p-ADIC STIRLING NUMBERS OF THE SECOND KIND

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Abstract. Let $S(n,k)$ denote the Stirling numbers of the second kind. We prove that the p-adic limit of $S(p^e a+c, p^e b+d)$ as $e\to\infty$ exists for any integers a, b, c, and d with $0\le b\le a$. We call the limiting p-adic integer $S(p^\infty a+c, p^\infty b+d)$. When $a\equiv b \bmod (p-1)$ or $d\le 0$, we express them in terms of p-adic binomial coefficients $\binom{p^\infty\alpha-1}{p^\infty\beta}$ introduced in a recent paper.

1. Main theorems

In [3], the author defined, for integers a, b, c, and d, with $0\le b\le a$, $\binom{p^\infty a+c}{p^\infty b+d}$ to be the p-adic integer which is the p-adic limit of $\binom{p^e a+c}{p^e b+d}$, and gave explicit formulas for these in terms of rational numbers and p-adic integers which, if p or n is even, could be considered to be $\mathrm{U}_p((p^\infty n)!) := \lim_e \mathrm{U}_p((p^e n)!)$. Here and throughout, $\nu_p(-)$ denotes the exponent of p in an integer or rational number and $\mathrm{U}_p(n) = n/p^{\nu_p(n)}$ denotes the unit factor in n. Here we do the same for Stirling numbers $S(n,k)$ of the second kind; i.e., we prove that the p-adic limit of $S(p^e a+c, p^e b+d)$ exists, and call it $S(p^\infty a+c, p^\infty b+d)$. If $a\equiv b \bmod (p-1)$ or $d\le 0$, we express these explicitly in terms of certain $\binom{p^\infty\alpha-1}{p^\infty\beta}$ together with certain Stirling-like rational numbers.

We now list our four main theorems, which will be proved in Sections 2 and 4. Let $\mathbb{Z}_p$ denote the p-adic integers with the usual metric.

Theorem 1.1. *Let p be a prime, and a, b, c, and d integers with $0\le a\le b$. Then the p-adic limit of $S(p^e a+c, p^e b+d)$ exists in $\mathbb{Z}_p$. We denote the limit as $S(p^\infty a+c, p^\infty b+d)$.*

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Theorem 1.2. *If p is any prime and $0 \le b \le a$, then $S(p^\infty a, p^\infty b) = 0$ if $a \not\equiv b$* mod $(p-1)$*, while*

$$S(p^\infty a, p^\infty b) = \binom{p^\infty \frac{pa-b}{p-1} - 1}{p^\infty \frac{p(a-b)}{p-1}} \text{ if } a \equiv b \mod (p-1).$$

These p-adic binomial coefficients are as introduced in [3].

Let $|s(n,k)|$ denote the unsigned Stirling numbers of the first kind.

Theorem 1.3. *If $0 \le b \le a$, then*

$$S(p^\infty a + c, p^\infty b + d) = \begin{cases} 0 & d = 0,\ c \ne 0 \\ 0 & d < 0,\ c \ge 0 \\ |s(|d|,|c|)| S(p^\infty a, p^\infty b) & c < 0,\ d < 0. \end{cases}$$

In particular, if $a \not\equiv b$ mod $(p-1)$, then $S(p^\infty a + c, p^\infty b + d) = 0$ whenever $d \le 0$.

For any prime number p, integer n, and nonnegative integer k, define the partial Stirling numbers $T_p(n,k)$ ([2]) by

$$T_p(n,k) = \frac{(-1)^k}{k!} \sum_{i \not\equiv 0\ (p)} (-1)^i \binom{k}{i} i^n. \tag{1.4}$$

Theorem 1.5. *If $a \equiv b$* mod $(p-1)$ *and $d \ge 1$, then*

$$S(p^\infty a + d - 1, p^\infty b + d) = T_p(d-1, d) \binom{p^\infty \frac{pa-b}{p-1} - 1}{p^\infty b}.$$

When $a \equiv b$ mod $(p-1)$, results for all $S(p^\infty a + c, p^\infty b + d)$ with $d > 0$ follow from these results and the standard formula

$$S(n,k) = kS(n-1,k) + S(n-1,k-1). \tag{1.6}$$

Explicit formulas are somewhat complicated and are relegated to Section 3.

2. Proofs when $a \equiv b$ mod $(p-1)$ or $d \le 0$

In this section, we prove Theorems 1.2, 1.3, and 1.5. If $a \equiv b$ mod $(p-1)$ or $d \le 0$, Theorem 1.1 follows immediately from Theorems 1.2, 1.3, and 1.5 and their proofs. These give explicit values for the limits when $d \le 0$ and for at least one value of c when $d > 0$. The existence of the limit for other values of c follows from (1.6) and induction. Examples are given in Section 3. We will prove Theorem 1.1 when $a \not\equiv b$ mod $(p-1)$ and $d > 0$ in Section 4.

We rely heavily on the following two results of Chan and Manna.

Theorem 2.1. ([1, 4.2,5.2]) *Suppose $n > p^m b$ with $m \geq 3$ if $p = 2$. Then, mod p^{m-1} if $p = 2$, and mod p^m if p is odd,*

$$S(n, p^m b) \equiv \begin{cases} \binom{n/2-2^{m-2}b-1}{n/2-2^{m-1}b} & \text{if } p = 2 \text{ and } n \equiv 0 \mod 2 \\ \binom{(n-p^{m-1}b)/(p-1)-1}{(n-p^m b)/(p-1)} & \text{if } p \text{ is odd and } n \equiv b \mod (p-1) \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.2. ([1, 4.3,5.3]) *Let p be any prime, and suppose $n \geq p^e b + d$. Then*

$$S(n, p^e b+d) \equiv \sum_{j\geq 0} S(p^e b+(p-1)j, p^e b) S(n-p^e b-(p-1)j, d) \mod p^e.$$

Proof of Theorem 1.2. The result follows from Theorem 2.1. If p is odd and $a \not\equiv b$ mod $(p-1)$, then $\nu_p(S(p^e a, p^e b)) \geq e$, while if $a \equiv b$ mod $(p-1)$, then

$$S(p^e a, p^e b) \equiv \binom{p^{e-1}\frac{pa-b}{p-1} - 1}{p^{e-1}\frac{p(a-b)}{p-1}} \mod p^e.$$

If $p = 2$, then

$$S(2^e a, 2^e b) \equiv \binom{2^{e-2}(2a-b) - 1}{2^{e-2}(2a-2b)} \mod 2^{e-1}.$$

□

Let $d_p(n)$ denote the sum of the digits in the p-ary expansion of a positive integer n.

Proof of Theorem 1.3. The first case follows readily Theorem 2.1. If $p = 2$, this says that $\nu(S(2^e a + c, 2^e b)) \geq e - 1$ if c is odd, while if $c = 2k$ is even, then, mod 2^{e-1},

$$S(2^e a + 2k, 2^e b) \equiv \binom{2^{e-1}a + k - 2^{e-2}b - 1}{2^{e-1}a + k - 2^{e-1}b}.$$

If $0 < k < 2^{e-1}$, this has 2-exponent

$$\nu_2 = d_2(a-b) + d_2(k) - (d_2(2a-b) + d_2(k-1)) + d_2(2^{e-2}b - 1) \to \infty$$

as $e \to \infty$, while if $k = -\ell < 0$, then

$$\nu_2 = e-1+d_2(a-b-1)-d_2(\ell-1)-(e-2+d_2(2a-b-1)-d_2(\ell))+d_2(2^{e-2}b-1) \to \infty.$$

The odd-primary case follows similarly.

The second case of the theorem follows from the result for $c = 0$ just established and (1.6) by induction. For the third case, write $c = -k$ and $d = -\ell$ and argue by induction on k and ℓ, starting with the fact that the result is true if $k = 0$ or $l = 0$. Then, mod p^e,

$$\begin{aligned} S(p^ea-k-1, p^eb-\ell-1) &= S(p^ea-k, p^eb-\ell) - (p^eb-\ell)S(p^ea-k-1, p^eb-\ell) \\ &\equiv S(p^ea, p^eb)(|s(\ell,k)| + \ell|s(\ell,k+1)|) \\ &= S(p^ea,p^eb)|s(\ell+1,k+1)|, \end{aligned}$$

implying the result. □

The proof of Theorem 1.5 will utilize the following two lemmas. We let $\lg_p(x) = [\log_p(x)]$.

Lemma 2.3. *If p is any prime and k and d are positive integers, then*

$$\nu_p\big(T_p((p-1)k+d-1,d) - T_p(d-1,d)\big) \ge \nu_p(k) - \lg_p(d).$$

Proof. We have

$$\begin{aligned} &|T_p((p-1)k+d-1,d) - T_p(d-1,d)| \\ = &\sum_{r=1}^{p-1}(-1)^r \tfrac{1}{d!}\sum_j (-1)^j \tbinom{d}{pj+r}(pj+r)^{d-1}((pj+r)^{(p-1)k}-1) \\ = &\sum_{r=1}^{p-1}(-1)^r \sum_{i>0,t\ge0} r^{(p-1)k+d-1-i-t}\tbinom{(p-1)k}{i}\tbinom{d-1}{t}\tfrac{1}{d!}\sum_j(-1)^j\tbinom{d}{pj+r}(pj)^{i+t}. \end{aligned}$$

Since $\binom{(p-1)k}{i} = \frac{(p-1)k}{i}\binom{(p-1)k-1}{i-1}$, we have $\nu_p\binom{(p-1)k}{i} \ge \nu_p(k) - \nu_p(i)$ for $i > 0$. Also

$$\nu_p\Big(\tfrac{1}{d!}\sum_j(-1)^j\tbinom{d}{pj+r}(pj)^{i+t}\Big) \ge \max(0, i+t-\nu_p(d!)),$$

with the first part following from [7, Thm 1.1]. Thus it will suffice to show

$$\lg_p(d) - \nu_p(i) + \max(0, i+t-\nu_p(d!)) \ge 0.$$

This is clearly true if $\nu_p(i) \le \lg_p(d)$, while if $\nu_p(i) > \lg_p(d) = \ell$, then $\nu_p(d!) \le \nu_p((p^{\ell+1}-1)!) = \frac{p^{\ell+1}-1}{p-1} - \ell - 1$ and $i - \nu_p(i) \ge p^{\ell+1} - \ell - 1$, implying the lemma. □

The following lemma is easily proved by induction on A.

Lemma 2.4. *If A and B are positive integers, then*

$$\sum_{i=0}^{A-1}\tbinom{i+B-1}{i}=\tbinom{A+B-1}{B}.$$

Now we can prove Theorem 1.5. We first prove it when $p=2$, and then indicate the minor changes required when p is odd. Using Theorem 2.2 at the first step and Theorem 2.1 at the second, we have

$$\begin{aligned}
&S(2^ea+d-1,2^eb+d)\\
\equiv\ &\sum_{i=2^eb}^{2^ea-1}S(i,2^eb)S(2^ea+d-1-i,d)\mod 2^e\\
\equiv\ &\sum_{j=2^{e-1}b}^{2^{e-1}a-1}\binom{j-2^{e-2}b-1}{j-2^{e-1}b}S(2^ea+d-1-2j,d)\mod 2^{e-1}\\
=\ &\sum_{k=0}^{2^{e-1}(a-b)-1}\binom{k+2^{e-2}b-1}{k}S(2^e(a-b)+d-1-2k,d)\\
=\ &\sum_{\ell=1}^{2^{e-1}(a-b)}\binom{2^{e-2}(2a-b)-1-\ell}{2^{e-2}b-1}S(2\ell+d-1,d)\\
=\ &\sum_{\ell=1}^{2^{e-1}(a-b)}\binom{2^{e-2}(2a-b)-1-\ell}{2^{e-2}b-1}\Big(T_2(2\ell+d-1,d)\pm\frac{1}{d!}\sum_j\binom{d}{2j}(2j)^{2\ell+d-1}\Big).
\end{aligned}$$

We have $\nu_2\binom{2^{e-2}(2a-b)-1-\ell}{2^{e-2}b-1}=f(a,b)+e-\nu_2(\ell)$, where $f(a,b)=\nu_2\binom{2a-b-1}{2a-2b}+\nu_2(a-b)-1$. By [4, Thm 1.5],

$$\nu_2\big(\tfrac{1}{d!}\textstyle\sum\binom{d}{2j}(2j)^{2\ell+d-1}\big)\ge 2\ell+\tfrac{d}{2}-1. \tag{2.5}$$

Thus, using Lemma 2.3 at the first step and Lemma 2.4 at the second, we obtain

$$\begin{aligned}
&S(2^ea+d-1,2^eb+d)\\
\equiv\ &T_2(d-1,d)\sum_{k=0}^{2^{e-1}(a-b)-1}\binom{k+2^{e-2}b-1}{k}\mod 2^{\min(e-1,e+f(a,b)-\lg(d))}\\
=\ &T_2(d-1,d)\binom{2^{e-1}(a-b)+2^{e-2}b-1}{2^{e-2}b}.
\end{aligned}$$

Letting $e\to\infty$ yields the claim of Theorem 1.5. In the congruence, we have also used that $\nu_2(T_2(d-1,d))\ge 0$. In fact, by (2.5) and $S(d-1,d)=0$, we have $\nu_2(T_2(d-1,d))\ge \frac{d}{2}-1$. See Table 2 for some explicit values of $T_2(d-1,d)$.

We now present the minor modifications required when p is odd and $a\equiv b$ mod $(p-1)$. Let $a=b+(p-1)t$. Then

$$\begin{aligned}
&S(p^ea+d-1,p^eb+d)\\
\equiv\ &\sum_{j=0}^{p^et-1}S(p^eb+(p-1)j,p^eb)S(p^e(a-b)-(p-1)j+d-1,d)\\
\equiv\ &\sum_{j=0}^{p^et-1}\binom{p^{e-1}b+j-1}{j}S(p^e(p-1)t-(p-1)j+d-1,d)\\
=\ &\sum_{\ell=1}^{p^et}\binom{p^et+p^{e-1}b-\ell-1}{p^{e-1}b-1}S((p-1)\ell+d-1,d)\\
\equiv\ &T_p(d-1,d)\sum_{j=0}^{p^et-1}\binom{p^{e-1}b+j-1}{j}\\
=\ &T_p(d-1,d)\binom{p^et+p^{e-1}b-1}{p^{e-1}b}.
\end{aligned}$$

3. More formulas and numerical values

In Theorem 1.3, we gave a simple formula for $S(p^\infty a+c,p^\infty b+d)$ when $d\le 0$. For $d>0$, all values can be written explicitly using (1.6) and the initial values given in Theorem 1.5, provided $a\equiv b$ mod $(p-1)$.

First assume $c\ge d-1$. For $i\ge 1$, define Stirling-like numbers $S_i(c,d)$ satisfying that for $d<i$ or $c\le d-1$ the only nonzero value is $S_i(i-1,i)=1$ and satisfying the analogue of (1.6) when $c\ge d$. Note that $S_1(c,d)=S(c,d)$ if $(c,d)\not\in\{(0,0),(0,1)\}$. The following result is easily obtained. Here we use that the binomial coefficient in Theorem 1.5 equals $\frac{p}{p-1}\frac{a-b}{b}S(p^\infty a,p^\infty b)$.

Proposition 3.1. *Assume $a\equiv b$ mod $(p-1)$. For $d\ge 1$, $c\ge d-1$, we have*

$$S(p^\infty a+c,p^\infty b+d)=S(p^\infty a,p^\infty b)\Big(S(c,d)+\sum_{i=1}^{d}S_i(c,d)T_p(i-1,i)\tfrac{p}{p-1}\tfrac{a-b}{b}\Big).$$

The reader may obtain a better feeling for these numbers from the table of values of $S(p^\infty a+c, p^\infty b+d)/S(p^\infty a, p^\infty b)$ in Table 1, in which T_i denotes $T_p(i-1,i)\frac{p}{p-1}\frac{a-b}{b}$.

Table 1. $S(p^\infty a+c, p^\infty b+d)/S(p^\infty a, p^\infty b)$ when $a \equiv b \mod (p-1)$

		d				
		1	2	3	4	5
	0	T_1				
	1	$1+T_1$	T_2			
c	2	$1+T_1$	$1+T_1+2T_2$	T_3		
	3	$1+T_1$	$3+3T_1+4T_2$	$1+T_1+2T_2$ $+3T_3$	T_4	
	4	$1+T_1$	$7+7T_1+8T_2$	$6+6T_1$ $+10T_2+9T_3$	$1+T_1+2T_2$ $+3T_3+4T_4$	T_5
	5	$1+T_1$	$15+15T_1$ $+16T_2$	$25+25T_1$ $+38T_2+27T_3$	$10+10T_1+18T_2$ $+21T_3+16T_4$	$1+T_1+2T_2$ $+3T_3+4T_4$ $+5T_5$

The first few values of $T_2(d-1,d)$ and $T_3(d-1,d)$ are given in Table 2.

Table 2. Some values of $T_2(d-1,d)$ and $T_3(d-1,d)$

d	1	2	3	4	5	6	7	8
$T_2(d-1,d)$	1	-1	2	$-\frac{14}{3}$	12	$-\frac{164}{5}$	$\frac{4208}{5}$	$-\frac{86608}{315}$
$T_3(d-1,d)$	1	0	$-\frac{3}{2}$	$\frac{9}{2}$	$-\frac{27}{4}$	$-\frac{81}{20}$	$\frac{4779}{80}$	$-\frac{15309}{80}$

For $c < d-1$, we use (1.6) to work backwards from $S(p^\infty a+d-1, p^\infty b+d)$, obtaining

Proposition 3.2. *Suppose $a \equiv b \mod (p-1)$. For $k \geq 1$, $d \geq 0$, let $Y(k,d) = S(p^\infty a+d-k, p^\infty b+d)$. Then $Y(1,d)$ is as in Theorem 1.5 for $d \geq 1$, $Y(k,0) = 0$*

for $k \ge 1$*, and, for* $k \ge 2$, $d \ge 1$,

$$Y(k,d) = \big(Y(k-1,d) - Y(k-1,d-1)\big)/d.$$

We illustrate these values in Table 3, where again T_i denotes $T_p(i-1,i)\frac{p}{p-1}\frac{a-b}{b}$.

TABLE 3. $S(p^\infty a + c, p^\infty b + d)/S(p^\infty a, p^\infty b)$ when $a \equiv b$ mod $(p-1)$

		d: 1	2	3	4
	-2	T_1	$\frac18 T_2 - \frac78 T_1$	$\frac{1}{81}T_3 - \frac{65}{648}T_2 + \frac{85}{216}T_1$	$\frac{1}{1024}T_4 - \frac{781}{82944}T_3 + \frac{865}{20736}T_2 - \frac{415}{3456}T_1$
	-1	T_1	$\frac14 T_2 - \frac34 T_1$	$\frac{1}{27}T_3 - \frac{19}{108}T_2 + \frac{11}{36}T_1$	$\frac{1}{256}T_4 - \frac{175}{6912}T_3 + \frac{115}{1728}T_2 - \frac{25}{288}T_1$
c	0	T_1	$\frac12 T_2 - \frac12 T_1$	$\frac19 T_3 - \frac{5}{18}T_2 + \frac16 T_1$	$\frac{1}{64}T_4 - \frac{37}{576}T_3 + \frac{13}{144}T_2 - \frac{1}{24}T_1$
	1		T_2	$\frac13 T_3 - \frac13 T_2$	$\frac{1}{16}T_4 - \frac{7}{48}T_3 + \frac{1}{12}T_2$
	2			T_3	$\frac14 T_4 - \frac14 T_3$

Note that since $S(d-1,d) = 0$ and $T_p(n,k) - S(n,k)$ is a sum like that in (1.4) taken over $i \equiv 0 \bmod p$, we deduce that $T_p(d-1,d) = 0$ if $1 < d < p$, which simplifies these results slightly.

4. THE CASE $a \not\equiv b$ MOD $(p-1)$

In this section, we complete the proof of Theorem 1.1 when $a \not\equiv b \bmod (p-1)$ by proving the following case.

Theorem 4.1. *Suppose* $0 \le b \le a$ *and* $d \ge 1$*. Then the* p*-adic limit of* $S(p^{e+1}a - (a-b), p^{e+1}b + d)$ *exists as* $e \to \infty$*.*

Then $\lim_e S(p^{e+1}a + c, p^{e+1}b + d)$ exists for all integers c by induction using (1.6).

Let $R_p(e)=(p^{e+1}-1)/(p-1)$. The proof of Theorem 4.1 begins with, mod p^e,

$$\begin{aligned}
&S(p^{e+1}a-(a-b),p^{e+1}b+d)\\
\equiv&\sum_{j=0}^{R_p(e)(a-b)}S(p^{e+1}b+(p-1)j,p^{e+1}b)S((p^{e+1}-1)(a-b)-(p-1)j,d)\\
\equiv&\sum_{j=0}^{R_p(e)(a-b)}\binom{p^eb+j-1}{j}\frac{(-1)^d}{d!}\sum_{i=0}^{d}(-1)^i\binom{d}{i}i^{(p^{e+1}-1)(a-b)-(p-1)j}\\
=&\sum_{i=0}^{d}(-1)^{i+d}\frac{1}{d!}\binom{d}{i}\sum_{j=0}^{R_p(e)(a-b)}\binom{p^eb+j-1}{j}i^{(p^{e+1}-1)(a-b)-(p-1)j}.
\end{aligned}$$

We show that for each i, the limit as $e\to\infty$ of

$$\sum_{j=0}^{R_p(e)(a-b)}\binom{p^eb+j-1}{j}i^{(p^{e+1}-1)(a-b)-(p-1)j}\tag{4.2}$$

exists in $\mathbb{Z}_p$. This will complete the proof of the theorem.

If $i\not\equiv 0$ mod p, write $i^{p-1}=Ap+1$, using Fermat's Little Theorem. Then (4.2) becomes

$$\begin{aligned}
&\sum_{\ell=0}^{R_p(e)(a-b)}(Ap)^\ell\sum_{j=0}^{R_p(e)(a-b)}\binom{p^eb+j-1}{j}\binom{R_p(e)(a-b)-j}{\ell}\\
=&\sum_{\ell=0}^{R_p(e)(a-b)}(Ap)^\ell\binom{p^eb+R_p(e)(a-b)}{p^eb+\ell}
\end{aligned}$$

by [6, p.9(3c)]. Lemma 4.5 says that for each ℓ, there exists a p-adic integer

$$z_\ell:=\lim_{e\to\infty}\binom{p^eb+R_p(e)(a-b)}{p^eb+\ell}.$$

Then $\sum_{\ell=0}^{\infty}(Ap)^\ell z_\ell$ is a p-adic integer, which is the limit of (4.2) as $e\to\infty$.

If $i=0$, since $0^0=1$ in (4.2) and the equations preceding it, (4.2) becomes

$$\binom{p^eb+R_p(e)(a-b)-1}{p^eb-1}=\frac{p^eb}{p^eb+R_p(e)(a-b)}\binom{p^eb+R_p(e)(a-b)}{p^eb}.$$

Since by the proof of Lemma 4.5 $\nu_p\binom{p^eb+R_p(e)(a-b)}{p^eb}$ is eventually constant, $\binom{p^eb+R_p(e)(a-b)-1}{p^eb-1}\to 0$ in $\mathbb{Z}_p$, due to the p^eb factor.

We complete the proof of Theorem 4.1 in the following lemma, which shows that the p-adic limit of (4.2) is 0 when $i \equiv 0 \bmod p$ and $i > 0$.

Lemma 4.3. *If* $0 \le j \le R_p(e)(a-b)$*, then*

$$\nu_p\binom{p^e b + j - 1}{j} + (p^{e+1}-1)(a-b) - (p-1)j \ge e - \log_p(a-b+p)$$

for e *sufficiently large.*

Proof. Let $\ell = R_p(e)(a-b) - j$ and $a - b = (p-1)t + \Delta$, $1 \le \Delta \le p-1$. The p-exponent of the binomial coefficient becomes

(4.4)

$$d_p(b-1)+e+d_p((p^{e+1}-1)t+R_p(e)\Delta-\ell)-d_p((p^{e+1}-1)t+R_p(e)\Delta+p^e b-\ell-1).$$

Choose s minimal so that $\frac{\Delta}{p-1}(p^s-1) - \ell - 1 - t \ge 0$. Then, if $e > s$, the p-ary expansion of $(p^{e+1}-1)t + R_p(e)\Delta - \ell$ splits as

$$p^e(pt+\Delta) \quad + \quad p^s\frac{p^{e-s}-1}{p-1}\Delta \quad + \quad \frac{p^s-1}{p-1}\Delta - \ell - t,$$

and there is a similar splitting for the expression at the end of (4.4). We obtain that (4.4) equals

$$e + \nu_p(b) + \nu_p\tbinom{pt+b+\Delta}{b} - \nu_p\big(\tfrac{\Delta}{p-1}(p^s-1) - \ell - t\big).$$

The expression in the lemma equals this plus $(p-1)\ell$. Since s was minimal, we have $\frac{\Delta}{p-1}(p^s-1)-\ell-t \le (p-1)(\ell+t)+p+\Delta$, and hence $\nu_p(\frac{\Delta}{p-1}(p^s-1)-\ell-t) \le \log_p((p-1)(\ell+t)+p+\Delta)$. The smallest value of $(p-1)\ell - \log_p((p-1)(\ell+t)+p+\Delta)$ occurs when $\ell = 0$. We obtain that the expression in the lemma is $\ge e - \log_p(a-b+p)$. $\square$

The following lemma was referred to above.

Lemma 4.5. *If* α *and* b *are positive integers and* $\ell \ge 0$*, then*

$$\lim_{e\to\infty}\binom{p^e b + R_p(e)\alpha}{p^e b + \ell}$$

exists in $\mathbb{Z}_p$*.*

This is another p-adic binomial coefficient, slightly different than those of [3], which we would call $\binom{p^\infty b + R_p(\infty)\alpha}{P^\infty b + \ell}$. The proof of the lemma breaks into two parts: showing that the p-exponents are eventually constant, and showing that the unit parts approach a limit.

The proof that the p-exponent is eventually constant is very similar to the proof of Lemma 4.3. Let $\alpha = (p-1)t + \Delta$ with $1 \le \Delta \le p-1$, and choose s minimal such that $\frac{\Delta}{p-1}(p^s-1) - t - \ell \ge 0$. Then the p-ary expansions split again into three parts and we obtain that for $e > s$, the desired p-exponent equals $\nu_p\binom{pt+b+\Delta}{b} + \nu_p\binom{\Delta(p^s-1)/(p-1)-t}{\ell}$, independent of e.

We complete the proof of Lemma 4.5 by showing that, if $\ell < \min(R_p(e-1)\alpha, p^e b)$ and $p^e > \alpha$, then

(4.6)
$$\mathrm{U}_p\binom{p^{e-1}b + R_p(e-1)\alpha}{p^{e-1}b+\ell} \equiv \mathrm{U}_p\binom{p^e b + R_p(e)\alpha}{p^e b+\ell} \mod p^{e+f(\alpha,b,\ell)-1},$$

where $f(\alpha,b,\ell) = \min(\nu_p(b) - \mathrm{lg}_p(\alpha), \nu_p(\alpha) - \mathrm{lg}_p(\ell), \nu_p(b) - \mathrm{lg}_p(\ell), 1)$. We write the second binomial coefficient in (4.6) as

$$(4.7) \qquad (-1)^{eb}\frac{(p^e b + R_p(e)\alpha)!}{(R_p(e)\alpha)!}\cdot\frac{(R_p(e)\alpha)!}{(R_p(e)\alpha-\ell)!}\cdot\frac{(p^e b)!}{(p^e b+\ell)!}\cdot\frac{(-1)^{eb}}{(p^e b)!}.$$

We show that the unit parts of these four factors are congruent to their $(e-1)$-analogue mod $p^{e+\nu_p(b)-\mathrm{lg}(\alpha)-1}$, $p^{e+\nu_p(\alpha)-\mathrm{lg}_p(\ell)-1}$, $p^{e+\nu_p(b)-\mathrm{lg}_p(\ell)-1}$, and p^e, respectively, which will imply the result. For the fourth factor, this was shown in [3]. For the second and third, the claim is clear, since each of the ℓ unit factors being multiplied will be congruent to their $(e-1)$-analogue modulo the specified amount.

To prove the first, we will prove

(4.8)
$$\mathrm{U}_p\left(\frac{(R_p(e)\alpha+1)\cdots(R_p(e)\alpha+p^e b)}{(R_p(e-1)\alpha+1)\cdots(R_p(e-1)\alpha+p^{e-1}b)}\right) \equiv (-1)^b \mod p^{e+\nu_p(b)-\mathrm{lg}_p(\alpha)-1}.$$

Since $\mathrm{U}_p(j) = \mathrm{U}_p(pj)$, we may cancel most multiples of p in the numerator with factors in the denominator. Using that $p\cdot R_p(e-1) = R_p(e)-1$, we obtain that the LHS of (4.8) equals $P\,\mathrm{U}_p(A)/\,\mathrm{U}_p(B)$, where P is the product of the units in the numerator, A is the product of all $j \equiv 0 \bmod p$ which satisfy

$$(R_p(e)-1)\alpha + p^e b < j \le R_p(e)\alpha + p^e b,$$

and B is the product of all integers k such that

$$(4.9) \qquad R_p(e-1)\alpha + 1 \le k \le R_p(e-1)\alpha + \left[\tfrac{\alpha}{p}\right].$$

Since the mod p^e values of the p-adic units in any interval of p^e consecutive integers are just a permutation of the set of positive p-adic units less than p^e, and by [5, Lemma 1] the product of these is -1 mod p^e, we obtain $P \equiv (-1)^b \mod p^e$. Thus (4.8) reduces to showing $\mathrm{U}_p(A)/\mathrm{U}_p(B) \equiv 1 \mod p^{e+\nu_p(b)-\mathrm{lg}_p(\alpha)-1}$.

We have
$$\frac{\mathrm{U}_p(A)}{\mathrm{U}_p(B)} = \prod \frac{\mathrm{U}_p(k+p^{e-1}b)}{\mathrm{U}_p(k)},$$
taken over all k satisfying (4.9). We show that if k satisfies (4.9), then

$$\nu_p(k) \le \mathrm{lg}_p(\alpha). \tag{4.10}$$

Then $\mathrm{U}_p(k) \equiv \mathrm{U}_p(k+p^{e-1}b) \mod p^{e+\nu_p(b)-\mathrm{lg}_p(\alpha)-1}$, establishing the result.

We prove (4.10) by showing that it is impossible to have $1 \le \alpha < p^t$, $1 \le i \le [\frac{\alpha}{p}]$, $t < e$, and

$$R_p(e-1)\alpha + i \equiv 0 \mod p^t. \tag{4.11}$$

From (4.11) we deduce $\alpha \equiv i(p-1) \mod p^t$. But $i(p-1) < \alpha$, so the only way to satisfy (4.11) would be with $\alpha = p^t$ and $i = 0$, but $\alpha < p^t$.

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