n)^j$ with $\|T\| = j(2^{e+1} - 1 - n)$, as large as it could possibly be, and less than $m_j(d) + 2^e$. Otherwise, since any multiset of 2-powers whose sum is $\geqslant 2^e$ has a subset whose sum equals 2^e, we can let $T = S \cup V$, where V is a subset of $Z(n)^j - S$ with $\|V\| = 2^e$. As before, no subset of $Z(n)^j$ can have size greater than that.

■

Now we wish to consider more general monomials. We claim that for any multiset S and positive integers m and n,

$$\phi(Z(m-1) \cup S, n) \leqslant \phi(Z(m) \cup S, n) + 1. \tag{2.6}$$

This follows from the fact that subtracting 1 from m can affect $Z(m)$ by adding 1, or changing $1, 2, \dots, 2^{t-1}$ to 2^t. These changes cannot add more than 1 to the largest subset of size $\leqslant n$. We show now that this implies that $h_k(n) = m_{k-1}(n) = \widetilde{h}_k(n)$, and hence (2.2) follows from (2.4).

Suppose that $x_1^{n-\varepsilon_1} \cdots x_{k-1}^{n-\varepsilon_{k-1}} x_k^{\ell}$ with $\varepsilon_i \geqslant 0$ and $\ell \leqslant n$ is a nonzero monomial in the expansion of $(x_1+x_k)^{n+b_1} \cdots (x_{k-1}+x_k)^{n+b_{k-1}}$. We wish to show that $\sum b_i \leqslant m_{k-1}(n)$. It follows from (2.6) that

$$\phi\Big(\bigcup_{i=1}^{k-1} Z(n-\varepsilon_i), n\Big) \leqslant \phi(Z(n)^{k-1}, n) + \sum \varepsilon_i = m_{k-1}(n) + \sum \varepsilon_i.$$

The odd binomial coefficients $\binom{n+b_i}{n-\varepsilon_i}$ imply that $P(b_i + \varepsilon_i) \subset Z(n - \varepsilon_i)$. Thus

$$\phi\Big(\bigcup_{i=1}^{k-1} P(b_i + \varepsilon_i), n\Big) \leqslant m_{k-1}(n) + \sum \varepsilon_i. \tag{2.7}$$

Since $\|P(b_i + \varepsilon_i)\| = b_i + \varepsilon_i$ and $\sum(b_i + \varepsilon_i) \leqslant n$, the left hand side of (2.7) equals $\sum(b_i + \varepsilon_i)$, hence $\sum b_i \leqslant m_{k-1}(n)$, as desired. ■

3. Examples and comparisons

In this section, we examine some special cases of our results (in Propositions 3.1 and 3.5) and make comparisons with some work in [1].

The numbers $z_3(n) = \mathrm{zcl}_3(P^n)$ are 1 less than a sequence which was listed by the author as A290649 at [3] in August 2017. They can be characterized as in Proposition 3.1, the proof of which is a straightforward application of the recursive formula

$$z_3(2^e + d) = \min(z_3(d) + 3 \cdot 2^e, 2(2^{e+1} - 1)) \text{ for } 0 \leqslant d < 2^e,$$

from Theorem 1.6.

Proposition 3.1. *For $n \geqslant 0$, $\mathrm{zcl}_3(n)$ is the largest even integer z satisfying $z \leqslant 3n$ and $\binom{z+1}{n} \equiv 1$ (2).*

We have not found similar characterizations for $z_k(n)$ when $k > 3$.

In [1, Thm 5.7], it is shown that our $g_k(n)$ in Theorem 1.6 is a decreasing function of k, and achieves a stable value of $2^{\nu(n+1)} - 1$ for sufficiently large k. They defined $s(n)$ to be the minimal value of k such that $g_k(n) = 2^{\nu(n+1)} - 1$. We obtain a formula for the precise value of $s(n)$ in our next result.

Let $S'(n)$ denote the set of integers i such that the 2^i position begins a string of two or more consecutive 1's in the binary expansion of n which stops prior to the 2^0 position. For example, $S'(187) = \{5\}$ since its binary expansion is 10111011.

Proposition 3.2. *Let $s(-)$ and $S'(-)$ be the functions just described. Then*

$$s(n) = \begin{cases} 2 & \text{if } n+1 \text{ is a 2-power} \\ 3 & \text{if } n+1 \text{ is not a 2-power and } S'(n) = \varnothing \\ \max\left\{\left\lceil \dfrac{2^{i+1} - 2^{\nu(n+1)}}{Z_i(n)} \right\rceil : i \in S'(n)\right\} & \text{otherwise.} \end{cases}$$

Proof. It is shown in [1, Expl 5.8] that $g_k(2^v - 1) = 2^v - 1$ for all $k \geqslant 2$, hence $s(2^v - 1) = 2$. This also follows readily from (1.7).

If the binary expansion of n has a string of $i + 1$ 1's at the end and no other consecutive 1's (so that $S(n) = \{i\}$ in (1.3)), then $Z_i(n) = 0$. Thus by (1.8) $g_k(n) = 2^{i+1} - 1 = 2^{\nu(n+1)} - 1$ for $k \geqslant 3$. If $n \neq 2^{i+1} - 1$, then $s(n) = 3$, since $g_2(n) > 2^{i+1} - 1$.

Now assume $S'(n)$ is nonempty. By (1.8), $s(n)$ is the smallest k such that

$$2^{i+1} - 1 - kZ_i(n) \leqslant 2^{\nu(n+1)} - 1 \tag{3.3}$$

for all $i \in S(n)$, which easily reduces to the claimed value. Note that if the string of 1's beginning at position 2^i goes all the way to the end, then (3.3) is satisfied; this case is omitted from $S'(n)$ in the theorem, because it would yield 0/0. ■

The following corollary is immediate.

Corollary 3.4. *If n is even and*

$$k \geqslant \max\{3, \left\lceil \frac{2^{i+1}-1}{Z_i(n)} \right\rceil : i \in S(n)\},$$

then $\mathrm{TC}_k(P^n) = kn$. *These are the only values of n and k for which* $\mathrm{zcl}_k(P^n) = kn$.

In [1, Def 5.10], a complicated formula was presented for numbers $r(n)$, and in [1, Thm 5.11], it was proved that $s(n) \leqslant r(n)$. It was conjectured there that $s(n) = r(n)$. However, comparison of the formula for $s(n)$ established in Proposition 3.2 with their formula for $r(n)$ showed that there are many values of n for which $s(n) < r(n)$. The first is $n = 50$, where we prove $s(50) = 5$, whereas their $r(50)$ equals 7. Apparently their computer program did not notice that

$$(x_1 + x_5)^{63}(x_2 + x_5)^{63}(x_3 + x_5)^{62}(x_4 + x_5)^{62}$$

contains the nonzero monomial $x_1^{50}x_2^{50}x_3^{50}x_4^{50}x_5^{50}$, showing that our $z_5(50) = 250$ and $g_5(50) = 0$, so $s(50) \leqslant 5$.

In Table 2, we present a table of some values of $s(-)$, omitting $s(2^v - 1) = 2$ and $s(2^v) = 3$ for $v > 0$.

Table 2. Some values of $s(n)$

n	5	6	9	10	11	12	13	14	17-21	22	23	24	25	26	27	28	29	30
$s(n)$	3	7	3	3	3	5	7	15	3	7	3	5	5	7	7	11	15	31

In [1], there seems to be particular interest in $\mathrm{TC}_k(P^{3\cdot 2^e})$. We easily read off from Theorem 1.2 the following result.

Proposition 3.5. *For $k \geqslant 2$ and $e \geqslant 1$, we have*

$$\mathrm{zcl}_k(P^{3\cdot 2^e}) = \begin{cases} (k-1)(2^{e+2}-1) & \textit{if } (e=1, k \leqslant 6) \textit{ or } (e \geqslant 2, k \leqslant 4) \\ k \cdot 3 \cdot 2^e & \textit{otherwise.} \end{cases}$$

This shows that the estimate $s(3 \cdot 2^e) \leqslant 5$ for $e \geqslant 2$ in [1] is sharp.

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