v_1-PERIODIC 2-EXPONENTS OF $SU(2^e)$ AND $SU(2^e+1)$

DONALD M. DAVIS

Abstract. We determine precisely the largest v_1-periodic homotopy groups of $SU(2^e)$ and $SU(2^e+1)$. This gives new results about the largest actual homotopy groups of these spaces. Our proof relies on results about 2-divisibility of restricted sums of binomial coefficients times powers proved by the author in a companion paper.

1. Main result

The 2-primary v_1-periodic homotopy groups, $v_1^{-1}\pi_i(X)$, of a topological space X are a localization of a first approximation to its 2-primary homotopy groups. They are roughly the portion of $\pi_*(X)$ detected by 2-local K-theory.([2]) If X is a sphere or compact Lie group, each v_1-periodic homotopy group of X is a direct summand of some actual homotopy group of X.([6])

Let

$$T_j(k)=\sum_{\text{odd } i}\binom{j}{i}i^k$$

denote one family of partial Stirling numbers. In [5], the author obtained several results about $\nu(T_j(k))$, where $\nu(n)$ denotes the exponent of 2 in n. Some of those will be used in this paper, and will be restated as needed.

Let

$$\mathbf{e}(k,n)=\min(\nu(T_j(k)):\ j\ge n).$$

It was proved in [1, 1.1] (see also [7, 1.4]) that $v_1^{-1}\pi_{2k}(SU(n))$ is isomorphic to $\mathbb{Z}/2^{\mathbf{e}(k,n)-\epsilon}$ direct sum with possibly one or two $\mathbb{Z}/2$'s. Here $\epsilon=0$ or 1, and $\epsilon=0$ if n is odd or if $k\equiv n-1 \bmod 4$, which are the only cases required here.

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Let

$$s(n)=n-1+\nu([n/2]!).$$

It was proved in [8] that $\mathbf{e}(n-1,n)\ge s(n)$. Let

$$\overline{\mathbf{e}}(n)=\max(\mathbf{e}(k,n):k\in\mathbb{Z}).$$

Thus $\overline{\mathbf{e}}(n)$ is what we might call the v_1-periodic 2-exponent of $SU(n)$. Then clearly

$$s(n)\le \mathbf{e}(n-1,n)\le \overline{\mathbf{e}}(n), \tag{1.1}$$

and calculations suggest that both of these inequalities are usually quite close to being equalities. In [4, p.22], a table is given comparing the numbers in (1.1) for $n\le 38$.

Our main theorem verifies a conjecture of [4] regarding the values in (1.1) when $n=2^e$ or 2^e+1.

Theorem 1.2.

a. *If* $e\ge 3$, *then* $\mathbf{e}(k,2^e)\le 2^e+2^{e-1}-1$ *with equality occurring iff* $k\equiv 2^e-1 \mod 2^{2^{e-1}+e-1}$.

b. *If* $e\ge 2$, *then* $\mathbf{e}(k,2^e+1)\le 2^e+2^{e-1}$ *with equality occurring iff* $k\equiv 2^e+2^{2^{e-1}+e-1} \mod 2^{2^{e-1}+e}$.

Thus the values in (1.1) for $n=2^e$ and 2^e+1 are as in Table 1.3.

Table 1.3. *Comparison of values*

n	$s(n)$	$\mathbf{e}(n-1,n)$	$\overline{\mathbf{e}}(n)$
2^e	$2^e+2^{e-1}-2$	$2^e+2^{e-1}-1$	$2^e+2^{e-1}-1$
2^e+1	$2^e+2^{e-1}-1$	$2^e+2^{e-1}-1$	2^e+2^{e-1}

Note that $\overline{\mathbf{e}}(n)$ exceeds $s(n)$ by 1 in both cases, but for different reasons. When $n=2^e$, the largest value occurs for $k=n-1$, but is 1 larger than the general bound established in [8]. When $n=2^e+1$, the general bound for $\mathbf{e}(n-1,n)$ is sharp, but a larger group occurs when $n-1$ is altered in a specific way. The numbers $\overline{\mathbf{e}}(n)$ are interesting, as they give what are probably the largest 2-exponents in $\pi_*(SU(n))$, and this is the first time that infinite families of these numbers have been computed precisely.

The homotopy 2-exponent of a topological space X, denoted $\exp_2(X)$, is the largest k such that $\pi_*(X)$ contains an element of order 2^k. An immediate corollary of Theorem 1.2 is

Corollary 1.4. *For $\epsilon \in \{0,1\}$ and $2^e+\epsilon \ge 5$,*

$$\exp_2(SU(2^e+\epsilon)) \ge 2^e + 2^{e-1} - 1 + \epsilon.$$

This result is 1 stronger than the result noted in [8, Thm 1.1].

Theorem 1.2 is implied by the following two results. The first will be proved in Section 2. The second is [5, Thm 1.1].

Theorem 1.5. *Let $e \ge 3$.*

i. *If $\nu(k) \ge e-1$, then $\nu(T_{2^e}(k)) = 2^e - 1$.*
ii. *If $j \ge 2^e$ and $\nu(k-(2^e-1)) \ge 2^{e-1}+e-1$, then $\nu(T_j(k)) \ge 2^e + 2^{e-1} - 1$.*
iii. *If $j \ge 2^e+1$ and $\nu(k-2^e) = 2^{e-1}+e-1$, then $\nu(T_j(k)) \ge 2^e + 2^{e-1}$.*

Theorem 1.6. ([5, 1.1]) *Let $e \ge 2$, $n = 2^e+1$ or 2^e+2, and $1 \le i \le 2^{e-1}$. There is a 2-adic integer $x_{i,n}$ such that for all integers x*

$$\nu(T_n(2^{e-1}x + 2^{e-1} + i)) = \nu(x - x_{i,n}) + n - 2.$$

Moreover

$$\nu(x_{i,2^e+1}) \begin{cases} = i & \text{if } i = 2^{e-2} \text{ or } 2^{e-1} \\ > i & \text{otherwise.} \end{cases}$$

and

$$\nu(x_{i,2^e+2}) \begin{cases} = i-1 & \text{if } 1 \le i \le 2^{e-2} \\ = i & \text{if } 2^{e-2} < i < 2^{e-1} \\ > i & \text{if } i = 2^{e-1}. \end{cases}$$

Regarding small values of e: [7, §8] and [5, Table 1.3] make it clear that the results stated in this section for $T_n(-)$, $\mathbf{e}(-,n)$ and $SU(n)$ are valid for small values of $n \ge 5$ but not for $n < 5$.

Proof that Theorems 1.5 and 1.6 imply Theorem 1.2. For part (a): Let $k \equiv 2^e - 1 \mod 2^{2^{e-1}+e-1}$. Theorem 1.5(ii) implies $\mathbf{e}(k, 2^e) \ge 2^e + 2^{e-1} - 1$, and 1.6 with $n = 2^e+2$, $i = 2^{e-1}-1$, and $\nu(x) \ge 2^{e-1}$ implies that equality is obtained for such k.

To see that $\mathbf{e}(k, 2^e) < 2^e + 2^{e-1} - 1$ if $k \not\equiv 2^e - 1 \mod 2^{2^{e-1}+e-1}$, we write $k = i + 2^{e-1}x + 2^{e-1}$ with $1 \le i \le 2^{e-1}$. We must show that for each k there exists some $j \ge 2^e$ for which $\nu(T_j(k)) < 2^e + 2^{e-1} - 1$.

- If $i = 2^{e-1}$, we use 1.5(i).
- If $i = 2^{e-2}$, we use 1.6 with $n = 2^e + 1$ if $\nu(x) < 2^{e-2}$ and with $n = 2^e + 2$ if $\nu(x) \ge 2^{e-2}$.
- For other values of i, we use 1.6 with $n = 2^e + 1$ if $\nu(x) \le i$ and with $n = 2^e + 2$ if $\nu(x) > i$, except in the excluded case $i = 2^{e-1} - 1$ and $\nu(x) > i$.

For part (b): Let $k \equiv 2^e + 2^{2^{e-1}+e-1} \mod 2^{2^{e-1}+e}$. Theorem 1.5(iii) implies $\mathbf{e}(k, 2^e + 1) \ge 2^e + 2^{e-1}$, and 1.6 with $n = 2^e + 2$, $i = 2^{e-1}$, and $\nu(x) = 2^{e-1}$ implies that equality is obtained for such k.

To see that $\mathbf{e}(k, 2^e) < 2^e + 2^{e-1}$ if $k \not\equiv 2^e + 2^{2^{e-1}+e-1} \mod 2^{2^{e-1}+e}$, we write $k = i + 2^{e-1}x + 2^{e-1}$ with $1 \le i \le 2^{e-1}$.

- If $i = 2^{e-1}$, we use 1.6 with $n = 2^e+1$ unless $\nu(x) = 2^{e-1}$, which case is excluded.
- If $i = 2^{e-2}$, we use 1.6 with $n = 2^e + 2$ if $\nu(x) = 2^{e-2}$ and with $n = 2^e + 1$ otherwise.
- If $1 \le i < 2^{e-2}$, we use 1.6 with $n = 2^e + 1$ if $\nu(x) = i - 1$ and with $n = 2^e + 2$ otherwise.
- If $2^{e-2} < i < 2^{e-1}$, we use 1.6 with $n = 2^e + 1$ if $\nu(x) = i$ and with $n = 2^e + 2$ otherwise.

■

The proof does not make it transparent why the largest value of $\mathbf{e}(k, n)$ occurs when $k = n - 1$ if $n = 2^e$ but not if $n = 2^e + 1$. The following example may shed some light. We consider the illustrative case $e = 4$. We wish to see why

- $\mathbf{e}(k, 16) \le 23$ with equality iff $k \equiv 15 \mod 2^{11}$, while
- $\mathbf{e}(k, 17) \le 24$ with equality iff $k \equiv 16 + 2^{11} \mod 2^{12}$.

Tables 1.7 and 1.8 give relevant values of $\nu(T_j(k))$.

Table 1.7. *Values of* $\nu(T_j(k))$ *relevant to* $\mathbf{e}(k,16)$

		j = 16	17	18	19
$\nu(k-15)$	7	24	**19**	20	20
	8	25	**20**	21	21
	9	26	**21**	22	22
	10	27	**22**	≥ 24	≥ 24
	11	≥ 29	≥ 24	**23**	23
	≥ 12	28	**23**	23	23

Table 1.8. *Values of* $\nu(T_j(k))$ *relevant to* $\mathbf{e}(k,17)$

		j = 17	18	19	20
$\nu(k-16)$	8	**20**	21	22	23
	9	**21**	22	23	24
	10	**22**	23	≥ 25	≥ 26
	11	≥ 24	**24**	24	25
	12	**23**	≥ 26	24	25
	≥ 13	**23**	25	24	25

The values $\mathbf{e}(k,16)$ and $\mathbf{e}(k,17)$ are the smallest entry in a row, and are listed in boldface. The tables only include values of k for which $\nu(k-(n-1))$ is rather large, as these give the largest values of $\nu(T_j(k))$. Larger values of j than those tabulated will give larger values of $\nu(T_j(n))$. Note how each column has the same general form, leveling off after a jump. This reflects the $\nu(x-x_{i,n})$ in Theorem 1.6. The prevalence of this behavior is the central theme of [5]. The phenomenon which we wish to illuminate here is how the bold values increase steadily until they level off in Table 1.7, while in Table 1.8 they jump to a larger value before leveling off. This is a consequence of the synchronicity of where the jumps occur in columns 17 and 18 of the two tables.

2. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. The proof uses the following results from [5].

Proposition 2.1. ([8, 3.4] *or* [5, 2.1]) *For any nonnegative integers n and k,*

$$\nu\Big(\sum_i \tbinom{n}{2i+1} i^k\Big) \ge \nu([n/2]!).$$

The next result is a refinement of Proposition 2.1. Here and throughout, $S(n,k)$ denote Stirling numbers of the second kind.

Proposition 2.2. ([5, 2.3]) *Mod* 4

$$\frac{1}{n!}\sum_i \tbinom{2n+\epsilon}{2i+b} i^k \equiv \begin{cases} S(k,n)+2nS(k,n-1) & \epsilon=0,\ b=0\\ (2n+1)S(k,n)+2(n+1)S(k,n-1) & \epsilon=1,\ b=0\\ 2nS(k,n-1) & \epsilon=0,\ b=1\\ S(k,n)+2(n+1)S(k,n-1) & \epsilon=1,\ b=1. \end{cases}$$

Proposition 2.3. ([5, 2.7]) *For $n\ge 3$, $j>0$, and $p\in\mathbb{Z}$,*

$$\nu\Big(\sum \tbinom{n}{2i+1}(2i+1)^p i^j\Big) \ge \max(\nu([\tfrac{n}{2}]!), n-\alpha(n)-j)$$

with equality if $n\in\{2^e+1, 2^e+2\}$ and $j=2^{e-1}$.

Other well-known facts that we will use are

$$(-1)^j j! S(k,j) = \sum \tbinom{j}{2i}(2i)^k - T_j(k) \tag{2.4}$$

and

$$S(k+i,k) \equiv \tbinom{k+2i-1}{k-1} \mod 2. \tag{2.5}$$

We also use that $\nu(n!) = n-\alpha(n)$, where $\alpha(n)$ denotes the binary digital sum of n, and that $\binom{m}{n}$ is odd iff, for all i, $m_i \ge n_i$, where these denote the ith digit in the binary expansions of m and n.

Proof of Theorem 1.5(i). Using (2.4), we have

$$T_{2^e}(2^{e-1}t) \equiv -S(2^{e-1}t, 2^e)(2^e)! \mod 2^{2^{e-1}t},$$

and we may assume $t\ge 2$ using the periodicity of $\nu(T_n(-))$ established in [3, 3.12]. But $S(2^{e-1}t,2^e) \equiv \binom{2^e t-2^{e+1}+2^e-1}{2^e-1} \equiv 1 \mod 2$. Since $\nu(2^e!) = 2^e-1 < 2^{e-1}t$, we are done. ■

Proof of parts (ii) and (iii) of Theorem 1.5. These parts follow from (a) and (b) below by letting $p = 2^e+\epsilon-1$ in (b), and adding.

(a) Let $\epsilon \in \{0,1\}$ and $n \ge 2^e+\epsilon$.

$$\nu(T_n(2^e+\epsilon-1)) \begin{cases} = 2^e+2^{e-1}-1 & \text{if } \epsilon=1 \text{ and } n=2^e+1 \\ \ge 2^e+2^{e-1}+\epsilon-1 & \text{otherwise.}\end{cases}$$

(b) Let $p \in \mathbb{Z}$, $n \ge 2^e$, and $\nu(m) \ge 2^{e-1}+e-1$. Then

$$\nu\Big(\sum \tbinom{n}{2i+1}(2i+1)^p\big((2i+1)^m-1\big)\Big) \begin{cases} = 2^e+2^{e-1}-1 & \text{if } n=2^e+1 \text{ and} \\ & \nu(m)=2^{e-1}+e-1 \\ \ge 2^e+2^{e-1} & \text{otherwise.}\end{cases}$$

First we prove (a). Using (2.4) and the fact that $S(k,j)=0$ if $k<j$, it suffices to prove

$$\nu\big(\sum \tbinom{n}{2i} i^{2^e+\epsilon-1}\big) \begin{cases} = 2^{e-1}-1 & \text{if } \epsilon=1 \text{ and } n=2^e+1 \\ \ge 2^{e-1} & \text{otherwise,}\end{cases}$$

and this is implied by Proposition 2.1 if $n \ge 2^e+4$. For $\epsilon=0$ and $2^e \le n \le 2^e+3$, by Proposition 2.2

$$\nu\big(\sum \tbinom{n}{2i} i^{2^e-1}\big) \ge 2^{e-1}-1+\min(1,\nu(S(2^e-1,2^{e-1}+\delta)))$$

with $\delta \in \{0,1\}$. The Stirling number here is easily seen to be even by (2.5).

Similarly $\nu(\sum \binom{2^e+1}{2i} i^{2^e}) = 2^{e-1}-1$ since $S(2^e,2^{e-1})$ is odd, and if $n-2^e \in \{2,3\}$, then $\nu\big(\sum \binom{n}{2i} i^{2^e}\big) \ge 2^{e-1}$ since $S(2^e,2^{e-1}+1)$ is even.

Now we prove part (b). The sum equals $\sum_{j>0} T_j$, where

$$T_j = 2^j \tbinom{m}{j} \sum_i \tbinom{n}{2i+1}(2i+1)^p i^j.$$

We show that $\nu(T_j) = 2^e+2^{e-1}-1$ if $n=2^e+1$, $j=2^{e-1}$, and $\nu(m)=2^{e-1}+e-1$, while in all other cases, $\nu(T_j) \ge 2^e+2^{e-1}$. If $j \ge 2^e+2^{e-1}$, we use the 2^j-factor. Otherwise, $\nu(\binom{m}{j}) = \nu(m)-\nu(j)$, and we use the first part of the max in Proposition 2.3 if $\nu(j) \ge e-1$, and the second part of the max otherwise. ■

References

[1] M. Bendersky and D. M. Davis, *2-primary v_1-periodic homotopy groups of* SU(n), Amer. J. Math. **114** (1991) 529–544.

[2] A. K. Bousfield, *On the 2-primary v_1-periodic homotopy groups of spaces*, Topology **44** (2005) 381-413.

[3] M. C. Crabb and K. Knapp, *The Hurewicz map on stunted complex projective spaces*, Amer Jour Math **110** (1988) 783-809.

[4] D. M. Davis, *Divisibility by 2 and 3 of certain Stirling numbers*, Integers **8** (2008) A56, 25 pp.

[5] ———, *Divisibility by 2 of partial Stirling numbers*, submitted, www.lehigh.edu/∼dmd1/partial5.pdf.

[6] D. M. Davis and M. Mahowald, *Some remarks on v_1-periodic homotopy groups*, London Math. Soc. Lect. Notes **176** (1992) 55–72.

[7] D. M. Davis and K. Potocka, *2-primary v_1-periodic homotopy groups of* $\mathrm{SU}(n)$ *revisited*, Forum Math **19** (2007) 783-822.

[8] D. M. Davis and Z. W. Sun, *A number-theoretic approach to homotopy exponents of $SU(n)$*, Jour Pure Appl Alg **209** (2007) 57-69.

Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA
E-mail address: dmd1@lehigh.edu