PROJECTIVE PRODUCT SPACES

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ABSTRACT. Let $\overline{n} = (n_1, \ldots, n_r)$. The quotient space $P_{\overline{n}} := S^{n_1} \times \cdots \times S^{n_r}/(\overline{x} \sim -\overline{x})$ is what we call a projective product space. We determine the integral cohomology ring $H^*(P_{\overline{n}})$ and the action of the Steenrod algebra on $H^*(P_{\overline{n}}; \mathbb{Z}_2)$. We give a splitting of $\Sigma P_{\overline{n}}$ in terms of stunted real projective spaces, and determine when S^{n_i} is a product factor of $P_{\overline{n}}$. We relate the immersion dimension and span of $P_{\overline{n}}$ to the much-studied sectioning question for multiples of the Hopf bundle over real projective spaces. We show that the immersion dimension of $P_{\overline{n}}$ depends only on $\min(n_i)$, $\sum n_i$, and r, and determine its precise value unless all $n_i \geq 10$. We also determine exactly when $P_{\overline{n}}$ is parallelizable.

1. INTRODUCTION

If $\overline{n} = (n_1, \ldots, n_r)$ with n_i positive integers, let

$$P_{\overline{n}} = S^{n_1} \times \cdots \times S^{n_r}/((x_1, \ldots, x_r) \sim (-x_1, \ldots, -x_r)),$$

where $x_i \in S^{n_i}$. This is a manifold of dimension $|\overline{n}| := n_1 + \cdots + n_r$, which we call a projective product space. If $\overline{n} = (n)$, then $P_{\overline{n}} = P^n$, the real projective space.

There is a 1-dimensional vector bundle $\xi_{\overline{n}}$ over $P_{\overline{n}}$ for which the k-fold Whitney sum $k\xi_{\overline{n}}$ has total space

$$S^{n_1} \times \cdots \times S^{n_r} \times \mathbb{R}^k/((x_1, \ldots, x_r, t_1, \ldots, t_k) \sim (-x_1, \ldots, -x_r, -t_1, \ldots, -t_k)),$$

and its sphere bundle clearly satisfies

$$(1.1) \qquad S(k\xi_{\overline{n}}) = P_{(n_1, \ldots, n_r, k-1)}.$$

Thus each space $P_{\overline{n}}$ can be built up iteratively as sphere bundles. For example, $P_{(n_1, n_2, n_3)} = S((n_3 + 1)\xi_{(n_1, n_2)})$ with $\xi_{(n_1, n_2)}$ the line bundle over $P_{(n_1, n_2)} = S((n_2 + 1)\xi_{n_1})$.

In Section 2, we determine the A-algebra $H^*(P_{\overline{n}}; \mathbb{Z}_2)$ (Theorem 2.1) and the algebra $H^*(P_{\overline{n}}; \mathbb{Z})$ (Theorem 2.15), and determine in Theorem 2.21 the ring $K^*(P_{\overline{n}})$. We show that $\Sigma P_{\overline{n}}$ splits as a wedge of desuspensions of stunted real projective spaces (Theorem 2.9), and determine when S^{n_i} is a product factor of $P_{\overline{n}}$ (Theorem 2.20). In Section 3, we relate the immersion dimension (Theorem 3.4) and span (Theorem 3.9) of $P_{\overline{n}}$ to results about sectioning multiples of the Hopf bundles over projective spaces, and we determine exactly when $P_{\overline{n}}$ is parallelizable (Theorem 3.12). In Section 4, we use known results for projective spaces to give some numerical results for the immersion dimension and span of $P_{\overline{n}}$. We give the precise numerical value of the immersion dimension of $P_{\overline{n}}$ unless all $n_i \geq 10$.

2. Cohomology of $P_{\overline{n}}$ and a splitting

The first property of the spaces $P_{\overline{n}}$ which we study is their mod-2 cohomology. Here and throughout, $\Lambda(-)$ denotes an exterior algebra, $\mathbb{Z}_2 = \mathbb{Z}/2$, A is the mod 2 Steenrod algebra, and $\mathrm{Sq} = \sum\limits_{n \geq 0} \mathrm{Sq}^n$.

Theorem 2.1. *Let $\overline{n} = (n_1, \ldots, n_r)$ with $n_1 \leq n_2 \leq \cdots \leq n_r$. If $n_1 < n_2$, or n_1 is odd, then there is an isomorphism of A-algebras*

$$H^*(P_{\overline{n}}; \mathbb{Z}_2) \approx \mathbb{Z}_2[y]/y^{n_1+1} \otimes \Lambda[x_{2,n_2}, \ldots, x_{r,n_r}]$$

with $|x_{i,n_i}| = n_i$, $|y| = 1$, $\mathrm{Sq}(x_{i,n_i}) = x_{i,n_i}(1+y)^{n_i+1}$, and $\mathrm{Sq}(y) = y(1+y)$. If n_1 is even and $n_1 = \cdots = n_k < n_{k+1}$ for some $k > 1$, then $H^*(P_{\overline{n}}; \mathbb{Z}_2)$ is as above, except that $x_{i,n_i}^2 = y^{n_1} x_{i,n_i}$ for $2 \leq i \leq k$.

We use the double subscripts for the cohomology classes x_{i,n_i} because we wish that the subscript indicate the grading, and yet the grading alone could be problematic due to possibly repeated values of n_i. We will consistently use x with a single subscript i for a point in the ith factor, while the double subscript will be used for cohomology (or K-theory) classes.

Proof. The proof proceeds by induction on r, with the case $r = 1$ being the well-known result for P^{n_1}. Let $\overline{m} = (n_1, \ldots, n_{r-1})$, and assume the result known for $P_{\overline{m}}$. Since $P_{\overline{n}} \approx S((n_r + 1)\xi_{\overline{m}})$, there is a cofibration

$$(2.2) \qquad\qquad P_{\overline{n}} \xrightarrow{\ p\ } P_{\overline{m}} \xrightarrow{\ i\ } T((n_r + 1)\xi_{\overline{m}}),$$

where $T(-)$ denotes the Thom space, and

$$p([x_1, \ldots, x_r]) = [x_1, \ldots, x_{r-1}]$$

for $x_i \in S^{n_i}$. Hence there is an exact sequence, with coefficients always in $\mathbb{Z}_2$,

(2.3)
$$\xleftarrow{i^*} H^{*+1}(T((n_r+1)\xi_{\overline{m}})) \xleftarrow{\delta} H^*(P_{\overline{n}}) \xleftarrow{p^*} H^*(P_{\overline{m}}) \xleftarrow{i^*} H^*(T((n_r+1)\xi_{\overline{m}})).$$

Since $n_{r-1} \leq n_r$, there is a map $P_{\overline{m}} \xrightarrow{j} P_{\overline{n}}$ defined by

$$j([x_1, \ldots, x_{r-1}]) = [x_1, \ldots, x_{r-1}, x_{r-1}].$$

In the last component here, $S^{n_{r-1}}$ is identified as the obvious subspace of S^{n_r}. Since $p \circ j$ is the identity map of $P_{\overline{m}}$, (2.3) splits, yielding

(2.4) $$H^*(P_{\overline{n}}) \approx H^*(P_{\overline{m}}) \oplus H^{*+1}(T((n_r + 1)\xi_{\overline{m}})),$$

and the splitting is as A-modules.

Let $x_{r,n_r} \in H^{n_r}(P_{\overline{n}})$ correspond to the Thom class $U \in H^{n_r+1}(T((n_r+1)\xi_{\overline{m}}))$ under (2.4). Then $\mathrm{Sq}(x_{r,n_r})$ corresponds to

$$\mathrm{Sq}(U) = W((n_r + 1)\xi_{\overline{m}})U = (1 + y)^{n_r+1}U.$$

Here Sq and W are the total Steenrod square and the total Stiefel-Whitney class, and we have used that the projection $P_{\overline{m}} \xrightarrow{p} P^{n_1}$ has $p^*(\xi_{n_1}) = \xi_{\overline{m}}$.

By the Thom isomorphism, the second summand of the RHS of (2.4) is $H^*(P_{\overline{m}}) \cdot x_{r,n_r}$. Thus (2.4) becomes

$$H^*(P_{\overline{n}}) \approx H^*(P_{\overline{m}}) \oplus H^*(P_{\overline{m}}) \cdot x_{r,n_r}$$

as graded abelian groups. The ring structure uses the multiplication of $H^*(P_{\overline{m}})$ on the RHS and

(2.5) $$x_{r,n_r}^2 = \mathrm{Sq}^{n_r} x_{r,n_r} = \binom{n_r+1}{n_r}y^{n_r}x_{r,n_r},$$

which is 0 if $n_1 < n_r$ or n_r is odd. The case described in the last sentence of the theorem is also immediate from (2.5). Thus the induction is extended. $\square$

Remark 2.6. We can give an explicit formula for the map $\Sigma P_{\overline{n}} \to T((n_r + 1)\xi_{\overline{m}})$ which splits the cofibration (2.2). If $\overline{x} \in S^{n_1} \times \cdots \times S^{n_{r-2}}$, then our map sends

(2.7) $$[t, \overline{x}, x_{r-1}, x_r] \mapsto [\overline{x}, x_{r-1}, tx_{r-1} + (1-t)x_r].$$

Here $t \in [0, 1]$, and $tx_{r-1} + (1-t)x_r$ takes place in D^{n_r+1}, the fiber of the disk bundle. It is a path between two points of the sphere bundle, which are identified to the basepoint in the Thom space.

We also need the following result about $H^*(P_{\overline{n}}; \mathbb{Q})$ and $H^*(P_{\overline{n}}; \mathbb{Z}/p)$ with p an odd prime. To set notation, recall that $H^*(S^{n_1} \times \cdots \times S^{n_r}; F)$ is the exterior algebra over F on classes x_{i,n_i}, $1 \leq i \leq r$, with $|x_{i,n_i}| = n_i$.

Theorem 2.8. *Let $F = \mathbb{Q}$ or $\mathbb{Z}/p$ with p an odd prime. The homomorphism*

$$H^*(P_{\overline{n}}; F) \xrightarrow{\rho^*} H^*(S^{n_1} \times \cdots \times S^{n_r}; F)$$

induced by the quotient map ρ sends $H^*(P_{\overline{n}}; F)$ isomorphically to the F-span of all products $x_{i_1, n_{i_1}} \cdots x_{i_k, n_{i_k}}$ such that $\sum_{j=1}^{k}(n_{i_j} + 1)$ is even.

Proof. Let $S_{\overline{n}} = S^{n_1} \times \cdots \times S^{n_r}$. The map $S_{\overline{n}} \xrightarrow{\rho} P_{\overline{n}}$ is a double cover, and so there is a fibration $S_{\overline{n}} \to P_{\overline{n}} \to K(\mathbb{Z}_2, 1) = RP^\infty$. This has a Serre spectral sequence with local coefficients

$$E_2^{p,q} = H^p(RP^\infty; \mathcal{H}^q(S_{\overline{n}}; F)) \Rightarrow H^*(P_{\overline{n}}; F).$$

The action of the generator of $\pi_1(RP^\infty)$ on $x_{i_1, n_{i_1}} \cdots x_{i_k, n_{i_k}}$ is by multiplication by $\prod(-1)^{n_{i_j}+1}$. Classes in $\mathcal{H}^q(S_{\overline{n}}; F)$ with trivial action will yield F in $E_2^{0,q}$ and nothing else, while those with nontrivial action ϕ yield nothing at all in E_2. This latter can be seen by noting, for example from [8, p.100], that $H^*(RP^\infty; \mathbb{Z}_\phi)$ is $\mathbb{Z}_2$ for odd positive values of $*$, and 0 for even values of $*$, including $* = 0$. By the Universal Coefficient Theorem, if $\mathbb{Z}$ is replaced by F, all groups become 0. Indeed, it is the cohomology of a cochain complex with F in each nonnegative grading and $\delta = 2 : C_{2i} \to C_{2i+1}$, $i \geq 0$. Thus the spectral sequence collapses, having as its only nonzero summands F generated by $x_{i_1, n_{i_1}} \cdots x_{i_k, n_{i_k}}$ in $E_2^{0,q}$ with $q = \sum n_{i_j}$ whenever $\sum(n_{i_j} + 1)$ is even. $\square$

Our next result is a splitting of $\Sigma P_{\overline{n}}$ as a wedge of desuspensions of stunted real projective spaces. Here $P_n^k = RP^k/RP^{n-1}$. This splitting has the potential to be used in studying span$(P_{\overline{n}})$ at the end of the next section. It is also useful in analyzing $K^*(P_{\overline{n}})$ in the proof of Theorem 2.21.

Theorem 2.9. *Let $\overline{n} = (n_1, \ldots, n_r)$ with $n_1 \leq n_i$ for all i. There is a homotopy equivalence*

$$(2.10) \qquad \Sigma P_{\overline{n}} \simeq \bigvee_{\overline{u} \subseteq (n_2, \ldots, n_r)} \Sigma^{1-\ell(\overline{u})} P_{|\overline{u}|+\ell(\overline{u})}^{n_1+|\overline{u}|+\ell(\overline{u})}.$$

Here, if $\overline{u} = (u_1, \ldots, u_s)$, then $|\overline{u}| := u_1 + \cdots + u_s$ and $\ell(\overline{u}) := s$.

Here we use subset notation for ordered subsets such as $\overline{n}$ and $\overline{u}$, which may have repeated entries. Note that there are 2^{r-1} wedge summands, each with $n_1 + 1$ cells, corresponding nicely to Theorem 2.1, as does the A-action. The desuspensions on the RHS of (2.10) exist for dimensional reasons.

Proof. The proof is by induction on r. Let $n_1 \leq \ldots \leq n_r$, and let $\overline{m} = (n_1, \ldots, n_{r-1})$, as in the proof of 2.1. Because of the map $P_{\overline{m}} \xrightarrow{j} P_{\overline{n}}$ such that $p \circ j = 1_{P_{\overline{m}}}$, i is null-homotopic in (2.2), and so there is a splitting

$$(2.11) \qquad \Sigma P_{\overline{n}} \simeq \Sigma P_{\overline{m}} \vee T((n_r+1)\xi_{\overline{m}}).$$

For each summand of the RHS of (2.10) such that $n_r \in \overline{u}$, we will construct a map

$$(2.12) \qquad T((n_r+1)\xi_{\overline{m}}) \longrightarrow \Sigma^{1-\ell(\overline{u})} P_{|\overline{u}|+\ell(\overline{u})}^{n_1+|\overline{u}|+\ell(\overline{u})}$$

such that, when preceded by the projection $\Sigma P_{\overline{n}} \to T((n_r+1)\xi_{\overline{m}})$, the composite has cohomology homomorphism injecting onto $\sigma(\mathbb{Z}_2[y]/y^{n_1+1}) \prod_{n_j \in \overline{u}} x_{j,n_j} \subset H^*(\Sigma P_{\overline{n}}; \mathbb{Z}_2)$.

Using these and (2.11) and the induction hypothesis applied to $P_{\overline{m}}$, we obtain maps from $\Sigma P_{\overline{n}}$ into all spaces in the wedge in (2.10), and hence, using the co-H-structure of $\Sigma P_{\overline{n}}$, we obtain the desired map in (2.10), which induces an isomorphism in $\mathbb{Z}_2$-cohomology.

To construct (2.12), we first construct a map

$$(2.13) \qquad T((n_r+1)\xi_{\overline{u}'}) \xrightarrow{f_{\overline{u}}} \Sigma^{1-\ell(\overline{u})} P_{|\overline{u}|+\ell(\overline{u})}^{n_1+|\overline{u}|+\ell(\overline{u})},$$

where $\overline{u}' = \overline{u} \cup \{n_1\} - \{n_r\}$, and then precede it by the projection $T((n_r+1)\xi_{\overline{m}}) \to T((n_r+1)\xi_{\overline{u}'})$. We obtain (2.13) by constructing a map of the $(\ell(\overline{u})-1)$-fold suspensions, and then noting that it desuspends to the desired map. Here we use that a map $\Sigma^t X \to \Sigma^t Y$ is a t-fold suspension if the dimension of X is less than twice the dimension of the bottom cell of Y. See, e.g., [15, 1.11].

Let $\overline{u} = (u_1, \ldots, u_s)$ with $u_s = n_r$. The suspended version of (2.13) is a map

$$(2.14) \qquad \Sigma^{s-1} T((u_s + 1)\xi_{(n_1, u_1, \ldots, u_{s-1})}) \to P_{u_1 + \cdots + u_s + s}^{u_1 + \cdots + u_s + s + n_1}$$

defined by

$$[t_1, \ldots, t_{s-1}, x, y, z_1, \ldots, z_{s-1}] \mapsto [\tfrac{M}{L} x, \tfrac{M}{L} t_1 z_1 \ldots, \tfrac{M}{L} t_{s-1} z_{s-1}, \sqrt{1 - M^2} y],$$

where $t_i \in [-1, 1]$, $x \in D^{u_s+1}$, $y \in S^{n_1} \subset \mathbb{R}^{n_1+1}$, and $z_i \in S^{u_i} \subset \mathbb{R}^{u_i+1}$. Also, $M = \max(|t_1|, \ldots, |t_{s-1}|, \|x\|)$, and $L = \sqrt{t_1^2 + \cdots + t_{s-1}^2 + \|x\|^2}$. Since $\|z_i\| = 1 = \|y\|$, one easily checks that the image point has norm 1 in $\mathbb{R}^{u_s+1} \times \mathbb{R}^{u_1+1} \times \cdots \times \mathbb{R}^{u_{s-1}+1} \times \mathbb{R}^{n_1+1}$. The map clearly respects the antipodal action, and if any $|t_i| = 1$ or $\|x\| = 1$, then the image point is in the subspace $P^{u_1 + \cdots + u_s + s - 1}$, which is collapsed in the target space. Thus the map is well-defined. One readily checks that it sends the interior of the top cell bijectively, and so the top $\mathbb{Z}_2$-cohomology class maps across. The map is natural with respect to decreasing values of n_1, and hence induces an injection in mod 2 cohomology, as claimed above.

Let $F = \mathbb{Q}$ or $\mathbb{Z}/p$ with p an odd prime. Since $H^*(P_n^k; F)$ has F in $* = n$ if n is even, and in $* = k$ if k is odd, and nothing else, one readily checks, using Theorem 2.8, that the two spaces in (2.10) have isomorphic F-cohomology groups. In (2.13), if $n_1 + |\overline{u}| + \ell(\overline{u})$ is odd, then, by the above observation about the top cell of the map just constructed, the F-cohomology homomorphism induced by (2.13) is nontrivial in the top dimension. If $|\overline{u}| + \ell(\overline{u})$ is even, then the F-cohomology homomorphism induced by (2.13) is nontrivial in dimension $|\overline{u}| + 1$ by consideration of the above construction when $n_1 = 0$. In our inductive construction of the map from the LHS of (2.10) to the RHS, all summands of the map ultimately come from (2.13). Thus the F-cohomology homomorphism induced by (2.10) is bijective. Since the map

$$\Sigma P_{\overline{n}} \to \bigvee_{\overline{u} \subseteq (n_2, \ldots, n_r)} \Sigma^{1 - \ell(\overline{u})} P_{|\overline{u}| + \ell(\overline{u})}^{n_1 + |\overline{u}| + \ell(\overline{u})}$$

of simply-connected spaces induces an isomorphism in $\mathbb{Q}$- and $\mathbb{Z}/p$-cohomology for all primes p, including $p = 2$, it is a homotopy equivalence. $\square$

Next we determine the integral cohomology ring $H^*(P_{\overline{n}}; \mathbb{Z})$.

Theorem 2.15. *Let $\overline{n} = (n_1, \ldots, n_r)$ with $n_1 \leq n_i$ for all i. Let $n_1 = 2m_1 + \epsilon$ with $\epsilon \in \{0, 1\}$. Let $E = \{i > 1 : n_i \text{ even}\}$. There are classes z with $|z| = 2$ and x_{i,n_i} with*

$|x_{i,n_i}| = n_i$ *and an isomorphism of graded rings*

$$H^*(P_{\overline{n}};\mathbb{Z}) \approx R \otimes \Lambda[x_{i,n_i} : i > 1, \ n_i \ odd]$$

with $R \approx A \oplus B \oplus C$ *as a graded abelian group, and*

$$A = (\mathbb{Z}[z]/(2z, z^{m_1+1})) \otimes \Big\langle \prod_{j\in S} x_{j,n_j} : S \subset E, \ |S| \ even \Big\rangle,$$

$$B = \Big\langle 2x_{1,n_1} \prod_{j\in S} x_{j,n_j} : S \subset E, \ |S| \not\equiv \epsilon \ (2) \Big\rangle,$$

$$C = (\mathbb{Z}_2[z]/(z^{m_1+\epsilon})) \otimes \Big\langle p_S : S \subset E, \ |S| \ odd \Big\rangle,$$

where $|p_S| = 1 + \sum_{j\in S} n_j$ *and if* $i,j \in E - S$ *with* $i \neq j$, *then* $x_{i,n_i} x_{j,n_j} p_S = p_{S\cup\{i,j\}}$. *The nontrivial products in* R *are the indicated products by* x_{i,n_i}*'s or by* z, *and also if* S *and* T *are disjoint subsets of* E *of odd cardinality, then* $p_S p_T = z \prod_{j\in S\cup T} x_{j,n_j}$. *Reduction* $H^*(P_{\overline{n}};\mathbb{Z}) \to H^*(P_{\overline{n}};\mathbb{Z}_2)$ *sends* z *to* y^2 *of 2.1,* x_{i,n_i} *to* x_{i,n_i}, *and* p_S *to* $y \prod_{j\in S} x_{j,n_j}$.

Here $|S|$ denotes the cardinality of the set S, and $\langle - \rangle$ denotes the span of a set of elements. Note that if n_i is even, x_{i,n_i} is not in $H^*(P_{\overline{n}};\mathbb{Z})$, but is involved in various product expressions. The reason for the factor 2 in the terms in B is to denote their image under $H^*(P_{\overline{n}}) \to H^*(S^{n_1} \times \cdots \times S^{n_r})$.

Proof. As in the proof of 2.8, we use the Serre spectral sequence of $S^{n_1} \times \cdots \times S^{n_r} \to P_{\overline{n}} \to RP^\infty$. The spectral sequence is a sum of two forms, and these vary with the parity of n_1. For every product $\prod_{i\in T} x_{i,n_i}$ with T a subset of $\{2,\ldots,r\}$, there is a portion of the spectral sequence as in one of the four diagrams below. In these diagrams, a dot represents $\mathbb{Z}_2$. Once these portions of the spectral sequence are noted, the conclusion of the theorem is mostly just bookkeeping. The product structure follows from images in $H^*(S^{n_1} \times \cdots \times S^{n_r};\mathbb{Z})$ and in $H^*(P_{\overline{n}};\mathbb{Z}_2)$. $\square$

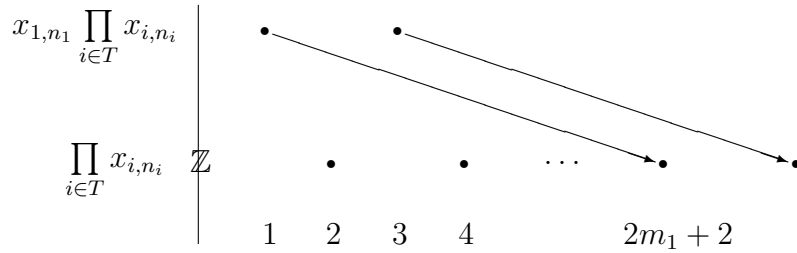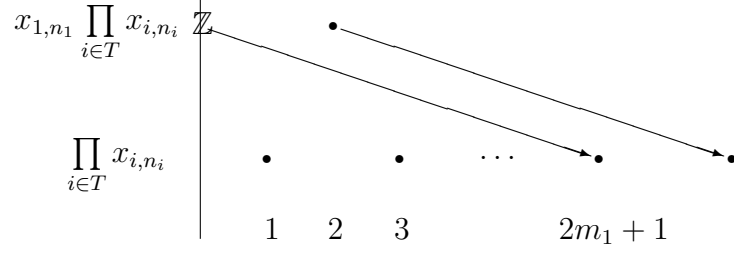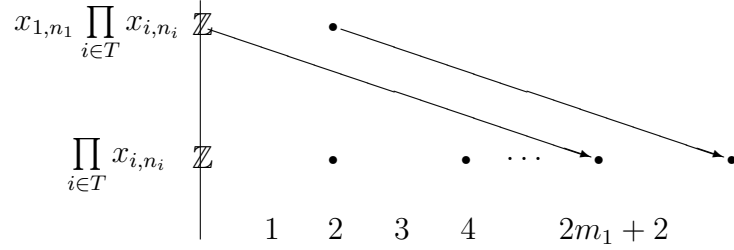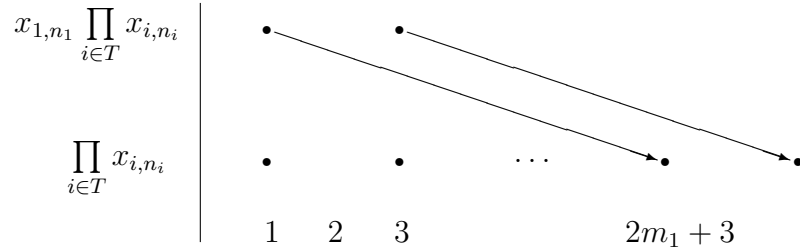**Diagram 2.16.** $n_1 = 2m_1$, $|T \cap E|$ *even, yielding* A *in 2.15*

Diagram 2.17. $n_1 = 2m_1$, $|T \cap E|$ *odd, yielding B and C in 2.15*

$$x_{1,n_1} \prod_{i \in T} x_{i,n_i} \quad \mathbb{Z}$$

$$\prod_{i \in T} x_{i,n_i}$$

$$1 \quad 2 \quad 3 \qquad 2m_1 + 1$$

Diagram 2.18. $n_1 = 2m_1 + 1$, $|T \cap E|$ *even, yielding A and B in 2.15*

$$x_{1,n_1} \prod_{i \in T} x_{i,n_i} \quad \mathbb{Z}$$

$$\prod_{i \in T} x_{i,n_i} \quad \mathbb{Z}$$

$$1 \quad 2 \quad 3 \quad 4 \qquad 2m_1 + 2$$

Diagram 2.19. $n_1 = 2m_1 + 1$, $|T \cap E|$ *odd, yielding C in 2.15*

$$x_{1,n_1} \prod_{i \in T} x_{i,n_i}$$

$$\prod_{i \in T} x_{i,n_i}$$

$$1 \quad 2 \quad 3 \qquad 2m_1 + 3$$

Theorem 2.15 suggests the possibility that the S^{n_i} with n_i odd might be product factors of $P_{\overline{n}}$. The following result shows the limited extent to which this is true. Here and throughout, $\nu(-)$ denotes the exponent of 2 in an integer, and $\phi(n)$ is the number of positive integers $\leq n$ which are congruent to 0, 1, 2, or 4 mod 8.

Theorem 2.20. *Let $n_1 \leq n_i$ for all i. Let $T = \{i > 1 : \nu(n_i + 1) \geq \phi(n_1)\}$.*

(1) *If $\overline{m}$ denotes the subtuple of $\overline{n}$ whose subscripts are not in T, then there is a homeomorphism $P_{\overline{n}} \approx P_{\overline{m}} \times \prod_{i \in T} S^{n_i}$.*

(2) *If $i \notin T$ and $\overline{m}$ is obtained from $\overline{n}$ by omitting n_i, then there does not exist a homotopy equivalence $P_{\overline{n}} \simeq P_{\overline{m}} \times S^{n_i}$.*

Proof. (1). For each $i \in T$, there is an action of S^{n_1} on S^{n_i} via Clifford modules ([3]). The homeomorphism

$$S^{n_1} \times \cdots \times S^{n_r} \xrightarrow{\Pi f_i} S^{n_1} \times \cdots \times S^{n_r}$$

defined by

$$f_i(x_1, \ldots, x_r) = \begin{cases} x_i & i \notin T \\ x_1 \cdot x_i & i \in T \end{cases}$$

passes to the desired homeomorphism

$$P_{\overline{m}} \times \prod_{i \in T} S^{n_i} \to P_{\overline{n}}.$$

(2) Assume such an equivalence exists, and, without loss of generality, that $i = 2$. Using 2.9, its suspension is an equivalence

$$\bigvee_{\overline{u} \subset (n_2, \ldots, n_r)} \Sigma^{1-\ell(\overline{u})} P_{|\overline{u}|+\ell(\overline{u})}^{n_1+|\overline{u}|+\ell(\overline{u})}$$

$$\simeq \quad S^{n_2+1} \vee \bigvee_{\overline{u} \subset (n_3, \ldots, n_r)} \left(\Sigma^{1-\ell(\overline{u})} P_{|\overline{u}|+\ell(\overline{u})}^{n_1+|\overline{u}|+\ell(\overline{u})} \vee \Sigma^{2+n_2-\ell(\overline{u})} P_{|\overline{u}|+\ell(\overline{u})}^{n_1+|\overline{u}|+\ell(\overline{u})} \right).$$

The wedge summands common to both sides, $\displaystyle\bigvee_{\overline{u} \subset (n_3, \ldots, n_r)} \Sigma^{1-\ell(\overline{u})} P_{|\overline{u}|+\ell(\overline{u})}^{n_1+|\overline{u}|+\ell(\overline{u})}$, might have some common summands of S^{n_2+1}, but the only way that the LHS could have one in addition to those is if the bottom cell of $P_{n_2+1}^{n_1+n_2+1}$ splits off, and this happens iff $\nu(n_2 + 1) \geq \phi(n_1)$. $\qquad\square$

Next we determine the ring $K^*(P_{\overline{n}})$. Here $K^*(-)$ denotes unreduced $\mathbb{Z}_2$-graded periodic complex K-theory. The result is quite similar to that for integral cohomology.

Theorem 2.21. *Let $\overline{n}$, m_1, ϵ, and E be as in 2.15. There is an isomorphism of $\mathbb{Z}_2$-graded rings,*

$$K^*(P_{\overline{n}}) \approx R_K \otimes \Lambda[x_{i,n_i} : i > 1, \ n_i \ odd],$$

with $R_K \approx A_K \oplus B_K \oplus C_K$ as graded abelian groups and

$$A_K = (\mathbb{Z} \oplus \mathbb{Z}/2^{m_1}) \otimes \left\langle \prod_{j \in S} x_{j,n_j} : S \subset E, \ |S| \ even \right\rangle$$

$$B_K = \left\langle 2x_{1,n_1} \prod_{j \in S} x_{j,n_j} : S \subset E, \ |S| \not\equiv \epsilon \ (2) \right\rangle$$

$$C_K = \left\langle p_S : S \subset E, \ |S| \ odd \right\rangle / 2^{m_1+\epsilon}.$$

Here $x_{j,n_j} \in K^{n_j}(P_{\overline{n}})$ with the superscript of K being considered mod 2. Also, $A_K \subset K^0(P_{\overline{n}})$, $B_K \subset K^\epsilon(P_{\overline{n}})$, and $C_K \subset K^1(P_{\overline{n}})$. The notation for C_K means that each p_S has order $2^{m_1+\epsilon}$. Note that $S = \emptyset$ is allowed in A_K, and so A_K contains the initial $\mathbb{Z} \oplus \mathbb{Z}/2^{m_1}$, whose generators are 1 and $g := (1 - \xi)$, satisfying $g^2 = 2g$. If $i, j \in E - S$ with $i \neq j$, then $x_{i,n_i}x_{j,n_j}p_S = p_{S\cup\{i,j\}}$. We have $g \cdot p_S = 2p_S$, and, if S and T are disjoint subsets of E of odd cardinality, then $p_S p_T = g \prod_{j\in S\cup T} x_{j,n_j}$. Other than obvious products with x_{i,n_i}'s, there are no other nontrivial products.

Proof. This mostly follows from the Atiyah-Hirzebruch spectral sequence (AHSS)

$$H^*(P_{\overline{n}}; K^*(pt)) \Rightarrow K^*(P_{\overline{n}}).$$

We use the description of $H^*(P_{\overline{n}}; \mathbb{Z})$ in 2.15 together with Diagrams 2.16, 2.17, 2.18, and 2.19. We use Theorem 2.9 and the well-known result for $K^*(P_n^k)$ ([2]) to see that there are no differentials in the AHSS and that the $\mathbb{Z}_2$'s along a row in one of the diagrams extend cyclically. Most of the product structure can be seen in the AHSS, including the products $g \cdot p_S$ and $p_S p_T$.

A more K-theoretic way to see $g \cdot p_S = 2p_S$ can be obtained using Theorem 2.9. First note that p_S can be interpreted as an element of $K^0(\Sigma P_{\overline{n}})$, and, using 2.9, as an element of $K^0(\Sigma^{1-\ell(\overline{u})} P_{|\overline{u}|+\ell(\overline{u})}^{|\overline{u}|+\ell(\overline{u})+n_1})$ with $\ell(\overline{u})$ odd, for $\overline{u}$ corresponding to S. Then p_S corresponds to $2^{(|\overline{u}|+\ell(\overline{u})-1)/2}(1 - \xi_{|\overline{u}|+\ell(\overline{u})+n_1}) \in K(P_{|\overline{u}|+\ell(\overline{u})}^{|\overline{u}|+\ell(\overline{u})+n_1})$. There is an action of $K(P^{n_1})$ on $K(P_{|\overline{u}|+\ell(\overline{u})}^{|\overline{u}|+\ell(\overline{u})+n_1}) \approx K(T(|\overline{u}| + \ell(\overline{u}))\xi_{n_1})$ using the action of $K(D(\theta))$ on $K(T(\theta))$, and an action of $K(P^{n_1})$ on $K(\Sigma P_{\overline{n}})$ using the projection map $P_{\overline{n}} \to P^{n_1}$. Using (2.7) and (2.14), one can show that the isomorphism of 2.9 is compatible with these actions. Thus $g \cdot p_S$ corresponds to $(1 - \xi) \cdot 2^e(1 - \xi) = 2(2^e(1 - \xi))$, for appropriate e. $\square$

3. Manifold properties

Two properties of manifolds M studied by algebraic topologists are $\mathrm{span}(M)$ and $\mathrm{imm}(M)$, defined by

Definition 3.1. *If M is a differentiable manifold, $\mathrm{span}(M)$ is the maximal number of linearly independent tangent vector fields on M, while $\mathrm{imm}(M)$ is the dimension of the smallest Euclidean space in which M can be immersed.*

In this section, we study $\mathrm{span}(P_{\overline{n}})$ and $\mathrm{imm}(P_{\overline{n}})$. We also determine exactly when $P_{\overline{n}}$ is parallelizable.

The answers are related to $\mathrm{span}(k\xi_n)$ and $\mathrm{gd}(\ell\xi_n)$, where k is a positive integer, ℓ an integer, and ξ_n the Hopf bundle over real projective space P^n. These much-studied quantities are defined in

Definition 3.2. *If θ is a vector bundle over a topological space X, $\mathrm{span}(\theta)$ is the maximal number of linearly independent (l.i.) sections of θ. If η is a stable vector bundle over X, then its geometric dimension, $\mathrm{gd}(\eta)$, is the smallest k such that there is a k-plane bundle over X stably equivalent to η.*

The following facts relating these concepts are well-known.

Proposition 3.3.   • *If M is a manifold with tangent bundle $\tau(M)$, then $\mathrm{span}(M) = \mathrm{span}(\tau(M))$.*

  • *If θ is a d-dimensional vector bundle over a finite-dimensional CW complex X with $d > \dim(X)$, then $\mathrm{gd}(\theta) + \mathrm{span}(\theta) = d$.*

  • *If $m > 0$, $\nu(L) \geq \phi(n)$, and $L - m > n$, then*

$$\mathrm{gd}(-m\xi_n) = \mathrm{gd}((L-m)\xi_n) = L - m - \mathrm{span}((L-m)\xi_n).$$

The immersion dimension of $P_{\overline{n}}$ is related to the geometric dimension of a stable vector bundle over a projective space in the following result. It seems somewhat strange that $\mathrm{imm}(P_{\overline{n}})$ does not depend on the values of most of the n_i.

Theorem 3.4. *If $\overline{n} = (n_1, \ldots, n_r)$ with $n_1 \leq n_i$ for all i, then*

$$\mathrm{imm}(P_{\overline{n}}) = |\overline{n}| + \max(\mathrm{gd}(-(|\overline{n}| + r)\xi_{n_1}), 1).$$

Proof. The tangent bundle $\tau(P_{\overline{n}})$ is given by

$$\{(\overline{u}, \overline{x}) \in \mathbb{R}^{n_1+1} \times \cdots \times \mathbb{R}^{n_r+1} \times S^{n_1} \times \cdots \times S^{n_r} : u_i \perp x_i \; \forall i\}/((\overline{u}, \overline{x}) \sim (-\overline{u}, -\overline{x})),$$

with $\overline{u} = (u_1, \ldots, u_r)$ and $\overline{x} = (x_1, \ldots, x_r)$. There is a vector bundle isomorphism

$$(3.5) \qquad \tau(P_{\overline{n}}) \oplus r\varepsilon \xrightarrow{\approx} (|\overline{n}| + r)\xi_{\overline{n}}$$

defined by

$$([u_1, \ldots, u_r, x_1, \ldots, x_r], t_1, \ldots, t_r) \mapsto [u_1 + t_1 x_1, \ldots, u_r + t_r x_r, \overline{x}].$$

Here and throughout $r\varepsilon$ denotes a trivial bundle of dimension r, and $t_i \in \mathbb{R}$.

The maps

$$P^{n_1} \xrightarrow{j} P_{\overline{n}} \xrightarrow{p} P^{n_1}$$

defined by $j([x]) = [x, \ldots, x]$ and $p([x_1, \ldots, x_r]) = [x_1]$ satisfy

(3.6) $\qquad\qquad p^*(\xi_{n_1}) = \xi_{\overline{n}} \quad \text{and} \quad j^*(\xi_{\overline{n}}) = \xi_{n_1}.$

By [9], $\mathrm{imm}(P_{\overline{n}})$ equals $|\overline{n}|$ plus the geometric dimension of the stable normal bundle of $P_{\overline{n}}$, unless this gd is 0, in which case $\mathrm{imm}(P_{\overline{n}}) = |\overline{n}|+1$. By (3.5), the stable normal bundle of $P_{\overline{n}}$ is $-(|\overline{n}| + r)\xi_{\overline{n}}$. By (3.6), we have $\mathrm{gd}(-(|\overline{n}| + r)\xi_{\overline{n}}) \le \mathrm{gd}(-(|\overline{n}| + r)\xi_{n_1})$ and $\mathrm{gd}(-(|\overline{n}| + r)\xi_{n_1}) \le \mathrm{gd}(-(|\overline{n}| + r)\xi_{\overline{n}})$, implying the result. $\qquad\square$

Since, by obstruction theory, if η is a stable vector bundle over a CW complex X, then $\mathrm{gd}(\eta) \le \dim(X)$, we obtain the following surprising corollary.

Corollary 3.7. *If $n_1 \le n_i$ for all i, then $P_{\overline{n}}$ can be immersed in $\mathbb{R}^{|\overline{n}|+n_1}$.*

An immediate corollary of (3.5) is

Corollary 3.8. *$P_{\overline{n}}$ is orientable if and only if $|\overline{n}| + r$ is even.*

Geometric dimension of multiples of the Hopf bundle over real projective spaces, sometimes called the generalized vector field problem, has been studied in many papers such as [1], [6], [7], [13], and [14]. One consequence of Theorem 3.4 is that every case of the generalized vector field problem is solving an immersion question for some manifold. In Section 4, we combine specific results on the generalized vector field problem with Theorem 3.4 to obtain numerical bounds on $\mathrm{imm}(P_{\overline{n}})$ for certain $\overline{n}$.

Our second manifold result, involving $\mathrm{span}(P_{\overline{n}})$, is similar, but not quite so complete. It is better expressed in terms of stable span, defined for a manifold M by $\mathrm{stablespan}(M) = \mathrm{span}(\tau(M) + \varepsilon) - 1$. It is a well-known consequence of obstruction theory that if $r \ge 1$, then $\mathrm{span}(\tau(M) + r\varepsilon) - r$ is independent of r, and hence equals $\mathrm{stablespan}(M)$. Clearly $\mathrm{span}(M) \le \mathrm{stablespan}(M)$.

Theorem 3.9. $\qquad$ (1) *If $\overline{n} = (n_1, \ldots, n_r)$ with $n_1 \le n_i$ for all i, then*

(3.10) $\qquad\qquad \mathrm{stablespan}(P_{\overline{n}}) = \mathrm{span}((|\overline{n}| + r)\xi_{n_1}) - r.$

(2) $\mathrm{span}(P_{\overline{n}}) = 0$ *if and only if all n_i are even.*

(3) *If $|\overline{n}|$ is even but not all n_i are even, then $\mathrm{span}(P_{\overline{n}}) = \mathrm{stablespan}(P_{\overline{n}})$.*

(4) *If $|\overline{n}| \equiv 3$ (mod 8) and $r \equiv 1$ (mod 4), then* $\text{span}(P_{\overline{n}}) = \text{stablespan}(P_{\overline{n}})$.

Proof.   (1) Both sides of (3.10) equal $\text{span}((|\overline{n}| + r)\xi_{\overline{n}}) - r$, one side using (3.5) and the other using (3.6).

(2) This is immediate from the classical theorem of Hopf that

$$\text{span}(M) > 0 \text{ iff } \chi(M) = 0,$$

together with the fact, from 2.1, that the Euler characteristic $\chi(P_{\overline{n}}) = \frac{1}{2}\prod(1 + (-1)^{n_i})$.

(3) Koschorke showed in [12, Theorem 20.1] that if $\dim(M)$ is even and $\chi(M) = 0$, then $\text{span}(M) = \text{stablespan}(M)$. As just noted, $\chi(P_{\overline{n}}) = 0$ if and only if some n_i is odd. This part of the theorem is now immediate from Koschorke's result.

(4) Koschorke also showed in [12, Corollary 20.10] that if M is a Spin manifold with $\dim(M) \equiv 3 \bmod 8$ and $\chi_2(M) = 0$, then $\text{span}(M) = \text{stablespan}(M)$. Here $\chi_2(M)$ is the Kervaire semicharacteristic, defined, for odd-dimensional manifolds, as the mod 2 value of the sum of the ranks of the even-dimensional mod-2 homology groups. Using Theorem 2.1, one easily shows that if $|\overline{n}|$ is odd, then $\chi_2(P_{\overline{n}}) = 0$ unless $r = 1$ and $n_1 \equiv 1 \bmod 4$, or $r = 2$, n_1 is even, and n_2 odd. Note that we needed $|\overline{n}| + r \equiv 0 \bmod 4$ in order that $P_{\overline{n}}$ be a Spin-manifold.

□

Similarly to Corollary 3.7, we have

Corollary 3.11. *If $n_1 \leq n_i$ for all i, then* $\text{stablespan}(P_{\overline{n}}) \geq |\overline{n}| - n_1$.

A closely-related result tells exactly when $P_{\overline{n}}$ is parallelizable. Here ν and ϕ are as defined prior to Theorem 2.20.

Theorem 3.12. *If $\overline{n} = (n_1, \ldots, n_r)$ with $n_1 \leq n_i$ for all i, then $P_{\overline{n}}$ is parallelizable if and only if $\nu(|\overline{n}| + r) \geq \phi(n_1)$ and not all n_i are even.*

Proof. Bredon and Kosinski proved in [4] that a stably parallelizable n-manifold M is parallelizable if and only if n is even and $\chi(M) = 0$ or n is odd and $\chi_2(M) = 0$. By (3.5) and (3.6), $\tau(P_{\overline{n}})$ is stably trivial iff $(|\overline{n}| + r)\xi_{n_1}$ is, and this is true iff $\nu(|\overline{n}| + r) \geq$

$\phi(n_1)$. The theorem now follows from our observations about $\chi(P_{\overline{n}})$ and $\chi_2(P_{\overline{n}})$ in the proof of 3.9. $\square$

An approach to showing that span equals stable span for an n-manifold M was presented in [10]. In the exact sequence of pointed sets

$$[\Sigma M, BO] \xrightarrow{\delta} [M, V_n] \to [M, BO(n)] \to [M, BO],$$

$[M, V_n]$ has two elements, with the nontrivial element being detected in $\mathbb{Z}_2$-cohomology. This is the cause of the possibility of there being an element in $[M, BO(n)]$ stably equivalent to the tangent bundle but not equal to it. If there is an element α in $[\Sigma M, BO] = \widetilde{KO}(\Sigma M)$ such that $\delta(\alpha) \neq 0$, then we can deduce that span equals stable span. Such an element α is specified in [10] by a condition on its Stiefel-Whitney classes. The splitting of $\Sigma P_{\overline{n}}$ in Theorem 2.9 enables us to understand $\widetilde{KO}(\Sigma P_{\overline{n}})$. However, it seems that there are no elements whose Stiefel-Whitney classes satisfy the required condition.

4. SOME NUMERICAL RESULTS FOR $\mathrm{imm}(P_{\overline{n}})$ AND $\mathrm{span}(P_{\overline{n}})$

In this section, we sample some of the known results about $\mathrm{gd}(k\xi_n)$ and discuss their implications for $P_{\overline{n}}$.

Using Stiefel-Whitney classes and construction of bilinear maps, Lam proved the following result in [13, Thm 1.1].

Proposition 4.1. $\mathrm{gd}(k\xi_n) \geq m_0$, where m_0 is the largest $m \leq n$ for which $\binom{k}{m}$ is odd. Equality occurs here if $\binom{[k/8]}{[n/8]}$ is odd.

The following well-known proposition is often useful in determining whether binomial coefficients are odd.

Proposition 4.2. If $k = 2^{e_0} + \cdots + 2^{e_t}$ with $e_0 < \cdots < e_t$, let $\mathrm{Bin}(k) = \{e_0, \ldots, e_t\}$. If $k > 0$, then $\binom{k}{n}$ is odd iff $\mathrm{Bin}(n) \subseteq \mathrm{Bin}(k)$, and $\binom{-k}{n}$ is odd iff $\mathrm{Bin}(k-1)$ and $\mathrm{Bin}(n)$ are disjoint.

Note that equality occurs in Proposition 4.1 if $n \leq 7$, and, using 3.4, we easily obtain the following corollary.

Corollary 4.3. If $\overline{n} = (n_1, \ldots, n_r)$ with $n_1 \leq n_i$ for all i, and $n_1 \leq 7$, then $\mathrm{imm}(P_{\overline{n}}) = |\overline{n}| + n_1 - \delta$, where δ is given in Table 4.4.

Table 4.4. *Values of δ*

		1	2	3	4	5	6	7	8
	0	0	0	0	0	0	0	0	0
n_1	1	0	1	0	1	0	1	0	1
	2	0	0	1	2	0	0	1	2
	3	0	1	2	3	0	1	2	3
	4	0	0	0	0	1	2	3	4
	5	0	1	0	1	2	3	4	5
	6	0	0	1	2	3	4	5	6
	7	0	1	2	3	4	5	6	7

(column header: $|\overline{n}| + r \pmod 8$)

Combining triviality of $16\xi_8$ and $32\xi_9$ with results in [13], we also have complete information about $\mathrm{gd}(k\xi_8)$ and $\mathrm{gd}(k\xi_9)$, which we state in Proposition 4.5. These, with 3.4, yield complete information about $\mathrm{imm}(P_{\overline{n}})$ when $n_1 = 8$ or 9. It is quite remarkable that whenever the smallest subscript n_1 is ≤ 9, the immersion dimension of $P_{\overline{n}}$ is precisely known. Other results about $\mathrm{imm}(P_{\overline{n}})$ can be obtained by combining 4.1 and 3.4, but we will not bother to state them.

Proposition 4.5. *If $0 \leq \Delta \leq 15$ and $i \geq 0$, then $\mathrm{gd}((16i + \Delta)\xi_8) = \min(\Delta, 8)$. If $i \geq 0$, then*

$$\mathrm{gd}((16i+\Delta)\xi_9) = \begin{cases} \Delta & \text{if } i \text{ even and } 0 \leq \Delta \leq 9, \text{ or } i \text{ odd and } \Delta = 6 \text{ or } 7 \\ 9 & \text{if } \Delta = 9, 11, 13, \text{ or } 15 \\ 8 & \text{if } \Delta = 8, 10, 12, \text{ or } 14 \\ 6 & \text{if } i \text{ is odd and } \Delta = 0, 2, 3, \text{ or } 5 \\ 5 & \text{if } i \text{ is odd and } \Delta = 1 \text{ or } 4. \end{cases}$$

The implications of Proposition 4.1 for $\mathrm{span}(P_{\overline{n}})$ are limited by the span-versus-stablespan conundrum. We readily obtain the following result about stable span.

Proposition 4.6. *Let $\overline{n} = (n_1, \ldots, n_r)$ with $n_1 \leq n_i$ for all i. Then $\mathrm{stablespan}(P_{\overline{n}}) \leq |\overline{n}| - m_1$, where m_1 is the largest $m \leq n_1$ such that $\binom{|\overline{n}|+r}{m}$ is odd. Equality is obtained if $\binom{[(|\overline{n}|+r)/8]}{[n_1/8]}$ is odd. If $n_1 \leq 7$, then $\mathrm{stablespan}(P_{\overline{n}}) = |\overline{n}| - n_1 + \delta$, where δ is as in Table 4.4 with column i replaced by $8 - i$; i.e., the column labels read $7, \ldots, 0$.*

Combining Theorem 3.9 with Proposition 4.6 yields results about $\mathrm{span}(P_{\overline{n}})$. However, even if $n_1 \leq 7$, we must be careful about trying to assert $\mathrm{span}(P_{\overline{n}}) = \mathrm{stablespan}(P_{\overline{n}})$ because of the situation described in part (2) of Theorem 3.9.

The second type of geometric dimension result on which we focus is vector bundles of low geometric dimension. These were first studied by Adams in [1], and Lam and Randall provide the current status in [14], some of which is described in the following theorem.

Theorem 4.7. ([1],[14]) *Assume $n \geq 18$.*

- *If $0 \leq d \leq 4$, then $\mathrm{gd}(k\xi_n) = d$ if and only if $k \equiv d \mod 2^{\phi(n)}$.*
- *If $\nu(k) = \phi(n) - 1$ and $n \not\equiv 7 \mod 8$ or if $\nu(k) = \phi(n) - 2$ and $n \equiv 2$ or $4 \mod 8$, then $\mathrm{gd}(k\xi_n) = 5$.*
- *The only other possible occurrences of $\mathrm{gd}(k\xi_n) = 5$ for $k \equiv 0$ $\mod 4$ are $(\nu(k) = \phi(n) - 1$ and $n \equiv 7 \mod 8)$ or $(\nu(k) = \phi(n) - 2$ and $n \equiv 1, 3, 5 \mod 8)$.*

This has the following immediate consequence for $\mathrm{imm}(P_{\bar{n}})$.

Corollary 4.8. *Let $\bar{n} = (n_1, \ldots, n_r)$ with $n_1 \leq n_i$ for all i.*

- $\mathrm{imm}(P_{\bar{n}}) = |\bar{n}| + 1$ *iff $\nu(|\bar{n}| + r) \geq \phi(n_1)$ or $\nu(|\bar{n}| + r + 1) \geq \phi(n_1)$.*
- *For $2 \leq d \leq 4$, $\mathrm{imm}(P_{\bar{n}}) = |\bar{n}| + d$ iff $\nu(|\bar{n}| + r + d) \geq \phi(n_1)$.*
- *If $\nu(|\bar{n}| + r) = \phi(n_1) - 1$ and $n_1 \not\equiv 7 \mod 8$, or if $\nu(|\bar{n}| + r) = \phi(n_1) - 2$ and $n_1 \equiv 2, 4 \mod 8$, then $\mathrm{imm}(P_{\bar{n}}) = |\bar{n}| + 5$.*

Results such as

"if $0 \leq d \leq 4$, then $\mathrm{stablespan}(P_{\bar{n}}) = |\bar{n}| - d$ iff $\nu(|\bar{n}| + r - d) \geq \phi(n_1)$"

can also be immediately read off from 3.9 and 4.7.

Next we recall the implications of Adams operations in K-theory for sectioning $k\xi_n$. Although slightly stronger results can be obtained using KO-theory, we prefer here the following simpler-to-state KU result.

Theorem 4.9. ([6]) *If $\binom{m-1}{n}$ is odd, then $m\xi_n$ has at most $m - n + 2\nu(m) + 1$ l.i. sections.*

The implication of this for $P_{\bar{n}}$ is given in the following result, which is immediate from 3.4, 3.9, and 4.9

Corollary 4.10. *Let $\bar{n} = (n_1, \ldots, n_r)$ with $n_1 \leq n_i$ for all i.*

- *If $\binom{-|\bar{n}| - r - 1}{n_1}$ is odd, then $\mathrm{imm}(P_{\bar{n}}) \geq |\bar{n}| + n_1 - 2\nu(|\bar{n}| + r) - 1$.*
- *If $\binom{|\bar{n}| + r - 1}{n_1}$ is odd, then $\mathrm{span}(P_{\bar{n}}) \leq |\bar{n}| - n_1 + 2\nu(|\bar{n}| + r) + 1$.*

Finally, we recall the strong implications of BP-theory for sectioning $k\xi_n$. Slightly stronger results have been recently obtained using tmf ([5]) or $ER(2)$ ([11]), but we list here the BP-result because it is much simpler to state.

Theorem 4.11. ([7]) *If* $\nu\binom{n+s}{k-s} = s$, *then* $\mathrm{gd}(2n\xi_{2k}) \geq 2k - 6s$.

In applying this, it is useful to note that $\nu\binom{\ell+m}{\ell} = \alpha(\ell) + \alpha(m) - \alpha(\ell + m)$, where $\alpha(m)$ is the number of 1's in the binary expansion of m. This implies that $\nu\binom{2\ell+2m}{2\ell} = \nu\binom{\ell+m}{\ell}$, which we will use in the next result. The implications for $P_{\overline{n}}$ are as follows, derived in the usual way.

Corollary 4.12. *Assume* $|\overline{n}| + r$ *and* n_1 *are even. Then*

- *If* $\nu\binom{-|\overline{n}|-r+2s}{n_1-2s} = s$, *then* $\mathrm{imm}(P_{\overline{n}}) \geq |\overline{n}| + n_1 - 6s$.
- *If* $\nu\binom{|\overline{n}|+r+2s}{n_1-2s} = s$, *then* $\mathrm{span}(P_{\overline{n}}) \leq |\overline{n}| - n_1 + 6s$.

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